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ON SOME RESTRICTED PLANE PARTITION FUNCTIONS

MIRCEA CIMPOEAŞ[1] AND ALEXANDRA TEODOR[2]

Abstract. We prove new formulas and congruences for $\mathrm{pp}_{r,c}(n)$, the number of plane partitions of n with at most r rows and c columns, $\mathrm{pp}_r^s(n)$, the number of strict plane partitions of n with at most r rows, and $\mathrm{pp}_r^{so}(n)$, the number of symmetric plane partitions of n with at most r rows.

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1. Introduction

Let n be a positive integer. We denote $[n]=\{1,2,\ldots,n\}$. A partition of n is a non-increasing sequence of positive integers whose sum equals n. We define $\mathrm{p}(n)$ as the number of partitions of n and, for convenience, we define $p(0)=1$.

A *plane partition* π of n is an array $(n_{ij})_{i,j\in[n]}$ of nonnegative integers such that

$$|\pi|=\sum_{i,j\in[n]} n_{ij}=n \text{ and } n_{ij}\geq n_{i'j'} \text{ for all } i,j,i',j'\in[n] \text{ such that } i\leq i' \text{ and } j\leq j'.$$

If $n_{ij}>n_{i(j+1)}$ whenever $n_{ij}\neq 0$, then we shall call such a partition *strict*. If $n_{ij}=n_{ji}$ for all i and j, then the partition is called *symmetric*.

For example, there are 6 plane partitions of $n=3$, namely:

$$\begin{matrix}3&0&0\\0&0&0\\0&0&0\end{matrix},\quad \begin{matrix}2&1&0\\0&0&0\\0&0&0\end{matrix},\quad \begin{matrix}2&0&0\\1&0&0\\0&0&0\end{matrix},\quad \begin{matrix}1&1&1\\0&0&0\\0&0&0\end{matrix},\quad \begin{matrix}1&1&0\\1&0&0\\0&0&0\end{matrix},\quad \begin{matrix}1&0&0\\1&0&0\\1&0&0\end{matrix}.$$

Note that four of them are strict partitions and two of them are symmetric. In 1910s, MacMahon [14] discovered the well-known generating function of $\mathrm{pp}(n)$ given by

$$\sum_{n=0}^{\infty}\mathrm{pp}(n)q^n=\prod_{n=1}^{\infty}\frac{1}{(1-q^n)^n}.$$

In particular, the above equality states that the number of plane partitions of n is equal to the number of partitions of n with parts in the multiset $\{1_1, 2_1, 2_2, 3_1, 3_2, 3_3, \cdots\}$, where each number j occurs exactly j times. See also [13] for further details.

More generally, let $\mathbf{a} := (a_1, \cdots, a_r)$ be a sequence of positive integers. The restricted partition function associated to $\mathbf{a}$ is the number of integer solutions $(x_1, \cdots, x_r)$ of $\sum_{n=0}^{\infty} a_i x_i = n$ with $x_i \geq 0$.

Let r, c and n be some positive integers. Let $\mathrm{pp}_{r,c}(n)$ be the number of plane partitions of n with at most r rows and c columns and set $\mathrm{pp}_{r,c}(0) = 1$. Note that $\mathrm{pp}_{1,n}(n) = \mathrm{p}(n)$ for all $n \geq 0$. Also, in the example above, it is easy to see, for instance, that $\mathrm{pp}_{2,2}(3) = 4$. Let $\mathrm{pp}^s_r(n)$ be the number of strict plane partitions of n with at most r rows and, $\mathrm{pp}^{so}_r(n)$, the number of strict plane partitions of n with at most r rows with odd entries and set $\mathrm{pp}^s_r(n) = 0$ and $\mathrm{pp}^{so}_r(n) = 0$. Note that $\mathrm{pp}^{so}_r(n)$ is equal to the number of symmetric plane partitions of n with at most r row. Also, in the example above, we have $\mathrm{pp}^s_3(3) = 4$ and $\mathrm{pp}^{so}_3(3) = 2$.

As it was noted in [12, Page 158], we can easily see that $\mathrm{pp}^{so}_r(n)$ counts also the number of symmetric plane partitions with at most r rows. The proof of this remark is a generalization of the well known argument which shows that the number of partitions with odd terms of n is the number of self-conjugate partitions of n.

As a continuation of [9], we study the functions $\mathrm{pp}_{r,c}(n)$, $\mathrm{pp}^s_r(n)$ and $\mathrm{pp}^{so}_r(n)$. The paper is structured as follows. In Section 2, we recall the definition and some basic properties of the restricted partition function $p_{\mathbf{a}}(n)$, where $\mathbf{a} = (a_1, \ldots, a_r)$ is a sequence of positive integers. Also, we recall several results which would be used later on.

In Section 3, we show the connections between the functions $\mathrm{pp}_{r,c}(n)$, $\mathrm{pp}^s_r(n)$, $\mathrm{pp}^{so}_r(n)$ and the restriction partition function; see Propositions 3.1, 3.4, 3.5. In particular, in Proposition 3.6 we present the expressions of $\mathrm{pp}_{r,c}(n)$, $\mathrm{pp}^s_r(n)$ and $\mathrm{pp}^{so}_r(n)$ as quasi-polynomials with periodic coefficients $d_{r,c,m}(n)$, $d^s_{r,m}(n)$ and $d^{so}_{r,m}(n)$ respectively. Seeing the functions $\mathrm{pp}_{r,c}(n)$, $\mathrm{pp}^s_r(n)$ and $\mathrm{pp}^{so}_r(n)$ as the restricted partition functions associated to the sequences

$$\begin{aligned}
\mathbf{a}_{r,c} &= (1^{[1]}, 2^{[2]}, \ldots, r^{[r]}, (r+1)^{[r]}, \ldots, c^{[r]}, (c+1)^{[r-1]}, \ldots, (r+c-1)^{[1]}) \\
\mathbf{a}^s_r &= (1^{[1]}, 2^{[2]}, 3^{[2]}, 4^{[3]}, \ldots, r^{[\lfloor \frac{r+2}{2} \rfloor]}, (r+1)^{[\lfloor \frac{r-1}{2} \rfloor]}, \ldots, (2r-2)^{[1]}) \text{ and} \\
\mathbf{a}^{so}_r &= (1^{[1]}, 3^{[1]}, 4^{[1]}, 5^{[1]}, 6^{[1]}, \ldots, (2r-1)^{[1]}, (2r)^{[\lfloor \frac{r}{2} \rfloor]}, (2r+2)^{[\lfloor \frac{r-1}{2} \rfloor]}, \ldots, (4r-4)^{[1]}),
\end{aligned}$$

where $\ell^{[k]}$ denotes k copies of ℓ, allows us to prove new results about them.

In Section 4, we give formulas for $d_{r,c,m}(n)$, $d^s_{r,m}(n)$ and $d^{so}_{r,m}(n)$; see Theorem 4.2. Also, we prove new formulas for $\mathrm{pp}_{r,c}(n)$, $\mathrm{pp}^s_r(n)$ and $\mathrm{pp}^{so}_r(n)$; see Theorem 4.3. Also, we show that if certain determinants are nonzero, we can express these functions in terms of Bernoulli numbers and values of Bernoulli polynomials; see Proposition 4.4.

In Section 5, we define $\mathrm{PP}_{r,c}(n)$, $\mathrm{PP}^s_r(n)$ and $\mathrm{PP}^{so}_r(n)$, the polynomial parts of $\mathrm{pp}_{r,c}(n)$, $\mathrm{pp}^s_r(n)$ and $\mathrm{pp}^{so}_r(n)$, respectively. In Theorem 5.3 and Theorem 5.4 we prove formulas for $\mathrm{PP}_{r,c}(n)$, $\mathrm{PP}^s_r(n)$ and $\mathrm{PP}^{so}_r(n)$. Also, similarly to the restricted partition function, we write $\mathrm{pp}_{r,c}(n)$, $\mathrm{pp}^s_r(n)$ and $\mathrm{pp}^{so}_r(n)$ as a sum of "waves", denoted $W_j(n,r,c)$, $W^s_j(n,r)$ and $W^{so}_j(n,r)$ and, in Theorem 5.5, we provide some formulas for them.

In Section 6, we study some arithmetic properties of $\mathrm{pp}_{r,c}(n)$, $\mathrm{pp}^s_r(n)$ and $\mathrm{pp}^{so}_r(n)$. In Proposition 6.2 we prove some congruences for these functions.

In Theorem 6.3, using a result from [11], we prove the following lower bounds:

$$\lim_{N\to\infty}\frac{\#\{n\leq N\ :\ \mathrm{pp}_{r,c}(n)\not\equiv 0(\mathrm{mod}\ m)\}}{N}\geq\frac{1}{rc(r+c)},$$
$$\lim_{N\to\infty}\frac{\#\{n\leq N\ :\ \mathrm{pp}^{s}_{r}(n)\not\equiv 0(\mathrm{mod}\ m)\}}{N}\geq\frac{2}{r(r^2+1)},$$
$$\lim_{N\to\infty}\frac{\#\{n\leq N\ :\ \mathrm{pp}^{so}_{r}(n)\not\equiv 0(\mathrm{mod}\ m)\}}{N}\geq\frac{1}{r^3},$$

where $m>1$ is an arbitrary integer.

2. Restricted partition function

Let $\mathbf{a}:=(a_1,a_2,\ldots,a_r)$ be a sequence of positive integers, $r\geq 1$. The *restricted partition function* associated to $\mathbf{a}$ is $p_{\mathbf{a}}:\mathbb{N}\to\mathbb{N}$, $p_{\mathbf{a}}(n)$ is the number of integer solutions $(x_1,\ldots,x_r)$ of $\sum_{i=1}^r a_ix_i=n$ with $x_i\geq 0$. Note that the generating function of $p_{\mathbf{a}}(n)$ is

$$\sum_{n=0}^{\infty}p_{\mathbf{a}}(n)q^n=\frac{1}{(1-z^{a_1})\cdots(1-z^{a_r})},\ |z|<1. \tag{2.1}$$

Let D be the common multiple of $a_1,\ a_2,\ldots,a_r$. We recall the following well known result:

Proposition 2.1. *(Bell [3])*
$p_{\mathbf{a}}(n)$ is a quasi-polynomial of degree $r-1$, with the period D, i.e.

$$p_{\mathbf{a}}(n)=d_{\mathbf{a},r-1}(n)n^{r-1}+\cdots+d_{\mathbf{a},1}(n)n+d_{\mathbf{a},0}(n),$$

where $d_{\mathbf{a},m}(n+D)=d_{\mathbf{a},m}(n)$ for $0\leq m\leq r-1$ and $n\geq 0$, and $d_{\mathbf{a},r-1}(n)$ is not identically zero.

Sylvester [16, 17, 18] decomposed the restricted partition in a sum of "waves",

$$p_{\mathbf{a}}(n)=\sum_{j\geq 1}W_j(n,\mathbf{a}), \tag{2.2}$$

where the sum is taken over all distinct divisors j of the components of $\mathbf{a}$ and showed that for each such j, $W_j(n,\mathbf{a})$ is the coefficient of t^{-1} in

$$\sum_{0\leq\nu<j,\ \gcd(\nu,j)=1}\frac{\rho_j^{-\nu n}e^{nt}}{(1-\rho_j^{\nu a_1}e^{-a_1t})\cdots(1-\rho_j^{\nu a_k}e^{-a_kt})},$$

where $\rho_j=e^{\frac{2\pi i}{j}}$ and $\gcd(0,0)=1$ by convention. Note that $W_j(n,\mathbf{a})$'s are quasi-polynomials of period j. Also, $W_1(n,\mathbf{a})$ is called the *polynomial part* of $p_{\mathbf{a}}(n)$ and it is denoted by $P_{\mathbf{a}}(n)$.

The *unsigned Stirling numbers* are defined by

$$\binom{n+r-1}{r-1}=\frac{(n+1)\cdots(n+r-1)}{(r-1)!}=\frac{1}{(r-1)!}\left(\begin{bmatrix}r\\r\end{bmatrix}n^{r-1}+\cdots+\begin{bmatrix}r\\2\end{bmatrix}n+\begin{bmatrix}r\\1\end{bmatrix}\right). \tag{2.3}$$

We recall several results which would be used later on:

Theorem 2.2. *([5, Theorem 2.8(1)] and [6]) For $0\leq m\leq r-1$ and $n\geq 0$ we have*

$$d_{\mathbf{a},m}(n)=\frac{1}{(r-1)!}\sum_{\substack{0\leq j_1\leq\frac{D}{a_1}-1,\ldots,0\leq j_r\leq\frac{D}{a_r}-1\\ a_1j_1+\cdots+a_rj_r\equiv n(\bmod D)}}\sum_{k=m}^{r-1}\begin{bmatrix}r\\k+1\end{bmatrix}(-1)^{k-m}\binom{k}{m}D^{-k}(a_1j_1+\cdots+a_rj_r)^{k-m}.$$

Theorem 2.3. *([5, Corollary 2.10]) We have*

$$p_{\mathbf{a}}(n) = \frac{1}{(r-1)!} \sum_{\substack{0\leq j_1\leq \frac{D}{a_1}-1,\dots,0\leq j_r\leq \frac{D}{a_r}-1 \\ a_1j_1+\cdots+a_rj_r\equiv n(\bmod D)}} \prod_{\ell=1}^{r-1} \left(\frac{n-a_1j_1-\cdots-a_rj_r}{D} + \ell \right).$$

Proposition 2.4. *([8, Proposition 4.2]) For any positive integer j with $j|a_i$ for some $1\leq i\leq r$, we have that:*

$$W_j(n,\mathbf{a}) = \frac{1}{D(r-1)!} \sum_{m=1}^{r} \sum_{\ell=1}^{j} \rho_j^{\ell} \sum_{k=m-1}^{r-1} \begin{bmatrix} r \\ k+1 \end{bmatrix} (-1)^{k-m+1} \binom{k}{m-1}$$

$$\cdot \sum_{\substack{0\leq j_1\leq \frac{D}{a_1}-1,\dots,0\leq j_r\leq \frac{D}{a_r}-1 \\ a_1j_1+\cdots+a_rj_r\equiv \ell(\bmod j)}} D^{-k}(a_1j_1+\cdots+a_rj_r)^{k-m+1} n^{m-1}.$$

Theorem 2.5. *([5, Corollary 3.6])*
For the polynomial part $P_{\mathbf{a}}(n)$ of the quasi-polynomial $p_{\mathbf{a}}(n)$ we have

$$P_{\mathbf{a}}(n) = \frac{1}{D(r-1)!} \sum_{0\leq j_1\leq \frac{D}{a_1}-1,\dots,0\leq j_r\leq \frac{D}{a_r}-1} \prod_{\ell=1}^{r-1} \left(\frac{n-a_1j_1-\cdots-a_rj_r}{D} + \ell \right).$$

The *Bernoulli numbers B_ℓ*'s are defined by the identity

$$\frac{t}{e^t-1} = \sum_{\ell=0}^{\infty} \frac{t^\ell}{\ell!} B_\ell.$$

$B_0=1$, $B_1=-\frac{1}{2}$, $B_2=\frac{1}{6}$, $B_4=-\frac{1}{30}$ and $B_n=0$ is n is odd and $n\geq 1$.

Theorem 2.6. *([5, Corollary 3.11] or [2, page 2])*
The polynomial part of $p_{\mathbf{a}}(n)$ is

$$P_{\mathbf{a}}(n) := \frac{1}{a_1\cdots a_r} \sum_{u=0}^{r-1} \frac{(-1)^u}{(r-1-u)!} \sum_{i_1+\cdots+i_r=u} \frac{B_{i_1}\cdots B_{i_r}}{i_1!\cdots i_r!} a_1^{i_1}\cdots a_r^{i_r} n^{r-1-u}.$$

Proposition 2.7. *([8, Corollary 2.4]) For $n\geq 0$ it holds that*

$$(r-1)! p_{\mathbf{a}}(n) \equiv 0 \mod (j+k+1)(j+k+2)\cdots(j+r-1),$$

where $k = \lfloor \frac{n}{D} \rfloor - \lceil \frac{n+\sigma}{D} \rceil + r$, $\sigma = a_1+\cdots+a_r$ and $\lceil \frac{n+\sigma}{D} \rceil - r \leq j \leq \lfloor \frac{n}{D} \rfloor$.

Theorem 2.8. *([11, Theorem 5.2]) Let $m>1$ be a positive integer. We have that*

$$\lim_{N\to\infty} \frac{\#\{n\leq N \ : \ p_{\mathbf{a}}(n) \not\equiv 0(\mathrm{mod}\ m)\}}{N} \geq \frac{1}{\sum_{i=1}^{r} a_i}.$$

3. Preliminaries

Now that we have defined plane partitions in Introduction, it is natural to attempt to enumerate them. Initially, it is convenient to restrict our consideration to a bounded size. We define $B(r,c,t)$ to be the set of all plane partitions with at most r rows and c columns and largest part at most t, that is $n_{11} \leq t$. According to [15, (7.109)], we have

$$\sum_{\pi \in B(r,c,t)} z^{|\pi|} = \prod_{i=1}^{r}\prod_{j=1}^{c}\prod_{k=1}^{t} \frac{1-z^{i+j+k-1}}{1-z^{i+j+k-2}},\ |z|<1. \tag{3.1}$$

Let $\mathrm{pp}_{r,c}(n)$ be the number of plane partitions of n with at most r rows and c columns. Let $\mathrm{pp}_{r,c}(0)=0$. We have that

$$\sum_{n=0} \mathrm{pp}_{r,c}(n) z^n = \lim_{t\to\infty} \sum_{\pi\in B(r,c,t)} z^{|\pi|}. \tag{3.2}$$

On the other hand, for $|z|<1$ we have

$$\lim_{t\to\infty}\prod_{k=1}^{t} \frac{1-z^{i+j+k-1}}{1-z^{i+j+k-2}} = \prod_{k=1}^{\infty} \frac{1-z^{i+j+k-1}}{1-z^{i+j+k-2}} = \frac{1-z^{i+j}}{1-z^{i+j-1}}\cdot\frac{1-z^{i+j+1}}{1-z^{i+j}}\cdots = \frac{1}{1-z^{i+j-1}}. \tag{3.3}$$

From (3.1), (3.2) and (3.3) it follows that

$$\sum_{n=0} \mathrm{pp}_{r,c}(n) z^n = \prod_{i=1}^{r}\prod_{j=1}^{c} \frac{1}{1-z^{i+j-1}} = \prod_{\ell=1}^{r+c-1} \frac{1}{(1-z^{\ell})^{\alpha_{r,c}(\ell)}}, \tag{3.4}$$

where

$$\alpha_{r,c}(\ell) = \#\{(i,j)\in\{1,\ldots,r\}\times\{1,\ldots,c\}\ :\ i+j-1=\ell\}.$$

Note that $\alpha_{r,c}(\ell) =$ the coefficient of t^{ℓ} in the polynomial

$$\varphi_{r,c}(t) = (1+t+\cdots+t^{r-1})(t+t+\cdots+t^{c}).$$

For convenience, we assume $r\leq c$. Henceforth, it is easy to see that

$$\alpha_{r,c}(\ell) = \begin{cases} \ell, & \ell\leq r-1 \\ r, & r\leq \ell\leq c, \\ r+c-\ell, & c+1\leq \ell\leq r+c-1 \end{cases}$$

Therefore, from (3.4) it follows that

$$\sum_{n=0} \mathrm{pp}_{r,c}(n) z^n = \frac{1}{(1-z)(1-z^2)^2\cdots(1-z^r)^r\cdots(1-z^c)^r(1-z^{c+1})^{r-1}\cdots(1-z^{c+r-1})}. \tag{3.5}$$

We consider the sequence

$$\mathbf{a}_{r,c} := (\ell^{[\alpha_{r,c}(\ell)]})_{1\leq\ell\leq r+c-1} = (1^{[1]},2^{[2]},\ldots,r^{[r]},(r+1)^{[r]},\ldots,c^{[r]},(c+1)^{[r-1]},\ldots,(r+c-1)^{[1]}), \tag{3.6}$$

where $\ell^{[k]}$ denotes k copies of ℓ.

Proposition 3.1. *With the above notation, we have that*

$$\mathrm{pp}_{r,c}(n) = \mathrm{p}_{\mathbf{a}_{r,c}}(n) \text{ for all } n\geq 0.$$

Proof. The result follows from (3.5), (3.6) and (2.1). ☐

We recall the following result:

Theorem 3.2. *([4, Theorem 1]) The generating function for strict plane partitions whose parts lie in a set S of positive integers is*

$$\prod_{i\in S}\frac{1}{1-z^i}\prod_{i<j\in S}\frac{1}{1-z^{i+j-1}}.$$

Let $\mathrm{pp}^s_r(n)$ be the number of strict plane partitions with at most r rows of n and $\mathrm{pp}^{so}_r(n)$, the number of strict plane partitions with at most r rows with odd entries. For convenience, we set $\mathrm{pp}^s_r(0)=0$ and $\mathrm{pp}^{so}_r(0)=0$.

We recall that $\mathrm{pp}^{so}_r(n)$ counts also the number of symmetric plane partitions with at most r rows.

Corollary 3.3. *With the above notations, for $|z|<1$ we have that:*

$$(1)\ \sum_{n=0}^{\infty}\mathrm{pp}^s_r(n)z^n=\prod_{i=1}^{r}\frac{1}{1-z^i}\prod_{1\le i<j\le r}\frac{1}{1-z^{i+j-1}},$$

$$(2)\ \sum_{n=0}^{\infty}\mathrm{pp}^{so}_r(n)z^n=\prod_{i=1}^{r}\frac{1}{1-z^{2i-1}}\prod_{1\le i<j\le r}\frac{1}{1-z^{2(i+j-1)}}.$$

Proof. (1) It is enough to note that $\mathrm{pp}^s_r(n)=$ the number of strict plane partitions with parts in $\{1,2,\ldots,r\}$ and then apply Theorem 3.2.

(2) Similarly, the result follows from Theorem 3.2 and the fact that $\mathrm{pp}^s_r(n)=$ the number of strict plane partitions with parts in $\{1,3,\ldots,2r-1\}$ □

According to Corollary 3.3(1), we have that

$$\sum_{n=0}^{\infty}\mathrm{pp}^s_r(n)z^n=\prod_{\ell=1}^{2r-2}\frac{1}{(1-z^\ell)^{\alpha^s_r(\ell)}}, \tag{3.7}$$

where $\alpha^s_r(\ell)=\#\{(i,j)\ :\ 1\le i<j\le r \text{ and } i+j-1=\ell\}+\begin{cases}1, & \ell\le r\\ 0, & \ell>r\end{cases}$. It is easy too see that $\alpha^s_r(\ell)=\begin{cases}\lfloor\frac{\ell+2}{2}\rfloor, & 1\le\ell\le r\\ \lfloor\frac{2r-\ell}{2}\rfloor, & r<\ell\le 2r-2\end{cases}$. Hence, we consider the sequence

$$\mathbf{a}^s_r:=(\ell^{[\alpha^s_r(\ell)]})_{1\le\ell\le 2r-2}=(1^{[1]},2^{[2]},3^{[2]},4^{[3]},\ldots,r^{[\lfloor\frac{r+2}{2}\rfloor]},(r+1)^{[\lfloor\frac{r-1}{2}\rfloor]},\ldots,(2r-2)^{[1]}). \tag{3.8}$$

Proposition 3.4. *With the above notation, we have that*

$$\mathrm{pp}^s_r(n)=\mathrm{p}_{\mathbf{a}^s_r}(n)\ \textit{for all}\ n\ge 0.$$

Proof. The result follows from (3.7), (3.8) and (2.1). □

According to Corollary 3.3(2), we have that

$$\sum_{n=0}^{\infty}\mathrm{pp}^{so}_r(n)z^n=\prod_{\ell=1}^{4r-4}\frac{1}{(1-z^\ell)^{\alpha^{so}_r(\ell)}}, \tag{3.9}$$

where $\alpha_r^{so}(\ell) = \begin{cases} 1, & \ell \text{ odd and } \ell \leq 2r-1 \\ \left\lfloor \frac{\ell}{4} \right\rfloor, & \ell \text{ even and } \ell \leq 2r \\ r - \left\lceil \frac{\ell}{4} \right\rceil, & \ell \text{ even and } 2r < \ell \leq 4r-4 \\ 0, & \text{otherwise} \end{cases}$. Hence, we consider the sequence

$$\mathbf{a}_r^{so} := (\ell^{[\alpha_r^{so}(\ell)]})_{1\leq \ell \leq 4r-4} = (1^{[1]}, 3^{[1]}, 4^{[1]}, 5^{[1]}, 6^{[1]}, \ldots, (2r-1)^{[1]}, (2r)^{[\lfloor \frac{r}{2} \rfloor]}, (2r+2)^{[\lfloor \frac{r-1}{2} \rfloor]}, \ldots, (4r-4)^{[1]}). \tag{3.10}$$

Proposition 3.5. *With the above notation, we have that*

$$\mathrm{pp}_r^{so}(n) = \mathrm{p}_{\mathbf{a}_r^{so}}(n) \text{ for all } n \geq 0.$$

Proof. The result follows from (3.9), (3.10) and (2.1). □

Let D_k be the least common multiple of $1, 2, \ldots, k$.

Proposition 3.6. *With the above notations, we have that:*

(1) $\mathrm{pp}_{r,c}(n)$ *is a quasi-polynomial of degree* $rc-1$*, with the period* D_{r+c-1}*, i.e.*

$$\mathrm{pp}_{r,c}(n) = d_{r,c,rc-1}(n)n^{rc-1} + \cdots + d_{r,c,1}(n)n + d_{r,c,0}(n).$$

(2) $\mathrm{pp}_r^s(n)$ *is a quasi-polynomial of degree* $\binom{r+1}{2}$*, with the period* D_{2r-2}*, i.e.*

$$\mathrm{pp}_r^s(n) = d^s_{r,\binom{r+1}{2}-1}(n)n^{\binom{r+1}{2}-1} + \cdots + d^s_{r,1}(n)n + d^s_{r,0}(n).$$

(3) $\mathrm{pp}_r^{so}(n)$ *is a quasi-polynomial of degree* $\binom{r+1}{2}$*, with the period* D_{2r-1}*, i.e.*

$$\mathrm{pp}_r^{so}(n) = d^{so}_{r,\binom{r+1}{2}-1}(n)n^{\binom{r+1}{2}-1} + \cdots + d^{so}_{r,1}(n)n + d^{so}_{r,0}(n).$$

Proof. (1) Note that $\mathbf{a}_{r,c}$ is a sequence of length rc. We denote $d_{r,c,m} := d_{\mathbf{a}_{r,c},m}$ for all $0 \leq m \leq rc-1$. The result follows from Proposition 3.1 and (2.1).

(2) Similarly, the result follows from Proposition 3.4 and (2.1).

(3) Similarly, the result follows from Proposition 3.5 and (2.1). □

4. New formulas for $\mathrm{pp}_{r,c}(n)$, $\mathrm{pp}_r^s(n)$ and $\mathrm{pp}_r^{so}(n)$

We fix two integers $N, s \geq 1$ and consider the numbers

$$f_{s,N,\ell} = \#\{(i_1, \ldots, i_s) \,:\, i_1 + \cdots + i_s = \ell,\ 0 \leq i_t \leq N-1\}, \text{ where } 0 \leq \ell \leq s(N-1). \tag{4.1}$$

It is clear that $f_{s,N,\ell}$ is the coefficient of t^ℓ of the polynomial

$$f_{s,N}(t) = (1 + t + \cdots + t^{N-1})^s. \tag{4.2}$$

Using the binomial expansion, we have

$$f_{s,N}(t) = (1-t^N)^s(1-t)^{-s} = \sum_{i=0}^{s}(-1)^i \binom{s}{i} t^{iN} \sum_{j=0}^{\infty} \binom{j+s-1}{j} t^j. \tag{4.3}$$

From (4.1), (4.2) and (4.3) we get:

Lemma 4.1. *With the above notations, we have that*

$$f_{s,N,\ell} = \sum_{i,j\geq 0,\ iN+j=\ell} (-1)^i \binom{s}{i}\binom{j+s-1}{j}.$$

We introduce the following notations:

$$
\begin{aligned}
\overline{\mathbf{A}} =&\{(\ell_1,\dots,\ell_{r+c-1})\ :\ 0\le \ell_1\le D-1,\ 0\le \ell_2\le D-2,\dots,0\le \ell_r\le D-r,\\
&0\le \ell_{r+1}\le r\left(\frac{D}{r+1}-1\right),\dots,0\le \ell_c\le r\left(\frac{D}{c}-1\right),\\
&0\le \ell_{c+1}\le (r-1)\left(\frac{D}{c+1}-1\right),\dots,0\le \ell_{r+c-1}\le \frac{D}{c+r-1}-1\},
\end{aligned} \tag{4.4}
$$

where $D = D_{r+c-1}$ and

$$
\mathbf{A}_n = \{(\ell_1,\dots,\ell_{r+c-1})\in\overline{\mathbf{A}}\ :\ \ell_1+2\ell_2+\cdots+(r+c-1)\ell_{r+c-1}\equiv n(\text{mod } D)\}. \tag{4.5}
$$

We also let:

$$
\begin{aligned}
\overline{\mathbf{B}} =&\{(\ell_1,\dots,\ell_{2r-2})\ :\ 0\le \ell_1\le D-1,\ 0\le \ell_2\le D-2, 0\le \ell_3\le D-2,\dots,\\
&0\le \ell_r\le \left\lfloor\frac{r+2}{2}\right\rfloor\left(\frac{D}{r}-1\right), 0\le \ell_{r+1}\le \left\lfloor\frac{r-1}{2}\right\rfloor\left(\frac{D}{r}-1\right),\dots,\\
&0\le \ell_{2r-2}\le\left(\frac{D}{2r-2}-1\right)\},
\end{aligned} \tag{4.6}
$$

where $D = D_{2r-2}$ and

$$
\mathbf{B}_n = \{(\ell_1,\dots,\ell_{2r-2})\in\overline{\mathbf{B}}\ : \ell_1+2\ell_2+\cdots+(2r-2)\ell_{2r-2}\equiv n(\text{mod } D)\}. \tag{4.7}
$$

Finally, we define:

$$
\begin{aligned}
\overline{\mathbf{C}} =&\{(\ell_1,\ell_3,\ell_4,\dots,\ell_{2r},\ell_{2r+2},\dots,\ell_{4r-4})\ :\ 0\le \ell_1\le D-1,\ 0\le \ell_3\le \frac{D}{3}-1,\dots,\\
&0\le \ell_{2r-1}\le \frac{D}{2r-1}-1,\ 0\le \ell_4\le \frac{D}{4}-1,\ 0\le \ell_6\le \frac{D}{6}-1,\ 0\le \ell_8\le \frac{D}{4}-2,\\
&\dots,0\le \ell_{2r}\le \left\lfloor\frac{r}{2}\right\rfloor\left(\frac{D}{2r}-1\right),\ 0\le \ell_{2r+2}\le \left\lfloor\frac{r-1}{2}\right\rfloor\left(\frac{D}{2r+2}-1\right),\\
&\dots,0\le \ell_{4r-4}\le \frac{D}{4r-4}-1\},
\end{aligned} \tag{4.8}
$$

where $D = D_{2r-1}$ and

$$
\begin{aligned}
\mathbf{C}_n =&\{(\ell_1,\ell_3,\ell_4,\dots,\ell_{2r},\ell_{2r+2},\dots,\ell_{4r-4})\in\overline{\mathbf{C}}\ :\\
&\ell_1+3\ell_3+\cdots+2r\ell_{2r}+(2r+2)\ell_{2r+2}+\cdots+(4r-4)\ell_{4r-4}\equiv n(\text{mod } D)\}.
\end{aligned} \tag{4.9}
$$

Theorem 4.2. *(1) Let $D := D_{r+c-1}$. For $n\ge 0$ and $0\le m\le rc-1$ we have that*

$$
\begin{aligned}
d_{r,c,m}(n) = \frac{1}{(rc-1)!}&\sum_{(\ell_1,\dots,\ell_{r+c-1})\in\mathbf{A}_n}\prod_{t=2}^{r-1}\sum_{i_t,j_t\ge 0,\ i_t(\frac{D}{t}-1)+j_t=\ell_t}(-1)^{i_t}\binom{t}{i_t}\binom{j_t+t-1}{j_t}\\
&\cdot\prod_{t=r}^{c}\sum_{i_t,j_t\ge 0,\ i_t(\frac{D}{t}-1)+j_t=\ell_t}(-1)^{i_t}\binom{r}{i_t}\binom{j_t+r-1}{j_t}\\
&\cdot\prod_{t=c+1}^{r+c-2}\sum_{i_t,j_t\ge 0,\ i_t(\frac{D}{t}-1)+j_t=\ell_t}(-1)^{i_t}\binom{r+c-t}{i_t}\binom{j_t+r+c-t-1}{j_t}
\end{aligned}
$$

$$\cdot \sum_{k=m}^{rc-1} \begin{bmatrix} rc \\ k+1 \end{bmatrix} (-1)^{k-m} \binom{k}{m} D^{-k} (\ell_1 + 2\ell_2 + \cdots + (r+c-1)\ell_{r+c-1})^{k-m}.$$

(2) Let $D := D_{2r-2}$. For $n \geq 0$ and $0 \leq m \leq \binom{r+1}{2} - 1$ we have that

$$d^s_{r,m}(n) = \frac{1}{(\binom{r+1}{2} - 1)!} \sum_{(\ell_1,\ldots,\ell_{2r-2}) \in \mathbf{B}_n} \prod_{t=2}^{r} \sum_{i_t,j_t \geq 0,\ i_t(\frac{D}{t}-1)+j_t=\ell_t} (-1)^{i_t} \binom{\frac{t+2}{2}}{i_t} \binom{j_t + \frac{t+2}{2} - 1}{j_t}$$

$$\cdot \prod_{t=r+1}^{2r-4} \sum_{i_t,j_t \geq 0,\ i_t(\frac{D}{t}-1)+j_t=\ell_t} (-1)^{i_t} \binom{\frac{2r-t}{2}}{i_t} \binom{j_t + \frac{2r-t}{2} - 1}{j_t}$$

$$\cdot \sum_{k=m}^{\binom{r+1}{2}-1} \begin{bmatrix} \binom{r+1}{2} \\ k+1 \end{bmatrix} (-1)^{k-m} \binom{k}{m} D^{-k} (\ell_1 + 2\ell_2 + \cdots + (2r-2)\ell_{2r-2})^{k-m}.$$

(3) Let $D := D_{2r-1}$. For $n \geq 0$ and $0 \leq m \leq \binom{r+1}{2} - 1$ we have that

$$d^{so}_{r,m}(n) = \frac{1}{(\binom{r+1}{2} - 1)!} \sum_{(\ell_1,\ell_3,\ldots,\ell_{4r-4}) \in \mathbf{C}_n} \prod_{t=4}^{r} \sum_{i_t,j_t \geq 0,\ i_t(\frac{D}{2t}-1)+j_t=\ell_{2t}} (-1)^{i_t} \binom{\frac{t}{2}}{i_t} \binom{j_t + \frac{t}{2} - 1}{j_t}$$

$$\cdot \prod_{t=r+1}^{2r-4} \sum_{i_t,j_t \geq 0,\ i_t(\frac{D}{2t}-1)+j_t=\ell_{2t}} (-1)^{i_t} \binom{\frac{2r-t}{2}}{i_t} \binom{j_t + \frac{2r-t}{2} - 1}{j_t}$$

$$\cdot \sum_{k=m}^{\binom{r+1}{2}-1} \begin{bmatrix} \binom{r+1}{2} \\ k+1 \end{bmatrix} (-1)^{k-m} \binom{k}{m} D^{-k} (\ell_1 + 3\ell_3 + \cdots + 2r\ell_{2r} + (2r+2)\ell_{2r+2} + \cdots + (4r-4)\ell_{4r-4})^{k-m}.$$

Proof. (1) From Proposition 3.1 and Theorem 2.2 it follows that

$$d_{r,c,m} = \frac{1}{(rc-1)!} \sum_{(j_1,\ldots,j_{rc}) \in \tilde{\mathbf{A}}_n} \sum_{k=m}^{rc-1} \begin{bmatrix} rc \\ k+1 \end{bmatrix} (-1)^{k-m} \binom{k}{m} D^{-k} \cdot (j_1 + 2j_2 + 2j_3 + \cdots$$

$$+ rj_{\binom{r}{2}+1} + \cdots + rj_{\binom{r+1}{2}} + \cdots + cj_{\frac{r(2c-r-1)}{2}+1} + \cdots + cj_{\frac{r(2c-r+1)}{2}} + \cdots + (r+c-1)j_{rc})^{k-m},$$

where $\tilde{\mathbf{A}}_n = \{(j_1, j_2, \ldots, j_{rc}) \ : \ 0 \leq j_1 \leq D-1,\ 0 \leq j_2 \leq \frac{D}{2}-1,\ 0 \leq j_3 \leq \frac{D}{2}-1, \ldots, 0 \leq j_{\binom{r}{2}+1} \leq \frac{D}{r} - 1, \ldots, 0 \leq j_{\binom{r+1}{2}} \leq \frac{D}{r} - 1, \ldots, 0 \leq j_{\frac{r(2c-r-1)}{2}+1} \leq \frac{D}{c} - 1, \ldots, 0 \leq j_{\frac{r(2c-r+1)}{2}} \leq \frac{D}{c} - 1, \ldots, 0 \leq j_{rc} \leq \frac{D}{r+c-1} - 1$ with $j_1 + 2j_2 + 2j_3 + \cdots + rj_{\binom{r}{2}+1} + \cdots + rj_{\binom{r+1}{2}} + \cdots + cj_{\frac{r(2c-r-1)}{2}+1} + \cdots + cj_{\frac{r(2c-r+1)}{2}} + \cdots + (r+c-1)j_{rc} \equiv n (\mathrm{mod}\ D)\}$.

We denote $\ell_1 = j_1,\ \ell_2 = j_2 + j_3, \ldots, \ell_r = j_{\binom{r}{2}+1} + \cdots + j_{\binom{r+1}{2}}, \ldots, \ell_c = j_{\frac{r(2c-r-1)}{2}+1} + \cdots + j_{\frac{r(2c-r+1)}{2}}, \ldots, \ell_{r+c-1} = j_{rc}$. From (4.1) and (4.5) it follows that

$$d_{r,c,m} = \frac{1}{(rc-1)!} \sum_{(\ell_1,\ldots,\ell_{r+c-1}) \in \mathbf{A}_n} \prod_{t=1}^{r+c-1} f_{s(t),N(t),\ell_t} \sum_{k=m}^{rc-1} \begin{bmatrix} rc \\ k+1 \end{bmatrix} (-1)^{k-m} \binom{k}{m} \cdot D^{-k} (\ell_1 + 2\ell_2 + \cdots + (r+c-1)\ell_{r+c-1})^{k-m}, \tag{4.10}$$

where $s(t) = \begin{cases} t, & 1 \le t \le r-1 \\ r, & r \le t \le c \\ c+r-t, & c+1 \le t \le c+r-1 \end{cases}$ and $N(t) = \frac{D}{t} - 1$. Since

$$\prod_{t=1}^{r+c-1} f_{s(t),N(t),\ell_t} = \prod_{t=1}^{r-1} f_{t,\frac{D}{t}-1,\ell_t} \prod_{t=r}^{c} f_{r,\frac{D}{t}-1,\ell_t} \prod_{t=c+1}^{r+c-1} f_{r+c-t,\frac{D}{t}-1,\ell_t},$$

from (4.10) and Lemma 4.1 we get the required results.

(2) From (3.8), Proposition 3.4 and Theorem 2.2 it follows that

$$d^{s}_{r,m}(n) = \frac{1}{(\binom{r+1}{2}-1)!} \sum_{(j_1,\dots,j_{\binom{r+1}{2}})\in\tilde{\mathbf{B}}_n} \sum_{k=m}^{\binom{r+1}{2}-1} \begin{bmatrix} \binom{r+1}{2} \\ k+1 \end{bmatrix} (-1)^{k-m} \binom{k}{m} D^{-k} \cdot (j_1 + 2j_2 + 2j_3 + \cdots + (2r-2)j_{\binom{r+1}{2}})^{k-m},$$

where $\tilde{\mathbf{B}}_n = \{(j_1, j_2, \dots, j_{\binom{r+1}{2}}) \ : \ 0 \le j_1 \le D-1,\ 0 \le j_2 \le \frac{D}{2}-1,\ 0 \le j_3 \le \frac{D}{2}-1, \dots,$ $0 \le j_{\binom{r+1}{2}} \le \frac{D}{2r-2} - 1$ with $j_1 + 2j_2 + 2j_3 + \cdots + (2r-2)j_{\binom{r+1}{2}} \equiv n (\mathrm{mod}\ D)\}$.

We denote $\ell_1 = j_1,\ \ell_2 = j_2 + j_3,\ \ell_3 = j_4 + j_5, \dots, \ell_{2r-2} = j_{\binom{r+1}{2}}$. From (4.1) and (4.7) it follows that

$$d^{s}_{r,m}(n) = \frac{1}{(\binom{r+1}{2}-1)!} \sum_{(\ell_1,\dots,\ell_{2r-2})\in\mathbf{B}_n} \prod_{t=1}^{2r-2} f_{s(t),N(t),\ell_t} \sum_{k=m}^{\binom{r+1}{2}-1} \begin{bmatrix} \binom{r+1}{2} \\ k+1 \end{bmatrix} (-1)^{k-m} \binom{k}{m} \cdot D^{-k} (\ell_1 + 2\ell_2 + 2\ell_3 + \cdots + (2r-2)\ell_{2r-2})^{k-m}, \tag{4.11}$$

where $s(t) = \begin{cases} \lfloor \frac{t+2}{2} \rfloor, & 1 \le t \le r \\ \lfloor \frac{2r-t}{2} \rfloor, & r < t \le 2r-2 \end{cases}$ and $N(t) = \frac{D}{t} - 1$.

Now, the conclusion follows from (4.11) and Lemma 4.1.

(3) From (3.10), Proposition 3.5 and Theorem 2.2 it follows that

$$d^{so}_{r,m}(n) = \frac{1}{(\binom{r+1}{2}-1)!} \sum_{(j_1,\dots,j_{\binom{r+1}{2}})\in\tilde{\mathbf{C}}_n} \sum_{k=m}^{\binom{r+1}{2}-1} \begin{bmatrix} \binom{r+1}{2} \\ k+1 \end{bmatrix} (-1)^{k-m} \binom{k}{m} D^{-k} \cdot (j_1 + 3j_2 + 4j_3 + \cdots + (4r-4)j_{\binom{r+1}{2}})^{k-m},$$

where $\tilde{\mathbf{C}}_n = \{(j_1, j_2, \dots, j_{\binom{r+1}{2}}) \ : 0 \le j_1 \le D-1,\ 0 \le j_2 \le \frac{D}{3}-1,\ 0 \le j_2 \le \frac{D}{4}-1, \dots,$ $0 \le j_{\binom{r+1}{2}} \le \frac{D}{4r-4} - 1$ with $j_1 + 3j_2 + 4j_3 + \cdots + (4r-4)j_{\binom{r+1}{2}} \equiv n (\mathrm{mod}\ D)\}$. From (4.1) and (4.9) it follows that

$$d^{so}_{r,m}(n) = \frac{1}{(\binom{r+1}{2}-1)!} \sum_{(\ell_1,\ell_3,\dots,\ell_{4r-4})\in\mathbf{C}_n} \prod_{t=1}^{4r-4} f_{s(t),N(t),\ell_t} \sum_{k=m}^{\binom{r+1}{2}-1} \begin{bmatrix} \binom{r+1}{2} \\ k+1 \end{bmatrix} (-1)^{k-m} \binom{k}{m} \cdot D^{-k} (\ell_1 + 3\ell_3 + \cdots + 2r\ell_{2r} + (2r+2)\ell_{2r+2} + \cdots + (4r-4)\ell_{4r-4})^{k-m}, \tag{4.12}$$

where $s(t)=\begin{cases}1, & \ell \text{ odd and } t\le 2r-1\\ \left\lfloor \frac{t}{4}\right\rfloor, & t \text{ even and } t\le 2r\\ r-\left\lceil \frac{t}{4}\right\rceil, & \ell \text{ even and } 2r<t\le 4r-4\\ 0, & \text{otherwise}\end{cases}$ and $N(t)=\frac{D}{t}-1$.

Now, the conclusion follows from (4.12) and Lemma 4.1. □

Theorem 4.3. *(1) Let $D:=D_{r+c-1}$. For $n\ge 0$ we have that*

$$\mathrm{pp}_{r,c}(n)=\frac{1}{(rc-1)!}\sum_{(\ell_1,\ldots,\ell_{r+c-1})\in\mathbf{A}_n}\prod_{t=2}^{r-1}\sum_{i_t,j_t\ge 0,\ i_t(\frac{D}{t}-1)+j_t=\ell_t}(-1)^{i_t}\binom{t}{i_t}\binom{j_t+t-1}{j_t}$$

$$\cdot\prod_{t=r}^{c}\sum_{i_t,j_t\ge 0,\ i_t(\frac{D}{t}-1)+j_t=\ell_t}(-1)^{i_t}\binom{r}{i_t}\binom{j_t+r-1}{j_t}$$

$$\cdot\prod_{t=c+1}^{r+c-2}\sum_{i_t,j_t\ge 0,\ i_t(\frac{D}{t}-1)+j_t=\ell_t}(-1)^{i_t}\binom{r+c-t}{i_t}\binom{j_t+r+c-t-1}{j_t}$$

$$\cdot\prod_{s=1}^{rc-1}\left(\frac{n-\ell_1-2\ell_2-\cdots-(r+c-1)\ell_{r+c-1}}{D}+s\right).$$

(2) Let $D:=D_{2r-2}$. For $n\ge 0$ we have that

$$\mathrm{pp}^{s}_{r}(n)=\frac{1}{(\binom{r+1}{2}-1)!}\sum_{(\ell_1,\ldots,\ell_{2r-2})\in\mathbf{B}_n}\prod_{t=2}^{r}\sum_{i_t,j_t\ge 0,\ i_t(\frac{D}{t}-1)+j_t=\ell_t}(-1)^{i_t}\binom{\frac{t+2}{2}}{i_t}\binom{j_t+\frac{t+2}{2}-1}{j_t}$$

$$\cdot\prod_{t=r+1}^{2r-4}\sum_{i_t,j_t\ge 0,\ i_t(\frac{D}{t}-1)+j_t=\ell_t}(-1)^{i_t}\binom{\frac{2r-t}{2}}{i_t}\binom{j_t+\frac{2r-t}{2}-1}{j_t}$$

$$\cdot\prod_{s=1}^{\binom{r+1}{2}-1}\left(\frac{n-\ell_1-2\ell_2-\cdots-(2r-2)\ell_{2r-2}}{D}+s\right).$$

(3) Let $D:=D_{2r-1}$. For $n\ge 0$ we have that

$$\mathrm{pp}^{so}_{r}(n)=\frac{1}{(\binom{r+1}{2}-1)!}\sum_{(\ell_1,\ell_3,\ldots,\ell_{4r-4})\in\mathbf{C}_n}\prod_{t=4}^{r}\sum_{i_t,j_t\ge 0,\ i_t(\frac{D}{2t}-1)+j_t=\ell_{2t}}(-1)^{i_t}\binom{\frac{t}{2}}{i_t}\binom{j_t+\frac{t}{2}-1}{j_t}$$

$$\cdot\prod_{t=r+1}^{2r-4}\sum_{i_t,j_t\ge 0,\ i_t(\frac{D}{2t}-1)+j_t=\ell_{2t}}(-1)^{i_t}\binom{\frac{2r-t}{2}}{i_t}\binom{j_t+\frac{2r-t}{2}-1}{j_t}$$

$$\cdot\sum_{k=m}^{\binom{r+1}{2}-1}\left[\begin{matrix}\binom{r+1}{2}\\ k+1\end{matrix}\right](-1)^{k-m}\binom{k}{m}D^{-k}(\ell_1+3\ell_3+\cdots+2r\ell_{2r}+(2r+2)\ell_{2r+2}+\cdots+(4r-4)\ell_{4r-4})^{k-m}$$

$$\cdot\prod_{s=1}^{\binom{r+1}{2}-1}\left(\frac{n-\ell_1-3\ell_2-\cdots-(4r-4)\ell_{4r-4}}{D}+s\right).$$

Proof. (1) From Proposition 3.1 and Theorem 2.3 it follows that

$$\mathrm{pp}_{r,c}(n) = \frac{1}{(rc-1)!} \sum_{(j_1,\dots,j_{rc})\in\tilde{\mathbf{A}}_n} \prod_{t=1}^{rc-1} \left(\frac{n - j_1 - 2j_2 - 2j_3 - \cdots}{D} \right.$$

$$\left. \frac{-rj_{\binom{r}{2}+1} - \cdots - rj_{\binom{r+1}{2}} - \cdots - cj_{\frac{r(2c-r-1)}{2}+1} - \cdots - cj_{\frac{r(2c-r+1)}{2}} - \cdots - (r+c-1)j_{rc}}{D} + t \right),$$

where $\tilde{\mathbf{A}}_n$ was defined in the proof of Theorem 4.2(1).

Denoting $\ell_1 = j_1$, $\ell_2 = j_2 + j_3, \dots, \ell_r = j_{\binom{r}{2}+1} + \cdots + j_{\binom{r+1}{2}}, \dots, \ell_c = j_{\frac{r(2c-r-1)}{2}+1} + \cdots + j_{\frac{r(2c-r+1)}{2}}, \dots, \ell_{r+c-1} = j_{rc}$, we get the required result, using a similar arguing as in the proof of aforementioned Theorem 4.2.

(2) The proof is similar, using Proposition 3.4 and Theorem 2.3.

(3) The proof is similar, using Proposition 3.5 and Theorem 2.3. □

The *Bernoulli polynomials* are defined by

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k.$$

For $\mathbf{a} = (a_1, \dots, a_k)$, the Bernoulli-Barnes numbers (see [1]) are

$$B_j(\mathbf{a}) = \sum_{i_1+\cdots+i_j=j} \binom{j}{i_1, \dots, i_k} B_{i_1} \cdots B_{i_k} a_1^{i_1} \cdots a_k^{i_k}.$$

Let $R, D \geq 1$ be two integers. We consider the $RD \times RD$ determinant:

$$\Delta(D,R) := \begin{vmatrix} \frac{B_1(\frac{1}{D})}{1} & \cdots & \frac{B_1(1)}{1} & \cdots & \frac{B_R(\frac{1}{D})}{R} & \cdots & \frac{B_R(1)}{R} \\ \frac{B_2(\frac{1}{D})}{2} & \cdots & \frac{B_2(1)}{2} & \cdots & \frac{B_{R+1}(\frac{1}{D})}{R+1} & \cdots & \frac{B_{R+1}(1)}{R+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{B_{RD}(\frac{1}{D})}{RD} & \cdots & \frac{B_{RD}(1)}{RD} & \cdots & \frac{B_{RD+R-1}(\frac{1}{D})}{RD+R-1} & \cdots & \frac{B_{RD+R-1}(1)}{RD+R-1} \end{vmatrix}. \tag{4.13}$$

To be more precise, the element on the row m and column $v + D\ell$ in $\Delta(D,R)$ is $\frac{B_{m+\ell-1}(\frac{v}{D})}{m+\ell-1}$, where $1 \leq m \leq RD$, $1 \leq v \leq D$ and $0 \leq \ell \leq R-1$.

Proposition 4.4. *(1) Let $D := D_{r+c-1}$. If $\Delta(D, rc) \neq 0$ then $\mathrm{pp}_{r,c}(n)$ can be expressed in terms of $B_j\left(\frac{v}{D}\right)$, where $1 \leq v \leq D$ and $1 \leq j \leq rcD + rc - 1$, and $B_j(\mathbf{a}_{r,c})$ with $rc \leq j \leq rcD + rc - 1$.*

(2) Let $D := D_{2r-2}$. If $\Delta(D, \binom{r+1}{2}) \neq 0$ then $\mathrm{pp}^s_r(n)$ can be expressed in terms of $B_j\left(\frac{v}{D}\right)$, where $1 \leq v \leq D$ and $1 \leq j \leq \binom{r+1}{2}D + \binom{r+1}{2} - 1$, and $B_j(\mathbf{a}^s_r)$ with $\binom{r+1}{2} \leq j \leq \binom{r+1}{2}D + \binom{r+1}{2} - 1$.

(3) Let $D := D_{2r-1}$. If $\Delta(D, \binom{r+1}{2}) \neq 0$ then $\mathrm{pp}^{so}_r(n)$ can be expressed in terms of $B_j\left(\frac{v}{D}\right)$, where $1 \leq v \leq D$ and $1 \leq j \leq \binom{r+1}{2}D + \binom{r+1}{2} - 1$, and $B_j(\mathbf{a}^{so}_r)$ with $\binom{r+1}{2} \leq j \leq \binom{r+1}{2}D + \binom{r+1}{2} - 1$.

Proof. (1) According to [7, (1.8)], we have that

$$\sum_{m=0}^{rc-1}\sum_{v=1}^{D} d_{r,c,m}(n)D^{n+m}\frac{B_{n+m+1}\left(\frac{v}{D}\right)}{n+m+1} = \frac{(-1)^{rc}n!}{(n+rc)!}B_{rc+n}(\mathbf{a}_{r,c}) - \delta_{0n}. \tag{4.14}$$

Taking $n=0,1,\ldots,rcD-1$ in (4.14) and seing $d_{r,c,m}$'s as unknowns, we obtain a linear system of type $rcD\times rcD$, whose determinant is $\Delta(D,rc)$. Therefore, if $\Delta(D,rc)\neq 0$, then $d_{r,c,m}(n)$'s are the solutions of the aforementioned system. Hence, since

$$\mathrm{pp}_{r,c}(n) = d_{r,c,m}(n)n^{rc-1}+\cdots+d_{r,c,1}(n)n+d_{r,c,0}(n),$$

we get the required result.

(2,3) The proofs are similar to the proof of (1). □

5. The polynomial part and Sylvester waves of $\mathrm{pp}_{r,c}(n)$, $\mathrm{pp}_r^s(n)$ and $\mathrm{pp}_r^{so}(n)$

We recall the following basic facts on quasi-polynomials [15, Proposition 4.4.1]:

Proposition 5.1. *The following conditions on a function $f:\mathbb{N}\to\mathbb{C}$ and integer $D>0$ are equivalent:*

(i) $f(n)$ is a quasi-polynomial of period D.

(ii) $\sum_{n=0}^{\infty} f(n)z^n = \frac{L(z)}{M(z)}$, where $L(z),M(z)\in\mathbb{C}[z]$, every zero λ of $M(z)$ satisfies $\lambda^D=1$ (provided $\frac{L(z)}{M(z)}$ has been reduced to lowest terms), and $\deg L(z)<\deg M(z)$.

(iii) For all $n\geq 0$, $f(n)=\sum_{\lambda^D=1}F_\lambda(n)\lambda^{-n}$, where each $F_\lambda(n)$ is a polynomial function. Moreover, $\deg F_\lambda(n)\leq m(\lambda)-1$, where $m(\lambda)=$ multiplicity of λ as a root of $M(z)$.

We define the *polynomial part* of $f(n)$ to be the polynomial function $F(n)=F_1(n)$, with the notation of Proposition 5.1. The polynomial part $F(n)$ of a quasi-polynomial $f(n)$ gives a rough approximation of $f(n)$, which is useful for studying the asymptotic behaviour of $f(n)$, when $n\gg 0$. If $\mathbf{a}=(a_1,\ldots,a_r)$ is a sequence of positive integers and $p_{\mathbf{a}}(n)$ is the restricted partition function associated to $\mathbf{a}$, we denote $P_{\mathbf{a}}(n)$, the polynomial part of $p_{\mathbf{a}}(n)$. Several formulas of $P_{\mathbf{a}}(n)$ were given in [2], [5] and [10].

Definition 5.2. *With the above notations, we define:*

(1) The polynomial part *of $\mathrm{pp}_{r,c}(n)$ is the function $\mathrm{PP}_{r,c}(n):=P_{\mathbf{a}_{r,c}}(n)$, $n\geq 0$.*

(2) The polynomial part *of $\mathrm{pp}_r^s(n)$ is the function $\mathrm{PP}_r^s(n):=P_{\mathbf{a}_r^s}(n)$, $n\geq 0$.*

(3) The polynomial part *of $\mathrm{pp}_{r,c}(n)$ is the function $\mathrm{PP}_r^{so}(n):=P_{\mathbf{a}_r^{so}}(n)$, $n\geq 0$.*

Theorem 5.3. *(1) Let $D:=D_{r+c-1}$. For $n\geq 0$ we have that*

$$\mathrm{PP}_{r,c}(n) = \frac{1}{D(rc-1)!}\sum_{(\ell_1,\ldots,\ell_{r+c-1})\in\overline{\mathbf{A}}}\prod_{t=2}^{r-1}\sum_{i_t,j_t\geq 0,\ i_t(\frac{D}{t}-1)+j_t=\ell_t}(-1)^{i_t}\binom{t}{i_t}\binom{j_t+t-1}{j_t}$$

$$\cdot\prod_{t=r}^{c}\sum_{i_t,j_t\geq 0,\ i_t(\frac{D}{t}-1)+j_t=\ell_t}(-1)^{i_t}\binom{r}{i_t}\binom{j_t+r-1}{j_t}$$

$$\cdot\prod_{t=c+1}^{r+c-2}\sum_{i_t,j_t\geq 0,\ i_t(\frac{D}{t}-1)+j_t=\ell_t}(-1)^{i_t}\binom{r+c-t}{i_t}\binom{j_t+r+c-t-1}{j_t}$$

$$\cdot \prod_{s=1}^{rc-1} \left(\frac{n - \ell_1 - 2\ell_2 - \cdots - (r+c-1)\ell_{r+c-1}}{D} + s \right).$$

(2) Let $D := D_{2r-2}$*. For* $n \geq 0$ *we have that*

$$\mathrm{PP}_r^s(n) = \frac{1}{D\left(\binom{r+1}{2} - 1\right)!} \sum_{(\ell_1, \dots, \ell_{2r-2}) \in \overline{\mathbf{B}}} \prod_{t=2}^{r} \sum_{i_t, j_t \geq 0,\ i_t(\frac{D}{t}-1)+j_t = \ell_t} (-1)^{i_t} \binom{\frac{t+2}{2}}{i_t} \binom{j_t + \frac{t+2}{2} - 1}{j_t}$$

$$\cdot \prod_{t=r+1}^{2r-4} \sum_{i_t, j_t \geq 0,\ i_t(\frac{D}{t}-1)+j_t = \ell_t} (-1)^{i_t} \binom{\frac{2r-t}{2}}{i_t} \binom{j_t + \frac{2r-t}{2} - 1}{j_t}$$

$$\cdot \prod_{s=1}^{\binom{r+1}{2}-1} \left(\frac{n - \ell_1 - 2\ell_2 - \cdots - (2r-2)\ell_{2r-2}}{D} + s \right).$$

(3) Let $D := D_{2r-1}$*. For* $n \geq 0$ *we have that*

$$\mathrm{PP}_r^{so}(n) = \frac{1}{D\left(\binom{r+1}{2} - 1\right)!} \sum_{(\ell_1, \ell_3, \dots, \ell_{4r-4}) \in \overline{\mathbf{C}}} \prod_{t=4}^{r} \sum_{i_t, j_t \geq 0,\ i_t(\frac{D}{2t}-1)+j_t = \ell_{2t}} (-1)^{i_t} \binom{\frac{t}{2}}{i_t} \binom{j_t + \frac{t}{2} - 1}{j_t}$$

$$\cdot \prod_{t=r+1}^{2r-4} \sum_{i_t, j_t \geq 0,\ i_t(\frac{D}{2t}-1)+j_t = \ell_{2t}} (-1)^{i_t} \binom{\frac{2r-t}{2}}{i_t} \binom{j_t + \frac{2r-t}{2} - 1}{j_t}$$

$$\cdot \sum_{k=m}^{\binom{r+1}{2}-1} \begin{bmatrix} \binom{r+1}{2} \\ k+1 \end{bmatrix} (-1)^{k-m} \binom{k}{m} D^{-k} (\ell_1 + 3\ell_3 + \cdots + 2r\ell_{2r} + (2r+2)\ell_{2r+2} + \cdots + (4r-4)\ell_{4r-4})^{k-m}$$

$$\cdot \prod_{s=1}^{\binom{r+1}{2}-1} \left(\frac{n - \ell_1 - 3\ell_2 - \cdots - (4r-4)\ell_{4r-4}}{D} + s \right).$$

Proof. (1) The proof is similar to the proof of Theorem 4.3, using Proposition 3.1, Theorem 2.5 and (4.4).

(2) The proof is similar to the proof of Theorem 4.3, using Proposition 3.4, Theorem 2.5 and (4.6).

(3) The proof is similar to the proof of Theorem 4.3, using Proposition 3.5, Theorem 2.5 and (4.8). □

Theorem 5.4. *(1) The polynomial part of* $\mathrm{pp}_{r,c}(n)$ *is*

$$\mathrm{PP}_{r,c}(n) = \frac{1}{1 \cdot 2^2 \cdots (r-1)^{r-1} (r \cdot (r+1) \cdots c)^r \cdot (c+1)^{r-1} \cdots (r+c-1)}$$
$$\cdot \sum_{u=0}^{rc-1} \frac{(-1)^u}{(rc-1-u)!} \sum_{i_1 + \cdots + i_{rc} = u} \frac{B_{i_1} \cdots B_{i_{rc}}}{i_1! \cdots i_{rc}!} 1^{i_1} 2^{i_2} 2^{i_3} \cdots (r+c-1)^{i_{rc}} n^{rc-1-u}.$$

(2) The polynomial part of $\mathrm{pp}_r^s(n)$ *is*

$$\mathrm{PP}_r^s(n) = \frac{1}{1 \cdot 2^2 \cdot 3^2 \cdots r^{\lfloor \frac{r+2}{2} \rfloor} \cdot (r+1)^{\lfloor \frac{r-1}{2} \rfloor} \cdots (2r-2)} \sum_{u=0}^{\binom{r+1}{2}-1} \frac{(-1)^u}{(\binom{r+1}{2}-1-u)!}$$
$$\cdot \sum_{i_1+\cdots+i_{\binom{r+1}{2}}=u} \frac{B_{i_1}\cdots B_{i_{\binom{r+1}{2}}}}{i_1!\cdots i_{\binom{r+1}{2}}!} 1^{i_1} 2^{i_2} 2^{i_3} \cdots (2r-2)^{i_{\binom{r+1}{2}}} n^{\binom{r+1}{2}-1-u}.$$

(3) The polynomial part of $\mathrm{pp}_r^{so}(n)$ *is*

$$\mathrm{PP}_r^{so}(n) = \frac{1}{1 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8^2 \cdots (2r-1) \cdot (2r)^{\lfloor \frac{r}{2} \rfloor} (2r+2)^{\lfloor \frac{r-1}{2} \rfloor} \cdots (4r-4)}$$
$$\cdot \sum_{u=0}^{\binom{r+1}{2}-1} \frac{(-1)^u}{(\binom{r+1}{2}-1-u)!} \sum_{i_1+\cdots+i_{\binom{r+1}{2}}=u} \frac{B_{i_1}\cdots B_{i_{\binom{r+1}{2}}}}{i_1!\cdots i_{\binom{r+1}{2}}!} 1^{i_1} 3^{i_2} 4^{i_3} \cdots (4r-4)^{i_{\binom{r+1}{2}}} n^{\binom{r+1}{2}-1-u}.$$

Proof. (1) It follows from (3.6), Proposition 3.1 and Theorem 2.6.
(2) It follows from (3.8), Proposition 3.4 and Theorem 2.6.
(3) It follows from (3.10), Proposition 3.5 and Theorem 2.6. □

As in (2.2), we can write

$$\mathrm{pp}_{r,c}(n) = \sum_{j\geq 1} W_j(n,r,c), \text{ where } W_j(n,r,c) = W_j(n,\mathbf{a}_{r,c}),$$
$$\mathrm{pp}_r^s(n) = \sum_{j\geq 1} W_j^s(n,r), \text{ where } W_j^s(n,r) = W_j(n,\mathbf{a}_r^s),$$
$$\mathrm{pp}_r^{so}(n) = \sum_{j\geq 1} W^{so}(n,r), \text{ where } W_j^{so}(n,r) = W_j(n,\mathbf{a}_r^{so}).$$

Also, for $1 \leq \ell \leq j$, we consider the sets:

$$\mathbf{A}_{\ell,j} = \{(\ell_1,\ldots,\ell_{r+c-1}) \in \overline{\mathbf{A}} \;:\; \ell_1 + 2\ell_2 + \cdots + (r+c-1)\ell_{r+c-1} \equiv \ell (\bmod\ j)\}.$$
$$\mathbf{B}_{\ell,j} = \{(\ell_1,\ldots,\ell_{2r-2}) \in \overline{\mathbf{B}} \;:\; \ell_1 + 2\ell_2 + \cdots + (2r-2)\ell_{2r-2} \equiv \ell (\bmod\ j)\}.$$
$$\mathbf{C}_{\ell,j} = \{(\ell_1,\ell_3,\ell_4,\ldots,\ell_{2r},\ell_{2r+2},\ldots,\ell_{4r-4}) \in \overline{\mathbf{C}} \;:\; \ell_1 + 3\ell_3 + \cdots +$$
$$+ 2r\ell_{2r} + (2r+2)\ell_{2r+2} + \cdots + (4r-4)\ell_{4r-4} \equiv \ell (\bmod\ j)\}.$$

Theorem 5.5. *With the above notations, we have that:*

(1) Let $D := D_{r+c-1}$. For $n \geq 0$ we have that

$$W_j(n,r,c) = \frac{1}{D(rc-1)!}\sum_{m=1}^{rc}\sum_{\ell=1}^{j} e^{\frac{2\pi\ell i}{j}} \sum_{k=m-1}^{rc-1} \begin{bmatrix} rc \\ k+1 \end{bmatrix} (-1)^{k-m+1}\binom{k}{m-1} D^{-k}$$

$$\cdot \sum_{(\ell_1,\ldots,\ell_{r+c-1})\in \mathbf{A}_{\ell,j}} \prod_{t=2}^{r-1} \sum_{i_t,j_t\geq 0,\ i_t(\frac{D}{t}-1)+j_t=\ell_t} (-1)^{i_t}\binom{t}{i_t}\binom{j_t+t-1}{j_t}$$

$$\cdot \prod_{t=r}^{c} \sum_{i_t,j_t\geq 0,\ i_t(\frac{D}{t}-1)+j_t=\ell_t} (-1)^{i_t}\binom{r}{i_t}\binom{j_t+r-1}{j_t}$$

$$\cdot \prod_{t=c+1}^{r+c-2} \sum_{i_t,j_t\geq 0,\ i_t(\frac{D}{t}-1)+j_t=\ell_t} (-1)^{i_t}\binom{r+c-t}{i_t}\binom{j_t+r+c-t-1}{j_t}$$

$$\cdot(\ell_1+2\ell_2+\cdots+(r+c-1)\ell_{r+c-1})^{k-m+1} n^{m-1}.$$

(2) Let $D := D_{2r-2}$. For $n \geq 0$ we have that

$$W_j^s(n,r) = \frac{1}{D(\binom{r+1}{2}-1)!}\sum_{m=1}^{\binom{r+1}{2}}\sum_{\ell=1}^{j} e^{\frac{2\pi\ell i}{j}} \sum_{k=m-1}^{\binom{r+1}{2}-1} \begin{bmatrix} \binom{r+1}{2} \\ k+1 \end{bmatrix} (-1)^{k-m+1}\binom{k}{m-1} D^{-k}$$

$$\cdot \sum_{(\ell_1,\ldots,\ell_{2r-2})\in \mathbf{B}_{\ell,j}} \prod_{t=2}^{r} \sum_{i_t,j_t\geq 0,\ i_t(\frac{D}{t}-1)+j_t=\ell_t} (-1)^{i_t}\binom{\frac{t+2}{2}}{i_t}\binom{j_t+\frac{t+2}{2}-1}{j_t}$$

$$\cdot \prod_{t=r+1}^{2r-4} \sum_{i_t,j_t\geq 0,\ i_t(\frac{D}{t}-1)+j_t=\ell_t} (-1)^{i_t}\binom{\frac{2r-t}{2}}{i_t}\binom{j_t+\frac{2r-t}{2}-1}{j_t}$$

$$\cdot(\ell_1+2\ell_2+\cdots+(2r-2)\ell_{2r-2})^{k-m} n^{m-1}.$$

(3) Let $D := D_{2r-1}$. For $n \geq 0$ we have that

$$W_j^{so}(n,r) = \frac{1}{D(\binom{r+1}{2}-1)!}\sum_{m=1}^{\binom{r+1}{2}}\sum_{\ell=1}^{j} e^{\frac{2\pi\ell i}{j}} \sum_{k=m-1}^{\binom{r+1}{2}-1} \begin{bmatrix} \binom{r+1}{2} \\ k+1 \end{bmatrix} (-1)^{k-m+1}\binom{k}{m-1} D^{-k}$$

$$\cdot \sum_{(\ell_1,\ell_3,\ell_4,\ldots,\ell_{2r},\ell_{2r+2},\ldots,\ell_{4r-4})\in \mathbf{C}_{\ell,j}} \prod_{t=4}^{r} \sum_{i_t,j_t\geq 0,\ i_t(\frac{D}{2t}-1)+j_t=\ell_{2t}} (-1)^{i_t}\binom{\frac{t}{2}}{i_t}\binom{j_t+\frac{t}{2}-1}{j_t}$$

$$\cdot \prod_{t=r+1}^{2r-4} \sum_{i_t,j_t\geq 0,\ i_t(\frac{D}{2t}-1)+j_t=\ell_{2t}} (-1)^{i_t}\binom{\frac{2r-t}{2}}{i_t}\binom{j_t+\frac{2r-t}{2}-1}{j_t}$$

$$\cdot(\ell_1+3\ell_3+\cdots+2r\ell_{2r}+(2r+2)\ell_{2r+2}+\cdots+(4r-4)\ell_{4r-4})^{k-m} n^{m-1}.$$

Proof. (1) The result follows from Proposition 3.1 and Proposition 2.4.

(2) The result follows from Proposition 3.4 and Proposition 2.4.

(3) The result follows from Proposition 3.5 and Proposition 2.4. □

6. Some congruences for $\operatorname{pp}_{r,c}(n)$, $\operatorname{pp}^s_r(n)$ and $\operatorname{pp}^{so}_r(n)$

Using the notations from (3.6), (3.8) and (3.10), we define:

$$\sigma_{r,c} := \sum_{\ell=1}^{r+c-1} \ell\alpha_{r,c}(\ell),\ \sigma^s_r := \sum_{\ell=1}^{2r-2} \ell\alpha^s_r(\ell),\ \sigma^{so}_r := \sum_{\ell=1}^{4r-4} \ell\alpha^{so}_r(\ell).$$

Lemma 6.1. *We have that:*

(1) $\sigma_{r,c} = \frac{rc(r+c)}{2}$.
(2) $\sigma^s_r = \frac{r(r^2+1)}{2}$.
(3) $\sigma^{so}_r = r^3$.

Proof. (1) According to (3.4) we have that

$$\sigma_{r,c} = \sum_{i=1}^{r}\sum_{j=1}^{c}(i+j-1) = \sum_{i=1}^{r}((i-1)c + \binom{c+1}{2}) = c\binom{r}{2} + r\binom{c+1}{2} = \frac{rc(r+c)}{2}.$$

(2) According to Corollary 3.3(1) and (3.7) we have that

$$\sigma^s_r = \sum_{i=1}^{r} i + \sum_{1\le i<j\le r}(i+j-1) = \binom{r+1}{2} + \sum_{i=1}^{r}\sum_{j=i+1}^{r}(i+j-1) =$$
$$= \binom{r+1}{2} + \sum_{i=1}^{r}(i(r-i) + \binom{r}{2} - \binom{i}{2}) =$$
$$= \binom{r+1}{2} + r\binom{r+1}{2} - \frac{r(r+1)(2r+1)}{4} + r\binom{r}{2} + \frac{1}{2}\binom{r+1}{2} = \frac{r(r^2+1)}{2}.$$

(3) First, note that, from the proof of (2) it follows that

$$\sum_{1\le i<j\le r}(i+j-1) = \frac{r(r^2+1)}{2} - \frac{r(r+1)}{2} = \frac{r^2(r-1)}{2}.$$

Therefore, according to Corollary 3.3(2) and (3.9), it follows that

$$\sigma^{so}_r = \sum_{i=1}^{r}(2i-1) + 2\sum_{1\le i<j\le r}(i+j-1) = r^2 + r^2(r-1) = r^3,$$

which concludes the proof. □

Proposition 6.2. *(1) For $n \ge 0$ it holds that*

$$(rc-1)! \cdot \operatorname{pp}_{r,c}(n) \equiv 0 \mod (j+k+1)(j+k+2)\cdots(j+rc-1),$$

where $D = D_{r+c-1}$, $k = \lfloor \frac{n}{D} \rfloor - \left\lceil \frac{2n+rc(r+c)}{2D} \right\rceil + rc$ and $\left\lceil \frac{2n+rc(r+c)}{2D} \right\rceil - rc \le j \le \lfloor \frac{n}{D} \rfloor$.

(2) For $n \ge 0$ it holds that

$$\left(\binom{r+1}{2} - 1\right)! \cdot \operatorname{pp}^s_r(n) \equiv 0 \mod (j+k+1)(j+k+2)\cdots(j+\binom{r+1}{2}-1),$$

where $D = D_{2r-2}$, $k = \lfloor \frac{n}{D} \rfloor - \left\lceil \frac{2n+r(r^2+1)}{2D} \right\rceil + \binom{r+1}{2}$ and $\left\lceil \frac{2n+r(r^2+1)}{2D} \right\rceil - \binom{r+1}{2} \le j \le \lfloor \frac{n}{D} \rfloor$.

(3) For $n \geq 0$ it holds that

$$\left(\binom{r+1}{2}-1\right)! \cdot \mathrm{pp}_r^s(n) \equiv 0 \mod (j+k+1)(j+k+2)\cdots(j+\binom{r+1}{2}-1),$$

where $D = D_{2r-1}$, $k = \lfloor \frac{n}{D} \rfloor - \lceil \frac{n+r^3}{D} \rceil + \binom{r+1}{2}$ *and* $\lceil \frac{n+r^3}{D} \rceil - \binom{r+1}{2} \leq j \leq \lfloor \frac{n}{D} \rfloor$.

Proof. (1) It follows from (3.6), Proposition 3.1, Lemma 6.1(1) and Proposition 2.7.

(2) It follows from (3.8), Proposition 3.4, Lemma 6.1(2) and Proposition 2.7.

(3) It follows from (3.10), Proposition 3.5, Lemma 6.1(3) and Proposition 2.7. □

Theorem 6.3. *If $m > 1$ is a positive integer, then*

$$(1)\ \lim_{N\to\infty} \frac{\#\{n \leq N \ :\ \mathrm{pp}_{r,c}(n) \not\equiv 0 (\mathrm{mod}\ m)\}}{N} \geq \frac{2}{rc(r+c)}.$$

$$(2)\ \lim_{N\to\infty} \frac{\#\{n \leq N \ :\ \mathrm{pp}_{r}^{s}(n) \not\equiv 0 (\mathrm{mod}\ m)\}}{N} \geq \frac{2}{r(r^2+1)}.$$

$$(3)\ \lim_{N\to\infty} \frac{\#\{n \leq N \ :\ \mathrm{pp}_{r}^{so}(n) \not\equiv 0 (\mathrm{mod}\ m)\}}{N} \geq \frac{1}{r^3}.$$

Proof. (1) The result follows from Theorem 2.8 and Lemma 6.1(1).

(2) The result follows from Theorem 2.8 and Lemma 6.1(2).

(3) The result follows from Theorem 2.8 and Lemma 6.1(3). □

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Faculty of Applied Sciences, National University of Science and Technology Politehnica Bucharest, Bucharest, Romania and Simion Stoilow Institute of Mathematics, Bucharest, Romania

Email address: mircea.cimpoeas@imar.ro

Faculty of Applied Sciences, National University of Science and Technology Politehnica Bucharest, Bucharest, Romania

Email address: alexandra.teodor@upb.ro