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SOME COEFFICIENT BOUNDS OF CERTAIN SUBCLASSES OF BI-UNIVALENT FUNCTIONS ASSOCIATED WITH LEMNISCATE OF BERNOULLI

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Abstract. In this current article, we introduced few subclasses of bi-univalent functions related to lemniscate of Bernoulli within the open unit disk $\mathbb{D}_0$. We investigate the estimates of the Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$, as well as the Fekete–Szezö functional problems $|a_3-\lambda a_2^2|$, for functions that fall within each of the these bi-univalent function classes. Furthermore, for special cases, corollaries are stated which some of them are new and have not been studied so far.

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1. Introduction

Let $\mathcal{H}$ represent the set of all holomorphic functions defined within the unit disk $\mathbb{D}_0$, which is expressed as $\mathbb{D}_0 := \{\varsigma : |\varsigma| < 1\}$. Additionally, let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ consisting of functions $f \in \mathcal{A}$ that can be expressed in the form

$$f(\varsigma) := \varsigma + \sum_{k=2}^{\infty} a_k \varsigma^k, \tag{1.1}$$

where $\varsigma \in \mathbb{D}_0$, and is subject to the normalization conditions $f(0) = f'(0) - 1 = 0$. Furthermore, let $\mathcal{S}$ be the subclass of $\mathcal{A}$ that includes univalent functions. Robertson [1] has introduced two well-known subclasses of $\mathcal{A}$, which are defined, for any $\delta \in [0, 1)$, as

$$\mathcal{S}^*(\delta) := \left\{ f \in \mathcal{A} : \Re\left(\frac{\varsigma f'(\varsigma)}{f(\varsigma)}\right) > \delta, \text{ for all } \varsigma \in \mathbb{D}_0 \right\},$$

$$\mathcal{C}(\delta) := \left\{ f \in \mathcal{A} : \Re\left(1 + \frac{\varsigma f''(\varsigma)}{f'(\varsigma)}\right) > \delta, \text{ for all } \varsigma \in \mathbb{D}_0 \right\},$$

and are referred to as starlike and convex functions of order δ. It is established that $\mathcal{S}^*(\delta)$ is a subset of $\mathcal{S}$, and $\mathcal{C}(\delta)$ is also a subset of $\mathcal{S}$. According to Alexander's relation, $f \in \mathcal{C}(\delta)$ if and only if $\varsigma f'(\varsigma) \in \mathcal{S}^*(\delta)$ for ς within the unit disk $\mathbb{D}_0$. When $\delta = 0$, the class $\mathcal{S}^*$, defined as $\mathcal{S}^*(0)$, simplifies to the well-known category of normalized starlike univalent functions, while $\mathcal{C}$, defined as $\mathcal{C}(0)$, corresponds to the normalized convex univalent functions.

A function $f(\varsigma)$ represented in the form (1.1) is classified as a starlike function with respect to symmetrical points if

$$\Re\left(\frac{2\varsigma f'(\varsigma)}{f(\varsigma) - f(-\varsigma)}\right) > 0, \quad \varsigma \in \mathbb{D}_0.$$

Let us define the set of all such functions as $\mathcal{S}_s^*$. According to Sakaguchi [2], if $f(\varsigma) \in \mathcal{S}_s^*$ and takes the form (1.1), it can be concluded that $|a_k| \leq 1$ for $k = 2, 3, \ldots$. It is evident that the class of starlike functions with respect to symmetrical points includes the class of convex functions with respect to symmetrical points, referred to as $\mathcal{C}_s$, which satisfies the following condition:

$$\Re\left(\frac{(\varsigma f'(\varsigma))'}{(f(\varsigma) - f(-\varsigma))'}\right) > 0, \quad \varsigma \in \mathbb{D}_0.$$

It is clear that for the classes $\mathcal{S}_s^*$ and $\mathcal{C}_s$, the Alexander relation is satisfied, specifically $f(\varsigma) \in \mathcal{C}_s$ if and only if $\varsigma f'(\varsigma) \in \mathcal{S}_s^*$.

Consider functions f and $\hbar$ that are analytic in $\mathbb{D}_0$. We say that f is subordinate to $\hbar$, represented as $f \prec \hbar$ in $\mathbb{D}_0$ or $f(\varsigma) \prec \hbar(\varsigma)$ for ς in $\mathbb{D}_0$, if there exists an analytic function κ defined in $\mathbb{D}_0$ with $\kappa(0) = 0$ and $|\kappa(\varsigma)| < 1$, such that $f(\varsigma)$ can be expressed as $f(\varsigma) = \hbar(\kappa(\varsigma))$ for all ς in $\mathbb{D}_0$. Consequently,

$$f(\varsigma) \prec \hbar(\varsigma), \quad \varsigma \in \mathbb{D}_0 \Rightarrow \quad f(0) = \hbar(0) \quad \textit{and} \quad f(\mathbb{D}_0) \subset \hbar(\mathbb{D}_0).$$

Notably, if $\hbar$ is univalent in $\mathbb{D}_0$, the following equivalence holds:

$$f(\varsigma) \prec \hbar(\varsigma), \quad \varsigma \in \mathbb{D}_0 \Leftrightarrow \quad f(0) = \hbar(0) \quad \textit{and} \quad f(\mathbb{D}_0) \subset \hbar(\mathbb{D}_0).$$

Based on the Koebe One-Quarter Theorem, every function $f \in \mathcal{S}$ has an inverse f^{-1} which complies with the following conditions:

$$f^{-1}(f(\varsigma)) = \varsigma, \quad \varsigma \in \mathbb{D}_0$$

and

$$f(f^{-1}(\vartheta)) = \vartheta, \quad \left(|\vartheta| < r_0(f); r_0(f) \geq \frac{1}{4}\right),$$

where

$$h(\vartheta) := f^{-1}(\vartheta) = \vartheta - a_2\vartheta^2 + (2a_2^2 - a_3)\vartheta^3 - (5a_2^2 - 5a_2a_3 + a_4)\vartheta^4 + \cdots . \tag{1.2}$$

A function $f \in \mathcal{A}$ is identified as bi-univalent in the area $\mathbb{D}_0$ if it satisfies the condition that both f and its inverse f^{-1} are univalent in $\mathbb{D}_0$. The set of bi-univalent functions in $\mathbb{D}_0$ is indicated by Σ, according to (1.2). For a brief overview of the history and notable examples of functions classified under Σ, see [3] and [4]. The concept of bi-univalent functions was first presented by Lewin [5] in 1967, who established an estimate for the second coefficient of functions within this category, stating that $|a_2| < 1.51$. This finding was subsequently refined by Brannan and Clunie [6], who demonstrated that $|a_2| \leq \sqrt{2}$. A variety of researchers have examined several captivating special families of Σ, as noted in [7, 8, 9, 10]. In contrast, Netanyahu [11] revealed that the maximum of $f \in \Sigma$, $|a_2| = 4/3$. The task of estimating the coefficient for each Taylor–Maclaurin coefficient $|a_k|$, for $k \in \mathbb{N}$ and $k \geq 3$ is still regarded as an open question. Another property that is widely researched in the context of the coefficient problems for $f \in \mathcal{A}$ is the Fekete-Szegö [12] functional, which is expressed as

$$|a_3 - \lambda a_2^2|, \quad \lambda \in \mathbb{R}.$$

Sokół and Thomas [13] presented and examined the class $\mathcal{S}_L^*$ within the unit disc $\mathbb{D}_0$. A function $f \in \mathcal{S}_L^*$, must satisfy the condition

$$\mathcal{G}(\varsigma) := \frac{\varsigma f'(\varsigma)}{f(\varsigma)} \prec \sqrt{1+\varsigma} = \xi(\varsigma),$$

with the square root branch selected such that $\xi(0) = 1$. The function $\mathcal{G}$ is situated in the domain defined by the right half of the lemniscate of Bernoulli, which is geometrically illustrated by the condition $|\mathcal{G}^2 - 1| < 1$ for all ς belonging to $\mathbb{D}_0$. Descriptive diagrams and further insights into the domain $|\mathcal{G}^2 - 1| < 1$ are available in [14]. It was also observed that the set $\xi(\mathbb{D}_0)$ is located within the area enclosed by the right loop of the Lemniscate of Bernoulli, denoted as $\Gamma : (x^2 + y^2)^2 - 2(x^2 - y^2) = 0$.

Define $\mathcal{P}$ as the set of functions ℓ belonging to $\mathcal{H}$, characterized by the normalization condition $\ell(0) = 1$. Such functions can be represented as

$$\ell(\varsigma) = 1 + \sum_{k=1}^{\infty} \ell_k \varsigma^k = 1 + \ell_1\varsigma + \ell_2\varsigma^2 + \ell_3\varsigma^3 + \cdots, \tag{1.3}$$

and it is essential that $\Re(\ell(\varsigma)) > 0$ for all ς in $\mathbb{D}_0$. In this context, $\ell(\varsigma)$ is referred to as a Carathéodory function. It is established that there is a relationship between the class $\mathcal{P}$ and the class of Schwarz functions κ, specifically that $\ell \in \mathcal{P}$ if and only if $\ell(\varsigma)$ can be expressed as $(1 + \kappa(\varsigma))/(1 - \kappa(\varsigma))$.

Lemma 1.1. [15, 16] *For $\ell \in \mathcal{P}$ represented as (1.3), the inequality*

$$|\ell_k| \le 2,\ k \ge 1$$

is satisfied, and this condition is sharp for each $k \in \mathbb{N}$.

Lemma 1.2. [17] *Establish that $\beta, \gamma \in \mathbb{R}$ and $\varsigma_1, \varsigma_2 \in \mathbb{C}$, with $|\varsigma_1| < B$, $|\varsigma_2| < B$, then*

$$|(\beta+\gamma)\varsigma_1 + (\beta-\gamma)\varsigma_2| \le \begin{cases} 2|\beta|B, & \text{if } |\beta| \ge |\gamma|, \\ 2|\gamma|B, & \text{if } |\beta| \le |\gamma|. \end{cases}$$

In this current article, we introduced few subclasses of bi-univalent functions related to lemniscate of Bernoulli within the open unit disk $\mathbb{D}_0$. We investigate the estimates of the Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$, as well as the Fekete–Szezö functional problems $|a_3 - \lambda a_2^2|$, for functions that fall within each of the these bi-univalent function classes. Furthermore, for special cases, corollaries are stated which some of them are new and have not been studied so far.

2. New Families of Analytic Functions

Sahoo and Patel [18] established the class $\tilde{\mathcal{R}}$ based on the Lemniscate of Bernoulli. A function $f(\varsigma)$ belonging to the class $\mathcal{A}$ is classified as part of the $\tilde{\mathcal{R}}$ class if and only if

$$\left|\left[f'(\varsigma)\right]^2 - 1\right| < 1. \tag{2.1}$$

Equivalently, based on equations (2.1), along with the definition of subordination, a function $f \in \tilde{\mathcal{R}}$ satisfies the following subordination conditions:

$$f'(\varsigma) \prec \xi(\varsigma).$$

The class $\tilde{\mathcal{R}}$ contains univalent functions in $\mathbb{D}_0$, so it contains the bi-univalent functions in the class $\mathcal{R}_\Sigma(\xi)$. Utilizing Bernoulli's Lemniscate, we have established a few new subclasses of bi-univalent functions.

Definition 2.1. A function $f(\varsigma)$ belonging to the class Σ is classified as part of the $\mathcal{R}_\Sigma(\xi)$ class if and only if

$$\left|\left[f'(\varsigma)\right]^2 - 1\right| < 1 \tag{2.2}$$

and

$$\left|\left[h'(\vartheta)\right]^2 - 1\right| < 1. \tag{2.3}$$

Equivalently, based on equations (2.2) and (2.3), along with the definition of subordination, a function $f \in \mathcal{R}_\Sigma(\xi)$ satisfies the following subordination conditions:

$$f'(\varsigma) \prec \xi(\varsigma)$$

and

$$h'(\vartheta) \prec \xi(\vartheta).$$

In this context, the function h is the inverse of the function f, as specified in equation (1.2).

Definition 2.2. Establish that $0 \le \alpha \le 1$. A function $f(\varsigma)$ belonging to the class Σ is classified as part of the $\mathcal{M}_\Sigma^\alpha(\xi)$ class if and only if

$$\left|\left[(1-\alpha)\frac{\varsigma f'(\varsigma)}{f(\varsigma)} + \alpha\left(1 + \frac{\varsigma f''(\varsigma)}{f'(\varsigma)}\right)\right]^2 - 1\right| < 1 \tag{2.4}$$

and

$$\left|\left[(1-\alpha)\frac{\vartheta h'(\vartheta)}{h(\vartheta)} + \alpha\left(1 + \frac{\vartheta h''(\vartheta)}{h'(\vartheta)}\right)\right]^2 - 1\right| < 1. \tag{2.5}$$

Equivalently, based on equations (2.4) and (2.5), along with the definition of subordination, a function $f \in \mathcal{M}_\Sigma^\alpha(\xi)$ satisfies the following subordination conditions:

$$(1-\alpha)\frac{\varsigma f'(\varsigma)}{f(\varsigma)} + \alpha\left(1 + \frac{\varsigma f''(\varsigma)}{f'(\varsigma)}\right) \prec \xi(\varsigma)$$

and

$$(1-\alpha)\frac{\vartheta h'(\vartheta)}{h(\vartheta)} + \alpha\left(1 + \frac{\vartheta h''(\vartheta)}{h'(\vartheta)}\right) \prec \xi(\vartheta).$$

In this context, the function h is the inverse of the function f, as specified in equation (1.2).

Remark 2.3. (i) If $\alpha = 0$, in Definition 2.2, then $\mathcal{M}_\Sigma^\alpha(\xi) \equiv \mathcal{M}_\Sigma^0(\xi) \equiv \mathcal{S}_\Sigma^*(\xi)$. A function $f(\varsigma)$ belonging to the class Σ is classified as part of the $\mathcal{S}_\Sigma^*(\xi)$ class if and only if

$$\left|\left[\frac{\varsigma f'(\varsigma)}{f(\varsigma)}\right]^2 - 1\right| < 1 \tag{2.6}$$

and

$$\left|\left[\frac{\vartheta h'(\vartheta)}{h(\vartheta)}\right]^2 - 1\right| < 1. \tag{2.7}$$

Equivalently, based on equations (2.6) and (2.7), along with the definition of subordination, a function $f \in \mathcal{S}^*_\Sigma(\xi)$ satisfies the following subordination conditions:

$$\frac{\varsigma f'(\varsigma)}{f(\varsigma)} \prec \xi(\varsigma)$$

and

$$\frac{\vartheta h'(\vartheta)}{h(\vartheta)} \prec \xi(\vartheta).$$

In this context, the function h is the inverse of the function f, as specified in equation (1.2).

(ii) If $\alpha = 1$, in Definition 2.2, then $\mathcal{M}^\alpha_\Sigma(\xi) \equiv \mathcal{M}^1_\Sigma(\xi) \equiv \mathcal{C}_\Sigma(\xi)$. A function $f(\varsigma)$ belonging to the class Σ is classified as part of the $\mathcal{C}_\Sigma(\xi)$ class if and only if

$$\left| \left[1 + \frac{\varsigma f''(\varsigma)}{f'(\varsigma)} \right]^2 - 1 \right| < 1 \tag{2.8}$$

and

$$\left| \left[1 + \frac{\vartheta h''(\vartheta)}{h'(\vartheta)} \right]^2 - 1 \right| < 1. \tag{2.9}$$

Equivalently, based on equations (2.8) and (2.9), along with the definition of subordination, a function $f \in \mathcal{C}_\Sigma(\xi)$ satisfies the following subordination conditions:

$$1 + \frac{\varsigma f''(\varsigma)}{f'(\varsigma)} \prec \xi(\varsigma)$$

and

$$1 + \frac{\vartheta h''(\vartheta)}{h'(\vartheta)} \prec \xi(\vartheta).$$

In this context, the function h is the inverse of the function f, as specified in equation (1.2).

Definition 2.4. Establish that $0 \le \mu \le 1$. A function $f(\varsigma)$ belonging to the class Σ is classified as part of the $\mathcal{LS}^{*,\mu}_{s,\Sigma}(\xi)$ class if and only if

$$\left| \left[(1-\mu)\frac{2\varsigma f'(\varsigma)}{f(\varsigma) - f(-\varsigma)} + \mu \frac{2[\varsigma f'(\varsigma)]'}{[f(\varsigma) - f(-\varsigma)]'} \right]^2 - 1 \right| < 1 \tag{2.10}$$

and

$$\left| \left[(1-\mu)\frac{2\vartheta h'(\vartheta)}{h(\vartheta) - h(-\vartheta)} + \mu \frac{2[\vartheta h'(\vartheta)]'}{[h(\vartheta) - h(-\vartheta)]'} \right]^2 - 1 \right| < 1. \tag{2.11}$$

Equivalently, based on equations (2.10) and (2.11), along with the definition of subordination, a function $f \in \mathcal{LS}^{*,\mu}_{s,\Sigma}(\xi)$ satisfies the following subordination conditions:

$$(1-\mu)\frac{2\varsigma f'(\varsigma)}{f(\varsigma) - f(-\varsigma)} + \mu \frac{2[\varsigma f'(\varsigma)]'}{[f(\varsigma) - f(-\varsigma)]'} \prec \xi(\varsigma)$$

and

$$(1-\mu)\frac{2\vartheta h'(\vartheta)}{h(\vartheta) - h(-\vartheta)} + \mu \frac{2[\vartheta h'(\vartheta)]'}{[h(\vartheta) - h(-\vartheta)]'} \prec \xi(\vartheta).$$

In this context, the function h is the inverse of the function f, as specified in equation (1.2).

Remark 2.5. (i) If $\mu = 0$, in Definition 2.4, then $\mathcal{LS}^{*,\mu}_{s,\Sigma}(\xi) \equiv \mathcal{LS}^{*,0}_{s,\Sigma}(\xi) \equiv \mathcal{S}^{*}_{s,\Sigma}(\xi)$. A function $f(\varsigma)$ belonging to the class Σ is classified as part of the $\mathcal{S}^{*}_{s,\Sigma}(\xi)$ class if and only if

$$\left|\left[\frac{2\varsigma f'(\varsigma)}{f(\varsigma) - f(-\varsigma)}\right]^2 - 1\right| < 1 \tag{2.12}$$

and

$$\left|\left[\frac{2\vartheta h'(\vartheta)}{h(\vartheta) - h(-\vartheta)}\right]^2 - 1\right| < 1. \tag{2.13}$$

Equivalently, based on equations (2.12) and (2.13), along with the definition of subordination, a function $f \in \mathcal{S}^{*}_{s,\Sigma}(\xi)$ satisfies the following subordination conditions:

$$\frac{2\varsigma f'(\varsigma)}{f(\varsigma) - f(-\varsigma)} \prec \xi(\varsigma)$$

and

$$\frac{2\vartheta h'(\vartheta)}{h(\vartheta) - h(-\vartheta)} \prec \xi(\vartheta).$$

In this context, the function h is the inverse of the function f, as specified in equation (1.2).

(ii) If $\mu = 1$, in Definition 2.4, then $\mathcal{LS}^{*,\mu}_{s,\Sigma}(\xi) \equiv \mathcal{LS}^{*,1}_{s,\Sigma}(\xi) \equiv \mathcal{C}_{s,\Sigma}(\xi)$. A function $f(\varsigma)$ belonging to the class Σ is classified as part of the $\mathcal{C}_{s,\Sigma}(\xi)$ class if and only if

$$\left|\left[\frac{2[\varsigma f'(\varsigma)]'}{[f(\varsigma) - f(-\varsigma)]'}\right]^2 - 1\right| < 1 \tag{2.14}$$

and

$$\left|\left[\frac{2[\vartheta h'(\vartheta)]'}{[h(\vartheta) - h(-\vartheta)]'}\right]^2 - 1\right| < 1. \tag{2.15}$$

Equivalently, based on equations (2.14) and (2.15), along with the definition of subordination, a function $f \in \mathcal{C}_{s,\Sigma}(\xi)$ satisfies the following subordination conditions:

$$\frac{2[\varsigma f'(\varsigma)]'}{[f(\varsigma) - f(-\varsigma)]'} \prec \xi(\varsigma)$$

and

$$\frac{2[\vartheta h'(\vartheta)]'}{[h(\vartheta) - h(-\vartheta)]'} \prec \xi(\vartheta).$$

In this context, the function h is the inverse of the function f, as specified in equation (1.2).

3. Coefficient Estimates and Fekete-Szegö Functional for the class $\mathcal{R}_\Sigma(\xi)$.

Theorem 3.1. *Consider the function $f(\varsigma)$ defined by (1.1) that belongs to the class $\mathcal{R}_\Sigma(\xi)$, then*

$$|a_2| \leq \frac{1}{\sqrt{26}} \approx 0.1961\ldots, \tag{3.1}$$

$$|a_3| \leq \frac{1}{6} \tag{3.2}$$

and

$$|a_3-\lambda a_2^2| \leq \begin{cases} \frac{1}{6}, & \text{if } \lambda \in \left[-\frac{10}{3}, \frac{16}{3}\right], \\ \frac{|1-\lambda|}{26}, & \text{if } \lambda \in \left(-\infty, -\frac{10}{3}\right) \cup \left(\frac{16}{3}, \infty\right). \end{cases} \tag{3.3}$$

Proof. If the function $f(\varsigma)$ is a member of the class $\mathcal{R}_\Sigma(\xi)$, then it follows:

$$f'(\varsigma) = \xi(\kappa_1(\varsigma)) \tag{3.4}$$

and

$$h'(\vartheta) = \xi(\kappa_2(\vartheta)), \tag{3.5}$$

where κ_1 and κ_2 are schwarz functions $\kappa_1(0) = \kappa_2(0) = 0$ and $|\kappa_1(\varsigma)| < 1$ and $|\kappa_2(\vartheta)| < 1$. Subsequently, utilizing the definition of class $\mathcal{P}$, we can derive the corresponding relation

$$\kappa_1(\varsigma) = \frac{u(\varsigma)-1}{u(\varsigma)+1} \quad \text{and} \quad \kappa_2(w) = \frac{s(\vartheta)-1}{s(\vartheta)+1},$$

where

$$u(\varsigma) = 1 + u_1\varsigma + u_2\varsigma^2 + u_3\varsigma^3 + \cdots \in \mathcal{P}$$

and

$$s(\vartheta) = 1 + s_1\vartheta + s_2\vartheta^2 + s_3\vartheta^3 + \cdots \in \mathcal{P}.$$

Therefore,

$$\xi(\kappa_1(\varsigma)) = \left(\frac{2u(\varsigma)}{u(\varsigma)+1}\right)^{\frac{1}{2}} = 1 + \frac{1}{4}u_1\varsigma + \left(\frac{1}{4}u_2 - \frac{5}{32}u_1^2\right)\varsigma^2 + \left(\frac{1}{4}u_3 - \frac{5}{16}u_1u_2 + \frac{13}{128}u_1^3\right)\varsigma^3 + \cdots \tag{3.6}$$

and

$$\xi(\kappa_2(\vartheta)) = \left(\frac{2s(\vartheta)}{s(\vartheta)+1}\right)^{\frac{1}{2}} = 1 + \frac{1}{4}s_1\vartheta + \left(\frac{1}{4}s_2 - \frac{5}{32}s_1^2\right)\vartheta^2 + \left(\frac{1}{4}s_3 - \frac{5}{16}s_1s_2 + \frac{13}{128}s_1^3\right)\vartheta^3 + \cdots. \tag{3.7}$$

Since,

$$f'(\varsigma) = 1 + 2a_2\varsigma + 3a_3\varsigma^2 + 4a_4\varsigma^3 + \cdots \tag{3.8}$$

and

$$h'(\vartheta) = 1 - 2a_2\vartheta + (6a_2^2 - 3a_3)\vartheta^2 - (20a_2^2 - 20a_2a_3 + 4a_4)\vartheta^3 + \cdots. \tag{3.9}$$

It follows from (3.6) and (3.8) that

$$2a_2 = \frac{1}{4}u_1 \tag{3.10}$$

and

$$3a_3 = \frac{1}{4}u_2 - \frac{5}{32}u_1^2. \tag{3.11}$$

Similarly, it follows from (3.7) and (3.9) that

$$-2a_2 = \frac{1}{4}s_1 \tag{3.12}$$

and

$$6a_2^2 - 3a_3 = \frac{1}{4}s_2 - \frac{5}{32}s_1^2. \tag{3.13}$$

It can be concluded from (3.10) and (3.12) that

$$u_1 + s_1 = 0 \tag{3.14}$$

and

$$a_2^2 = \frac{u_1^2 + s_1^2}{128}. \tag{3.15}$$

By summing the equalities (3.11) and (3.13), we get

$$6a_2^2 = \frac{u_2 + s_2}{4} - \frac{5}{32}(u_1^2 + s_1^2). \tag{3.16}$$

Substituting the value of $u_1^2 + s_1^2$ from (3.15) in (3.16), we get

$$a_2^2 = \frac{u_2 + s_2}{104}. \tag{3.17}$$

By applying Lemma 1.1, in conjunction with the triangle inequality in the equations (3.15) and (3.17), it can be concluded that

$$|a_2| \leq \frac{1}{4} \quad \textit{and} \quad |a_2| \leq \frac{1}{\sqrt{26}}. \tag{3.18}$$

This validates the initial findings presented in (3.1). In addition, subtracting (3.13) from (3.11), yields

$$6a_3 - 6a_2^2 = \frac{u_2 - s_2}{4} + \frac{5}{32}(s_1^2 - u_1^2). \tag{3.19}$$

The above relation (3.19) combined with (3.14) and (3.17), leads that

$$a_3 = \frac{16u_2 - 10s_2}{312}. \tag{3.20}$$

By applying Lemma 1.1, in conjunction with the triangle inequality in the equation (3.20), it can be concluded that

$$|a_3| \leq \frac{1}{6}. \tag{3.21}$$

This validates the initial findings presented in (3.2). From equations (3.17) and (3.20), we get

$$a_3 - \lambda a_2^2 = \left(\frac{1}{24} + g(\lambda)\right) u_2 - \left(\frac{1}{24} - g(\lambda)\right) s_2, \tag{3.22}$$

where

$$g(\lambda) = \frac{1 - \lambda}{104}.$$

In view of Lemma 1.1 and Lemma 1.2, in equation (3.22), we get

$$|a_3 - \lambda a_2^2| \leq \begin{cases} \frac{1}{6}, & \text{if } |g(\lambda)| \leq \frac{1}{24}, \\ 4|g(\lambda)|, & \text{if } |g(\lambda)| \geq \frac{1}{24}. \end{cases} \tag{3.23}$$

This validates the initial findings presented in (3.3), which completes the proof of Theorem 3.1. □

4. Coefficient Estimates and Fekete-Szegö Functional for the class $\mathcal{M}_{\Sigma}^{\alpha}(\xi)$.

Theorem 4.1. *Establish that $0 \leq \alpha \leq 1$. Consider the function $f(\varsigma)$ defined by (1.1) that belongs to the class $\mathcal{M}_{\Sigma}^{\alpha}(\xi)$, then*

$$|a_2| \leq \frac{1}{\sqrt{(1+\alpha)(7+5\alpha)}}, \tag{4.1}$$

$$|a_3| \leq \frac{1}{4(1+2\alpha)} \tag{4.2}$$

and

$$|a_3 - \lambda a_2^2| \leq \begin{cases} \dfrac{1}{4(1+2\alpha)}, & \text{if } \lambda \in \left[-\dfrac{3+4\alpha+5\alpha^2}{4(1+2\alpha)}, \dfrac{11+20\alpha+5\alpha^2}{4(1+2\alpha)}\right], \\ \dfrac{|1-\lambda|}{(1+\alpha)(7+5\alpha)}, & \text{if } \lambda \in \left(-\infty, -\dfrac{3+4\alpha+5\alpha^2}{4(1+2\alpha)}\right) \cup \left(\dfrac{11+20\alpha+5\alpha^2}{4(1+2\alpha)}, \infty\right). \end{cases} \tag{4.3}$$

Proof. If the function $f(\varsigma)$ is a member of the class $\mathcal{M}_{\Sigma}^{\alpha}(\xi)$, then it follows:

$$(1-\alpha)\frac{\varsigma f'(\varsigma)}{f(\varsigma)} + \alpha\left(1 + \frac{\varsigma f''(\varsigma)}{f'(\varsigma)}\right) = \xi(\kappa_1(\varsigma)) \tag{4.4}$$

and

$$(1-\alpha)\frac{\vartheta h'(\vartheta)}{h(\vartheta)} + \alpha\left(1 + \frac{\vartheta h''(\vartheta)}{h'(\vartheta)}\right) = \xi(\kappa_2(\vartheta)), \tag{4.5}$$

Since,

$$(1-\alpha)\frac{\varsigma f'(\varsigma)}{f(\varsigma)} + \alpha\left(1 + \frac{\varsigma f''(\varsigma)}{f'(\varsigma)}\right) = 1 + (1+\alpha)a_2\varsigma + [2(1+2\alpha)a_3 - (1+3\alpha)a_2^2]\varsigma^2 + \cdots \tag{4.6}$$

and

$$(1-\alpha)\frac{\vartheta h'(\vartheta)}{h(\vartheta)} + \alpha\left(1 + \frac{\vartheta h''(\vartheta)}{h'(\vartheta)}\right) = 1 - (1+\alpha)a_2\vartheta + [(3+5\alpha)a_2^2 - 2(1+2\alpha)a_3]\vartheta^2 + \cdots. \tag{4.7}$$

It follows from (3.6) and (4.6) that

$$(1+\alpha)a_2 = \frac{1}{4}u_1 \tag{4.8}$$

and

$$2(1+2\alpha)a_3 - (1+3\alpha)a_2^2 = \frac{1}{4}u_2 - \frac{5}{32}u_1^2. \tag{4.9}$$

Similarly, it follows from (3.7) and (4.7) that

$$-(1+\alpha)a_2 = \frac{1}{4}s_1 \tag{4.10}$$

and

$$(3+5\alpha)a_2^2 - 2(1+2\alpha)a_3 = \frac{1}{4}s_2 - \frac{5}{32}s_1^2. \tag{4.11}$$

It can be concluded from (4.8) and (4.10) that

$$u_1 + s_1 = 0 \tag{4.12}$$

and

$$a_2^2 = \frac{u_1^2 + s_1^2}{32(1+\alpha)^2}. \tag{4.13}$$

By summing the equalities (4.9) and (4.11), we get

$$2(1+\alpha)a_2^2 = \frac{u_2 + s_2}{4} - \frac{5}{32}(u_1^2 + s_1^2). \tag{4.14}$$

Substituting the value of $u_1^2 + s_1^2$ from (4.13) in (4.14), we get

$$a_2^2 = \frac{u_2 + s_2}{4(1+\alpha)(7+5\alpha)}. \tag{4.15}$$

By applying Lemma 1.1, in conjunction with the triangle inequality in the equations (4.13) and (4.15), it can be concluded that

$$|a_2| \leq \frac{1}{2(1+\alpha)} \quad \textit{and} \quad |a_2| \leq \frac{1}{\sqrt{(1+\alpha)(7+5\alpha)}}. \tag{4.16}$$

This validates the initial findings presented in (4.1). In addition, subtracting (4.11) from (4.9), yields

$$4(1+2\alpha)a_3 - 4(1+2\alpha)a_2^2 = \frac{u_2 - s_2}{4} + \frac{5}{32}(s_1^2 - u_1^2). \tag{4.17}$$

The above relation (4.17) combined with (4.12) and (4.15), leads that

$$a_3 = \frac{(11 + 20\alpha + 5\alpha^2)u_2 - (3 + 4\alpha + 5\alpha^2)s_2}{16(1+\alpha)(1+2\alpha)(7+5\alpha)}. \tag{4.18}$$

By applying Lemma 1.1, in conjunction with the triangle inequality in the equation (4.18), it can be concluded that

$$|a_3| \leq \frac{1}{4(1+2\alpha)}. \tag{4.19}$$

This validates the initial findings presented in (4.2). From equations (4.15) and (4.18), we get

$$a_3 - \lambda a_2^2 = \left(\frac{1}{16(1+2\alpha)} + g(\lambda)\right) u_2 - \left(\frac{1}{16(1+2\alpha)} - g(\lambda)\right) s_2, \tag{4.20}$$

where

$$g(\lambda) = \frac{1-\lambda}{4(1+\alpha)(7+5\alpha)}.$$

In view of Lemma 1.1 and Lemma 1.2, in equation (4.20), we get

$$|a_3 - \lambda a_2^2| \leq \begin{cases} \dfrac{1}{4(1+2\alpha)}, & \text{if } |g(\lambda)| \leq \dfrac{1}{16(1+2\alpha)}, \\ 4|g(\lambda)|, & \text{if } |g(\lambda)| \geq \dfrac{1}{16(1+2\alpha)}. \end{cases} \tag{4.21}$$

This validates the initial findings presented in (4.3), which completes the proof of Theorem 4.1. □

If we select $\alpha = 0$, in Theorem 4.1, we get the following corollary.

Corollary 4.2. *Consider the function* $f(\varsigma)$ *defined by* (1.1) *that belongs to the class* $\mathcal{S}^*_\Sigma(\xi)$, *then*

$$|a_2| \leq \frac{1}{\sqrt{7}} \approx 0.3779\ldots,$$

$$|a_3| \leq \frac{1}{4} = 0.25$$

and

$$|a_3 - \lambda a_2^2| \leq \begin{cases} \frac{1}{4}, & \text{if } \lambda \in \left[-\frac{3}{4}, \frac{11}{4}\right], \\ \frac{|1-\lambda|}{7}, & \text{if } \lambda \in \left(-\infty, -\frac{3}{4}\right) \cup \left(\frac{11}{4}, \infty\right). \end{cases}$$

If we select $\alpha = 1$, in Theorem 4.1, we get the following corollary.

Corollary 4.3. *Consider the function* $f(\varsigma)$ *defined by* (1.1) *that belongs to the class* $\mathcal{C}_\Sigma(\xi)$, *then*

$$|a_2| \leq \frac{1}{\sqrt{24}} \approx 0.2041\ldots,$$

$$|a_3| \leq \frac{1}{12} = 0.0833\ldots$$

and

$$|a_3 - \lambda a_2^2| \leq \begin{cases} \frac{1}{12}, & \text{if } \lambda \in [-1, 3], \\ \frac{|1-\lambda|}{24}, & \text{if } \lambda \in (-\infty, -1) \cup (3, \infty). \end{cases}$$

5. Coefficient Estimates and Fekete-Szegö Functional for the class $\mathcal{LS}^{*,\mu}_{s,\Sigma}(\xi)$.

Theorem 5.1. *Establish that* $0 \leq \mu \leq 1$. *Consider the function* $f(\varsigma)$ *defined by* (1.1) *that belongs to the class* $\mathcal{LS}^{*,\mu}_{s,\Sigma}(\xi)$, *then*

$$|a_2| \leq \frac{1}{2\sqrt{6 + 12\mu + 5\mu^2}}, \tag{5.1}$$

$$|a_3| \leq \frac{1}{4(1+2\mu)} \tag{5.2}$$

and

$$|a_3 - \lambda a_2^2| \leq \begin{cases} \frac{1}{4(1+2\mu)}, & \text{if } \lambda \in \left[-\frac{5(1+\mu)^2}{1+2\mu}, \frac{7+14\mu+5\mu^2}{1+2\mu}\right], \\ \frac{|1-\lambda|}{4(6+12\mu+5\mu^2)}, & \text{if } \lambda \in \left(-\infty, -\frac{5(1+\mu)^2}{1+2\mu}\right) \cup \left(\frac{7+14\mu+5\mu^2}{1+2\mu}, \infty\right). \end{cases} \tag{5.3}$$

Proof. If the function $f(\varsigma)$ is a member of the class $\mathcal{LS}^{*,\mu}_{s,\Sigma}(\xi)$, then it follows:

$$(1-\mu)\frac{2\varsigma f'(\varsigma)}{f(\varsigma) - f(-\varsigma)} + \mu \frac{2[\varsigma f'(\varsigma)]'}{[f(\varsigma) - f(-\varsigma)]'} = \xi(\kappa_1(\varsigma)) \tag{5.4}$$

and

$$(1-\mu)\frac{2\vartheta h'(\vartheta)}{h(\vartheta)-h(-\vartheta)}+\mu\frac{2[\vartheta h'(\vartheta)]'}{[h(\vartheta)-h(-\vartheta)]'}=\xi(\kappa_2(\vartheta)), \tag{5.5}$$

Since,

$$(1-\mu)\frac{2\varsigma f'(\varsigma)}{f(\varsigma)-f(-\varsigma)}+\mu\frac{2[\varsigma f'(\varsigma)]'}{[f(\varsigma)-f(-\varsigma)]'}=1+2(1+\mu)a_2\varsigma+2(1+2\mu)a_3\varsigma^2+\cdots \tag{5.6}$$

and

$$(1-\mu)\frac{2\vartheta h'(\vartheta)}{h(\vartheta)-h(-\vartheta)}+\mu\frac{2[\vartheta h'(\vartheta)]'}{[h(\vartheta)-h(-\vartheta)]'}=1-2(1+\mu)a_2\vartheta+[4(1+2\mu)a_2^2-2(1+2\mu)a_3]\vartheta^2+\cdots. \tag{5.7}$$

It follows from (3.6) and (5.6) that

$$2(1+\mu)a_2=\frac{1}{4}u_1 \tag{5.8}$$

and

$$2(1+2\mu)a_3=\frac{1}{4}u_2-\frac{5}{32}u_1^2. \tag{5.9}$$

Similarly, it follows from (3.7) and (5.7) that

$$-2(1+\mu)a_2=\frac{1}{4}s_1 \tag{5.10}$$

and

$$4(1+2\mu)a_2^2-2(1+2\mu)a_3=\frac{1}{4}s_2-\frac{5}{32}s_1^2. \tag{5.11}$$

It can be concluded from (5.8) and (5.10) that

$$u_1+s_1=0 \tag{5.12}$$

and

$$a_2^2=\frac{u_1^2+s_1^2}{128(1+\mu)^2}. \tag{5.13}$$

By summing the equalities (5.9) and (5.11), we get

$$4(1+2\mu)a_2^2=\frac{u_2+s_2}{4}-\frac{5}{32}(u_1^2+s_1^2). \tag{5.14}$$

Substituting the value of $u_1^2+s_1^2$ from (5.13) in (5.14), we get

$$a_2^2=\frac{u_2+s_2}{16(6+12\mu+5\mu^2)}. \tag{5.15}$$

By applying Lemma 1.1, in conjunction with the triangle inequality in the equations (5.13) and (5.15), it can be concluded that

$$|a_2|\le\frac{1}{4(1+\mu)}\quad \textit{and}\quad |a_2|\le\frac{1}{2\sqrt{6+12\mu+5\mu^2}}. \tag{5.16}$$

This validates the initial findings presented in (5.1). In addition, subtracting (5.11) from (5.9), yields

$$4(1+2\mu)a_3-4(1+2\mu)a_2^2=\frac{u_2-s_2}{4}+\frac{5}{32}(s_1^2-u_1^2). \tag{5.17}$$

The above relation (5.17) combined with (5.12) and (5.15), leads that

$$a_3 = \frac{(7+14\mu+5\mu^2)u_2 - (5+10\mu+5\mu^2)s_2}{16(1+2\mu)(6+12\mu+5\mu^2)}. \tag{5.18}$$

By applying Lemma 1.1, in conjunction with the triangle inequality in the equation (5.18), it can be concluded that

$$|a_3| \le \frac{1}{4(1+2\mu)}. \tag{5.19}$$

This validates the initial findings presented in (5.2). From equations (5.15) and (5.18), we get

$$a_3 - \lambda a_2^2 = \left(\frac{1}{16(1+2\mu)} + g(\lambda)\right) u_2 - \left(\frac{1}{16(1+2\mu)} - g(\lambda)\right) s_2, \tag{5.20}$$

where

$$g(\lambda) = \frac{1-\lambda}{16(6+12\mu+5\mu^2)}.$$

In view of Lemma 1.1 and Lemma 1.2, in equation (5.20), we get

$$|a_3 - \lambda a_2^2| \le \begin{cases} \dfrac{1}{4(1+2\mu)}, & \text{if } |g(\lambda)| \le \dfrac{1}{16(1+2\mu)}, \\ 4|g(\lambda)|, & \text{if } |g(\lambda)| \ge \dfrac{1}{16(1+2\mu)}. \end{cases} \tag{5.21}$$

This validates the initial findings presented in (5.3), which completes the proof of Theorem 5.1. □

If we select $\mu = 0$, in Theorem 5.1, we get the following corollary.

Corollary 5.2. *Consider the function $f(\varsigma)$ defined by (1.1) that belongs to the class $\mathcal{S}^*_{s,\Sigma}(\xi)$, then*

$$|a_2| \le \frac{1}{2\sqrt{6}} \approx 0.2041\ldots,$$

$$|a_3| \le \frac{1}{4} = 0.25$$

and

$$|a_3 - \lambda a_2^2| \le \begin{cases} \dfrac{1}{4}, & \text{if } \lambda \in [-5, 7], \\ \dfrac{|1-\lambda|}{24}, & \text{if } \lambda \in (-\infty, -5) \cup (7, \infty). \end{cases}$$

If we select $\mu = 1$, in Theorem 5.1, we get the following corollary.

Corollary 5.3. *Consider the function $f(\varsigma)$ defined by (1.1) that belongs to the class $\mathcal{C}_{s,\Sigma}(\xi)$, then*

$$|a_2| \le \frac{1}{2\sqrt{23}} \approx 0.1042\ldots,$$

$$|a_3| \le \frac{1}{12} \approx 0.0833\ldots$$

and

$$|a_3 - \lambda a_2^2| \leq \begin{cases} \dfrac{1}{12}, & \text{if } \lambda \in \left[-\dfrac{20}{3}, \dfrac{26}{3}\right], \\ \dfrac{|1-\lambda|}{92}, & \text{if } \lambda \in \left(-\infty, -\dfrac{20}{3}\right) \cup \left(\dfrac{26}{3}, \infty\right). \end{cases}$$

Conclusion

In this current work, we introduce three new subclasses of the class of bi-univalent functions Σ, namely $\mathcal{R}_\Sigma(\xi)$, $\mathcal{M}_\Sigma^\alpha(\xi)$ and $\mathcal{LS}_{s,\Sigma}^{*,\mu}(\xi)$, by using lemniscate of Bernoulli. We investigate the estimates of the Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$, as well as the Fekete–Szezö functional problems $|a_3 - \lambda a_2^2|$, by using subordination principle. Additionally, we can expand these types of studies to include bounded boundary rotation and bounded radius rotation (see [8, 9]).

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