MIXED PROBLEM FOR A NONLINEAR BARENBLATT-ZHELTOV-KOCHINA EQUATION WITH HILFER FRACTIONAL OPERATOR

T. K. YULDASHEV, B. J. KADIRKULOV AND A. A. MATCHANOVA

Abstract. The issues of unique classical solvability and construction of a solution to a mixed problem for a nonlinear differential equation containing a Hilfer fractional analogue of the Barenblatt-Zheltov-Kochina operator are studied. Taking into account the peculiarities of the fractional Hilfer operator, the Fourier series method was used. Eigenvalues, eigenfunctions and associated functions are found for the spectral problem and for the conjugate spectral problem. Obtained: a scalar nonlinear fractional equation and three countable systems of nonlinear fractional differential equations with initial value conditions. The study of the existence and uniqueness of a solution to a mixed problem is reduced to the study of the existence and uniqueness of a nonlinear ordinary integral equation and countable systems of nonlinear ordinary integral equations in the corresponding Banach spaces. Sufficient coefficient conditions for unique classical solvability of a mixed problem are established.

Mathematics Subject Classification (2010): 35G05, 35J08, 35J25
Key words: Hilfer fractional operator, associated functions, conjugate spectral problem, classical solvability.

Article history:
Received: May 17, 2024
Received in revised form: November 28, 2024
Accepted: December 1, 2024

1. Introduction. Formulation of the problem

Problems of mechanics often turn out to be initial-boundary (mixed) problems. In particular, many mixed problems arise in hydrodynamics when solving problems of hydroelasticity [1]. In [8, 10] mixed problems for second-order linear differential equations of parabolic and hyperbolic types were studied. The main provisions of the theory of non-stationary filtration in fractured-pore formations are formulated in the work of G. I. Barenblatt, Yu. P. Zheltov and I. N. Kochina [4] (see also [5]). The theory and applications of fractional calculus have been developed by many authors (see, for example, [7, 9, 11, 12, 14]).

In our work we consider a mixed problem for a nonlinear differential equation with Hilfer operator of fractional integro-differentiation. Thus, in the domain $\Omega \equiv (0,T)\times(0,1)$ we consider the equation

$$\left(D^{\alpha,\gamma} - D^{\alpha,\gamma}\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x^2}\right) U(t,x) = f\left(t, x, \int_0^T\int_0^1 G(s,y)\,U(s,y)\,dy\,ds\right) \tag{1.1}$$

with mixed conditions

$$\lim_{t\to+0} J_{0t}^{1-\gamma} U(t,x) = \varphi(x), \tag{1.2}$$

$$U(t,0) = 0, \tag{1.3}$$

$$U_x(t,1) = U_x(t,x_0), \quad 0 \le t \le T, \quad 0 < x_0 < 1, \tag{1.4}$$

where $D^{\alpha,\gamma} = D_{0t}^{\alpha,\beta} = J_{0t}^{\beta(1-\alpha)} \frac{d}{dt} J_{0t}^{(1-\beta)(1-\alpha)} = J_{0t}^{\gamma-\alpha} \frac{d}{dt} J_{0t}^{1-\gamma}$ is Hilfer fractional operator, $0 \le \beta \le 1$, $\gamma = \alpha + \beta - \alpha\beta$, $\varphi(x) \in C^6[0,1]$, $0 < \alpha \le \gamma \le 1$,

$$\frac{d^i \varphi(x)}{dx^i}\,|_{x=0} = 0, \quad \frac{d^j \varphi(x)}{dx^j}\,|_{x=1} = \frac{d^j \varphi(x)}{dx^j}\,|_{x=x_0}, \quad i = 0,2,4,6, \quad j = 1,3,5,$$

$$\frac{d^i f(t,x,\cdot)}{dx^i}\,|_{x=0} = 0, \quad \frac{d^j f(t,x,\cdot)}{dx^j}\,|_{x=1} = \frac{d^j f(t,x,\cdot)}{dx^j}\,|_{x=x_0}, \quad i = 0,2,4,6, \quad j = 1,3,5,$$

$f(t,x,u) \in C_{t,x,u}^{0,6,0}(\bar{\Omega} \times R)$, $0 < G(t,x) \in C(\bar{\Omega})$, $J_{0t}^{\alpha}\psi(t) = \frac{1}{\Gamma(\alpha)} \int\limits_0^t \frac{\psi(s)\,ds}{(t-s)^{1-\alpha}}$ is Riemann-Liouville integral operator, $\bar{\Omega} \equiv [0,T] \times [0,1]$, $0 < T < \infty$.

Problem. *It is required to find a function $U(t,x)$, that satisfies the nonlinear differential equation* (1.1)*, the initial value condition* (1.2)*, the boundary value conditions* (1.3)*,* (1.4) *and belongs to the class of smoothness*

$$t^{1-\gamma} D^{\alpha,\gamma} U \in C(\overline{\Omega}), \quad D^{\alpha,\gamma} U_{xx} \in C(\Omega), \quad U_{xx} \in C(\Omega). \tag{1.5}$$

We note that the study of the existence and uniqueness of a solution to a mixed problem is reduced to the study of the existence and uniqueness of a nonlinear ordinary integral equation in the space $C[0,T]$ and countable systems of nonlinear ordinary integral equations in the Banach space $B_2[0,T]$ (see, Section 4).

This problem is studied in the case of linear equation in the work [13].

2. Eigenvalues and eigenfunctions of spectral problem

First, we consider homogeneous differential equation

$$D^{\alpha,\gamma} U(t,x) - D^{\alpha,\gamma} U_{xx}(t,x) - U_{xx}(t,x) = 0. \tag{2.1}$$

We will look for a non-trivial particular solution of this problem in the form $U(t,x) = u(t) \cdot \vartheta(x)$. Substituting this product into equation (2.1), we obtain

$$\frac{D^{\alpha,\gamma} u(t)}{D^{\alpha,\gamma} u(t) + u(t)} = \frac{\vartheta''(x)}{\vartheta(x)}.$$

Hence, equating second fraction into $-\lambda$ we obtain the ordinary differential equation of second order

$$\vartheta''(x) + \lambda \vartheta(x) = 0. \tag{2.2}$$

According to the conditions (1.3) and (1.4), we have boundary value conditions

$$\vartheta(0) = 0, \quad \vartheta'(1) = \vartheta'(x_0), \quad 0 < x_0 < 1. \tag{2.3}$$

For $\lambda < 0$, problem (2.2), (2.3) has only a trivial solution, so consider the case $\lambda \ge 0$. Solving the spectral problem (2.2), (2.3), we derive eigenvalues

$$\lambda_0 = 0, \quad \lambda_{1,n} = \left(\frac{2n\pi}{1 + x_0} \right)^2, \quad \lambda_{2,n} = \left(\frac{2n\pi}{1 - x_0} \right)^2, \quad n \in \mathbb{N} \tag{2.4}$$

and eigenfunctions

$$\vartheta_0(x) = x, \quad \vartheta_{i,n}(x) = \sin \sqrt{\lambda_{i,n}} x, \quad i = 1,2, \quad n \in \mathbb{N}. \tag{2.5}$$

The spectral problem (2.2), (2.3) was studied in detail in [2]. For ease of readability of the article, we present some results obtained in these works. Let us denote

$$\Delta_{n,m} = \frac{n-m}{n+m}, \quad m,n \in \mathbb{N}, \; n > m.$$

Lemma 2.1. *The system of functions* (2.5) *has associated functions only for those eigenvalues* $\lambda_{1,n},\ \lambda_{2,m},\ n,m\in\mathbb{N}$ *of problem* (2.2), (2.3) *for which relation* $x_0=\Delta_{n,m}$ *holds. For each such pair* (n,m), *there is only one associated function.*

Lemma 2.2. *Let* x_0 *be a rational number from the interval* $(0,1)$ *such that* $x_0=\frac{p}{q},\ p<q,\ p$ *and* q *be coprime natural numbers. Then there exists a countable values of* n *and* m *such that for two series of eigenvalues from* (2.4), *we have* $\lambda_{1,n}=\lambda_{2,m}$, *in addition* n *and* m *have the form* $m=s(q-p)$ *and* $n=s(q+p)$. *Here* $s\in\mathbb{N}$, *when* $q-p$ *is odd, and* $2s\in\mathbb{N}$, *when* $q-p$ *is even.*

Corollary 2.3. *Let* x_0 *be a rational number from the interval* $(0,1)$ *such that* $x_0=\frac{p}{q},\ p<q,\ p$ *and* q *be coprime natural numbers. Then* $m=s\in\mathbb{N}$ *and therefore for two sets of eigenvalues* $\left\{\lambda_{1,n}\right\}_{n=1}^{\infty}$ *and* $\left\{\lambda_{2,m}\right\}_{m=1}^{\infty}$ *from* (2.4), *the inclusion* $\left\{\lambda_{2,m}\right\}_{m=1}^{\infty}\subset\left\{\lambda_{1,n}\right\}_{n=1}^{\infty}$ *takes place, i.e., the set of eigenvalues* $\left\{\lambda_{2,m}\right\}_{m=1}^{\infty}$ *is contained in the set* $\left\{\lambda_{1,n}\right\}_{n=1}^{\infty}$.

Along with problem (2.2), (2.3), we also consider adjoint problem. It is not difficult to determine that the following problem will be adjoint to problem (2.2), (2.3):

$$\omega''(x)+\lambda\,\omega(x)=0,\quad \lambda\geq 0,\quad x\in(0,x_0)\cup(x_0,1), \tag{2.6}$$

$$\omega(0)=0,\quad \omega'(1)=0, \tag{2.7}$$

$$\omega'(x_0+0)=\omega'(x_0-0),\quad \omega(x_0+0)-\omega(x_0-0)=\omega(1). \tag{2.8}$$

Consider the case when x_0 is an irrational number from $(0,1)$. We obtain in this case two series of eigenvalues of the form (2.4), which correspond to eigenfunctions of the form (2.5), and all these functions are different and not orthogonal. Problem (2.6)-(2.8) also has eigenvalues of the form (2.4). Solving this problem, it is not difficult to see that the eigenfunctions have the form

$$\omega_0(x)=\begin{cases}0, & x\in[0,x_0),\\ \frac{2}{1-x_0^2}, & x\in(x_0,1],\end{cases}\quad \omega_{1,n}(x)=\begin{cases}\frac{4\sin\sqrt{\lambda_{1,n}}x}{1+x_0}, & x\in[0,x_0),\\ \frac{2\cos\sqrt{\lambda_{1,n}}(1-x)}{(1+x_0)\sin\sqrt{\lambda_{1,n}}}, & x\in(x_0,1],\end{cases}$$

$$\omega_{2,n}(x)=\begin{cases}0, & x\in[0,x_0),\\ \frac{4\cos\sqrt{\lambda_{2,n}x}}{(1-x_0)\sin\sqrt{\lambda_{2,n}x}}, & x\in(x_0,1].\end{cases}$$

Lemma 2.4. *Let the number* x_0 *be irrational. Then there is a sequence of* $\{n_m\}$, *for which* $\|\omega_{i,n_m}(x)\|_{L_2(0,1)}\to\infty,\ \ i=1,2.$

Corollary 2.5. *Let* x_0 *be any irrational number from the interval* $(0,1)$. *Then the system of root functions of problem* (2.6)-(2.8) *does not form Riesz basis in* $L_2[0,1]$.

More detailed information on Riesz bases can be found in [6]. We consider our mixed problem (1.1)-(1.5) when the following condition is fulfilled.

Condition A. Let x_0 be a rational number from the interval $(0,1)$ such that $x_0=\frac{p}{q},\ p<q,\ q-p=1$, p and q be positive integers.

If condition A is satisfying, then solving problem (2.2), (2.3), instead (2.4) we obtain

$$\lambda_0=0,\quad \lambda_{1,n}=\left(\frac{2qn\pi}{p+q}\right)^2,\quad \lambda_{2,m}=(2qm\pi)^2,\quad n,m\in\mathbb{N},\ \ n\neq m(p+q). \tag{2.9}$$

These eigenvalues correspond to eigenfunctions in (2.5). For each value of $\lambda_{2,m}$, there also exist associated functions of the form

$$\tilde{\vartheta}_{2,m}(x)=x\cos\sqrt{\lambda_{2,m}}x. \tag{2.10}$$

Problem (2.6)–(2.8) also has eigenvalues of the form (2.9). Solving this problem, it is not difficult to see that the eigenfunctions have the form

$$\{\omega_0(x);\ \omega_{1,n}(x);\ \omega_{2,m}(x)\},\ n,m\in\mathbb{N},\ n\neq m(p+q). \tag{2.11}$$

where

$$\omega_0(x)=\begin{cases}0, & x\in[0,x_0),\\ \frac{2}{1-x_0^2}, & x\in(x_0,1],\end{cases}\quad \omega_{1,n}(x)=\begin{cases}\frac{4\sin\sqrt{\lambda_{1,n}}x}{1+x_0}, & x\in[0,x_0),\\ \frac{2\cos\sqrt{\lambda_{1,n}}(1-x)}{(1+x_0)\sin\sqrt{\lambda_{1,n}}}, & x\in(x_0,1],\end{cases}$$

$$\omega_{2,m}(x)=\begin{cases}0, & x\in[0,x_0),\\ \frac{4\cos\sqrt{\lambda_{2,m}}x}{1-x_0}, & x\in(x_0,1].\end{cases}$$

There also exist associated functions of the form

$$\tilde{\omega}_{2,m}(x)=\begin{cases}\frac{4\sin\sqrt{\lambda_{2,m}}x}{1+x_0}, & x\in[0,x_0),\\ \frac{4(1-x)\sin\sqrt{\lambda_{2,m}}x}{1-x_0^2}, & x\in(x_0,1].\end{cases} \tag{2.12}$$

We note that systems of eigenfunctions (2.5), (2.10) and (2.11), (2.12) are biorthonormal in the space $L_2[0,1]$, that is

$$(\vartheta_0(x),\,\omega_0(x))=1,\quad (\vartheta_{1,n}(x),\,\omega_{1,k}(x))=\begin{cases}1, & n=k,\\ 0, & n\neq k,\end{cases}$$

$$(\vartheta_{2,m}(x),\,\tilde{\omega}_{2,k}(x))=\begin{cases}1, & m=k,\\ 0, & m\neq k,\end{cases}\qquad \left(\tilde{\vartheta}_{2,m}(x),\,\omega_{2,k}(x)\right)=\begin{cases}1, & m=k,\\ 0, & m\neq k,\end{cases}$$

$$(\vartheta_0(x),\,\omega_{1,n}(x))=(\vartheta_0(x),\,\omega_{2,m}(x))=(\vartheta_0(x),\,\tilde{\omega}_{2,m}(x))=0,$$

$$(\vartheta_{1,n}(x),\,\omega_0(x))=(\vartheta_{1,n}(x),\,\omega_{2,m}(x))=(\vartheta_{1,n}(x),\,\tilde{\omega}_{2,m}(x))=0,$$

$$(\vartheta_{2,m}(x),\,\omega_0(x))=(\vartheta_{2,m}(x),\,\omega_{1,n}(x))=(\vartheta_{2,m}(x),\,\omega_{2,k}(x))=0,$$

$$\left(\tilde{\vartheta}_{2,m}(x),\,\omega_0(x)\right)=\left(\tilde{\vartheta}_{2,m}(x),\,\tilde{\omega}_{2,k}(x)\right)=0,$$

where by $(\cdot,\,\cdot)$ is denoted the inner product in $L_2[0,1]$.

We note that if the condition A is satisfying, then the systems of root functions of problems (2.2), (2.3) and (2.6)-(2.8) form a Riesz basis in the space $L_2[0,1]$ (see, [2, 3]). Therefore, taking into account the formulas (2.5), (2.10) and (2.11), (2.12), we look for a solution

$$U(t,x)=U_0(t,x)+U_1(t,x)+U_2(t,x)+\tilde{U}_2(t,x) \tag{2.13}$$

to the problem (1.1)-(1.5) in the following form of Fourier series:

$$U(t,x)=u_0(t)\,\vartheta_0(x)+\sum_{n=1}^{\infty *}u_{1,n}(t)\,\vartheta_{1,n}(x)+\sum_{m=1}^{\infty}\left(u_{2,m}(t)\,\vartheta_{2,m}(x)+\tilde{u}_{2,m}(t)\,\tilde{\vartheta}_{2,m}(x)\right), \tag{2.14}$$

where

$$u_0(t)=\int_0^1 U_0(t,y)\omega_0(y)dy,\quad u_{1,n}(t)=\int_0^1 U_1(t,y)\omega_{1,n}(y)dy,$$

$$u_{2,m}(t)=\int_0^1 U_2(t,y)\tilde{\omega}_{2,m}(y)dy,\quad \tilde{u}_{2,m}(t)=\int_0^1 \tilde{U}_2(t,y)\omega_{2,m}(y)dy.$$

Here "*" means that the sum is taken over $n\in\mathbb{N}$, different from $k(q+p),\ k\in\mathbb{N}$.

3. Reducing the solution of problem to a countable systems of integral equations

Let the condition A be satisfied and a function $U(t,x)$ be a solution to the mixed problem (1.1)-(1.5). Then, applying representation (2.14) into equation (1.1) and taking (2.5) and (2.13) into account, we obtain

$$D^{\alpha,\gamma}\left\{x\,u_0(t)+\sum_{n=1}^{\infty *}u_{1,n}(t)\,\vartheta_{1,n}(x)+\sum_{m=1}^{\infty}\Big(u_{2,m}(t)\,\vartheta_{2,m}(x)+\tilde{u}_{2,m}(t)\,\tilde{\vartheta}_{2,m}(x)\Big)+\right.$$

$$\left.+\sum_{n=1}^{\infty *}\lambda_{1,n}u_{1,n}(t)\,\vartheta_{1,n}(x)+\sum_{m=1}^{\infty}\Big[\lambda_{2,m}u_{2,m}(t)\,\vartheta_{2,m}(x)+\tilde{u}_{2,m}(t)\Big(2\sqrt{\lambda_{2,m}}\vartheta_{2,m}(x)+\lambda_{2,m}\tilde{\vartheta}_{2,m}(x)\Big)\Big]\right\}+$$

$$+\sum_{n=1}^{\infty *}\lambda_{1,n}u_{1,n}(t)\,\vartheta_{1,n}(x)+\sum_{m=1}^{\infty}\Big[\lambda_{2,m}u_{2,m}(t)\,\vartheta_{2,m}(x)+\tilde{u}_{2,m}(t)\Big(2\sqrt{\lambda_{2,m}}\,\vartheta_{2,m}(x)+\lambda_{2,m}\tilde{\vartheta}_{2,m}(x)\Big)\Big]=$$

$$=x\,f_0(t,u)+\sum_{n=1}^{\infty *}f_{1,n}(t,u)\,\vartheta_{1,n}(x)+\sum_{m=1}^{\infty}\Big(f_{2,m}(t,u)\,\vartheta_{2,m}(x)+\tilde{f}_{2,m}(t,u)\,\tilde{\vartheta}_{2,m}(x)\Big),$$

where

$$f_0(t,u)=\int_0^1 f_0\left(t,y,\int_0^T\int_0^1 G(s,z)\,z\,u_0(s)dzds\right)\omega_0(y)dy,$$

$$f_{1,n}(t,u)=\int_0^1 f_1\left(t,y,\int_0^T\int_0^1 G(s,z)\sum_{i=1}^{\infty *}u_{1,i}(s)\,\vartheta_{1,i}(z)dzds\right)\omega_{1,n}(y)dy,$$

$$f_{2,m}(t,u)=\int_0^1 f_2\left(t,y,\int_0^T\int_0^1 G(s,z)\sum_{j=1}^{\infty}u_{2,j}(s)\,\vartheta_{2,j}(z)\,dzds\right)\tilde{\omega}_{2,m}(y)dy,$$

$$\tilde{f}_{2,m}(t,u)=\int_0^1 \tilde{f}_2\left(t,y,\int_0^T\int_0^1 G(s,z)\sum_{j=1}^{\infty}\tilde{u}_{2,j}(s)\,\tilde{\vartheta}_{2,j}(z)\,dzds\right)\omega_{2,m}(y)dy.$$

Hence, taking (2.11) and (2.12) into account, we obtain four fractional differential equations

$$D^{\alpha,\gamma}u_0(t)=f_0(t,u), \tag{3.1}$$

$$D^{\alpha,\gamma}u_{1,n}(t)+\mu_{1,n}u_{1,n}(t)=\frac{1}{1+\lambda_{1,n}}f_{1,n}(t,u), \tag{3.2}$$

$$D^{\alpha,\gamma}u_{2,m}(t)+\mu_{2,m}u_{2,m}(t)=-\frac{2\sqrt{\lambda_{2,m}}}{1+\lambda_{2,m}}\left(D^{\alpha,\gamma}\tilde{u}_{2,m}(t)+\tilde{u}_{2,m}(t)\right)+\frac{1}{1+\lambda_{2,m}}f_{2,m}(t,u), \tag{3.3}$$

$$D^{\alpha,\gamma}\tilde{u}_{2,m}(t)+\mu_{2,m}\tilde{u}_{2,m}(t)=\frac{1}{1+\lambda_{2,m}}\tilde{f}_{2,m}(t,u), \tag{3.4}$$

$$\mu_{1,n}=\frac{\lambda_{1,n}}{1+\lambda_{1,n}},\quad \mu_{2,m}=\frac{\lambda_{2,m}}{1+\lambda_{2,m}},$$

$\lambda_{1,n}$ and $\lambda_{2,m}$ $(n,m\in\mathbb{N},\ n\neq m(p+q))$ are defined from (2.9). The equation (3.1) is scalar fractional differential equation. The equations (3.2)-(3.4) are countable systems (CS) of fractional differential equations. We note here that CS of fractional differential equations (3.3) consists two unknown functions $u_{2,m}(t)$ and $\tilde{u}_{2,m}(t)$. So, we will solve it only after solving the CS (3.4). To solve the equations (3.1)-(3.4) we define initial value conditions [20, 21].

Taking into account the formulas (2.5), (2.10) and (2.11), (2.12), we consider the function $\varphi(x)$ as a function in the case of (2.13):

$$\varphi(x) = \varphi_0(x) + \varphi_1(x) + \varphi_2(x) + \tilde{\varphi}_2(x).$$

So, from the condition (1.2) we determine the initial value conditions

$$\lim_{t\to+0} J_{0t}^{1-\gamma} u_0(t) = \lim_{t\to+0} J_{0t}^{1-\gamma} \int\limits_0^1 U_0(t,y)\omega_0(y)dy = \int\limits_0^1 \varphi_0(y)\omega_0(y)dy = \varphi_0, \tag{3.5}$$

$$\lim_{t\to+0} J_{0t}^{1-\gamma} u_{1,n}(t) = \lim_{t\to+0} J_{0t}^{1-\gamma} \int\limits_0^1 U_1(t,y)\omega_{1,n}(y)dy = \int\limits_0^1 \varphi_1(y)\omega_{1,n}(y)dy = \varphi_{1,n}, \tag{3.6}$$

$$\lim_{t\to+0} J_{0t}^{1-\gamma} u_{2,m}(t) = \lim_{t\to+0} J_{0t}^{1-\gamma} \int\limits_0^1 U_2(t,y)\tilde{\omega}_{2,m}(y)dy = \int\limits_0^1 \varphi_2(y)\tilde{\omega}_{2,m}(y)dy = \varphi_{2,m}, \tag{3.7}$$

$$\lim_{t\to+0} J_{0t}^{1-\gamma} \tilde{u}_{2,m}(t) = \lim_{t\to+0} J_{0t}^{1-\gamma} \int\limits_0^1 \tilde{U}_2(t,y)\omega_{2,m}(y)dy = \int\limits_0^1 \tilde{\varphi}_2(x)\omega_{2,m}(y)dy = \tilde{\varphi}_{2,m}. \tag{3.8}$$

Thus, we have reduced the solvability issues of the mixed problem (1.1)-(1.4) to the study of the unique solvability of the fractional equation (3.1) and countable systems of fractional equations (3.2)-(3.4) with the corresponding initial conditions (3.5)-(3.8). Since our equations are nonlinear, we apply the method of successive approximations in combination with the method of contracting mappings. First, we reduce these initial problems to a nonlinear integral equation and to countable systems of nonlinear integral equations [19]. Further, we formulate the corresponding theorems.

The solving methods for a fractional differential equations (3.1) and CS of fractional differential equations (3.2)-(3.4) are the same. So, we show the scheme of solving only for the equations (3.1) and (3.2). First, we solve the equation (3.1). In this order we rewrite (3.1) in the form

$$D^{\alpha,\gamma} u_0(t) = f_0(t,u), \tag{3.9}$$

Applying the Riemann-Liouville integral operator J_{0t}^{α} to both sides of the equation (3.9), we have

$$u_0(t) = \frac{C_0}{\Gamma(\gamma)}\, t^{\gamma-1} + \frac{1}{\Gamma(\gamma)} \int\limits_0^t (t-s)^{\alpha-1} f_0(s,u)\, ds, \tag{3.10}$$

where C_0 is arbitrary constant.

Using initial value condition (3.5), we represent (3.10) as the solution of the equation (3.9)

$$u_0(t) = \frac{\varphi_0}{\Gamma(\gamma)}\, t^{\gamma-1} + \frac{1}{\Gamma(\gamma)} \int\limits_0^t (t-s)^{\,\alpha-1} f_0(s,u)\, ds. \tag{3.11}$$

The equation (3.11) is a nonlinear Volterra type integral equation with singularity at the point $t=0$. So, we multiply it to the function $t^{1-\gamma}$:

$$t^{1-\gamma} u_0(t) = I_0\big(t; u_0\big) \equiv \frac{\varphi_0}{\Gamma(\gamma)} +$$

$$+ \int\limits_0^t K_0(t,s) \int\limits_0^1 f_0\left(s, y, \int\limits_0^T \int\limits_0^1 G(\theta,z)\, z\, u_0(\theta) dz d\theta\right) \omega_0(y) dy\, ds, \tag{3.12}$$

where

$$K_0(t,s)=\frac{t^{1-\gamma}(t-s)^{\alpha-1}}{\Gamma(\gamma)}.$$

Smoothness conditions S_0**.** Let $\varphi_0(x)\in C[0,1]$, $f_0(t,x,\cdot)\in C(\Omega\times\mathbb{R})$ be fulfilled.

We use the norm $\|u(t)\|_C=\max_{0\le t\le T}|u(t)|$ in the space $C[0,T]$ of continuous functions.

Theorem 3.1. *Let the smoothness conditions* S_0 *be fulfilled and:*

1). $\int\limits_0^T\int\limits_0^1\frac{G(\theta,z)}{\theta^{1-\gamma}}zdzd\theta\le G_0<\infty,\ \ 0<G_0=\text{const}$;

2). $\left\|f_0\big(t,x,xu_0^0(t)\big)\right\|_C\le\delta_0,\ \ 0<\delta_0=\text{const}$;

3). $\|f_0(t,x,u_1)-f_0(t,x,u_2)\|_C\le l_0(x)\,\|\,u_1-u_2\,\|_C\,,\ \ 0<l_0(x)\in L_2[0,1]$;

4). $\rho_0=\tilde{l}_0G_0\frac{2\delta_0}{1-x_0^2}\frac{T^{1+\alpha-\gamma}}{\alpha\Gamma(\gamma)}<1,\ \ \tilde{l}_0=\int\limits_0^1 l_0(x)dx$.

Then nonlinear Volterra integral equation of second kind (3.12) *has a unique solution in the class of continuous functions on the segment* $[0,T]$.

The solution $u_0(t)\in C[0,T]$ *of the nonlinear integral equation* (3.12) *can be found by the following iteration process*

$$\left[\begin{array}{l} t^{1-\gamma}u_0^{\tau+1}(t)=I_0(t;u_0^{\tau}),\quad \tau=0,1,2,...\\ t^{1-\gamma}u_0^0(t)=\frac{\varphi_0}{\Gamma(\gamma)}.\end{array}\right.$$

Remark 3.2. The last inequality in the theorem is equivalent to the inequality:

$$T<\left[\frac{\left(1-x_0^2\right)\alpha\,\Gamma(\gamma)}{2\delta_0\tilde{l}_0G_0}\right]^{\gamma-\alpha-1}.$$

Moreover, there are opportunities to choose the constants x_0, δ_0, $\tilde{l}_0$, G_0 here.

Proof. We consider the following operator [18]

$$I_0\big(t;u_0\big):C([0,T];\mathbb{R})\to C([0,T];\mathbb{R}),$$

defined by the right-hand side of integral equation (3.12). Using the principle of contracting mappings, we show that the operator $I_0\big(t;u_0\big)$, defined by equality (3.12), has a unique fixed point.

Indeed, for this iteration process we have estimates

$$\left\|\,t^{1-\gamma}u_0^0(t)\,\right\|_C\le\frac{|\,\varphi_0\,|}{\Gamma(\gamma)}<\infty,$$

$$\left\|\,t^{1-\gamma}\left[u_0^1(t)-u_0^0(t)\right]\,\right\|_C\le$$

$$\le\max_{0\le t\le T}\int\limits_0^t K_0(t,s)\int\limits_0^1\left|f_0\left(s,y,\int\limits_0^T\int\limits_0^1 G(\theta,z)\,z\,u_0^0(\theta)dzd\theta\right)\right|\cdot|\,\omega_0(y)\,|\,dy\,ds\le$$

$$\le\frac{2T^{1+\alpha-\gamma}}{1-x_0^2}\frac{\|\,f_0(t,x,u)\,\|_C}{\alpha\,\Gamma\,(\gamma)}\le\frac{2\delta_0}{1-x_0^2}\frac{T^{1+\alpha-\gamma}}{\alpha\,\Gamma(\gamma)}<\infty,$$

$$\left\|\,t^{1-\gamma}\left[u_0^{\tau+1}(t)-u_0^{\tau}(t)\right]\,\right\|_C\le$$

$$\le\max_{0\le t\le T}\int\limits_0^t K_0(t,s)\int\limits_0^1 l_0(y)\omega_0(y)\int\limits_0^T\int\limits_0^1 G(\theta,z)z\,\left|\,u_0^{\tau}(\theta)-u_0^{\tau-1}(\theta)\,\right|dzd\theta\,dy\,ds\le$$

$$\le G_0\frac{\max\limits_{0\le\theta\le T}\left|\theta^{1-\gamma}\left[u_0^{\tau}(\theta)-u_0^{\tau-1}(\theta)\right]\right|}{\Gamma(\gamma)}\max_{0\le t\le T}\int\limits_0^t t^{1-\gamma}(t-s)^{\alpha-1}\int\limits_0^1 l_0(y)\,|\omega_0(y)|\,dyds\le$$

$$\leq \left\| t^{1-\gamma}\left[u_0^\tau(t)-u_0^{\tau-1}(t)\right]\right\|_C \frac{2}{1-x_0^2}\frac{\tilde{l}_0 G_0}{\Gamma(\gamma)}\max_{0\leq t\leq T}\int_0^t t^{1-\gamma}(t-s)^{\alpha-1}\,ds\leq$$

$$\leq \rho_0\cdot\left\| t^{1-\gamma}\left[u_0^\tau(t)-u_0^{\tau-1}(t)\right]\right\|_C,$$

where

$$\rho_0=\tilde{l}_0 G_0\frac{2}{1-x_0^2}\frac{T^{1+\alpha-\gamma}}{\alpha\,\Gamma(\gamma)}<1,\quad \tilde{l}_0=\int_0^1 l_0(y)dy.$$

From these estimates implies that the right-hand side of the fractional differential equation (3.12) as an operator $I_0\big(t;u_0\big)$ is contracting. Therefore, there is a unique fixed point. Hence, we deduce that there is unique solution $u_0(t)\in C[0,T]$ of the equation (3.12). Theorem 3.1 is proved. ☐

From the representation (3.12) one can find that

$$t^{1-\gamma}U_0(t,x)=x\,t^{1-\gamma}u_0(t)=x\,\frac{\varphi_0}{\Gamma(\gamma)}+$$

$$+\int_0^t x\,K_0(t,s)\int_0^1 f_0\left(s,y,\int_0^T\int_0^1 G(\theta,z)\,U_0(\theta,z)dzd\theta\right)\omega_0(y)dy\,ds. \tag{3.13}$$

Corollary 3.3. *The function* $t^{1-\gamma}U_0(t,x)\in C\big(\Omega\big)$ *in* (3.13) *is the unique solution of the problem* (1.1)–(1.4) *corresponding for eigenvalue* $\lambda_0=0$ *and eigenfunction* $\vartheta_0(x)=x$.

Indeed,

$$|\,t^{1-\gamma}U_0(t,x)\,|\leq|\,x\,|\cdot\|\,t^{1-\gamma}u_0(t)\,\|_C<\infty.$$

Now we solve the fractional order CS of ordinary differential equations (3.2) with initial value condition (3.6). In this purpose we rewrite the CS (3.2) as

$$D^{\alpha,\gamma}u_{1,n}(t)=-\mu_{1,n}u_{1,n}(t)+g_{1,n}(t), \tag{3.14}$$

where

$$g_{1,n}(t)=\frac{1}{1+\lambda_{1,n}}f_{1,n}(t,u). \tag{3.15}$$

Applying the Riemann-Liouville integral operator J_{0t}^α to both sides of this equation and taking into account the linearity of this operator and the formula [15]:

$$J_{0t}^\gamma D_{0t}^\gamma u_{1,n}(t)=u_{1,n}(t)-\frac{1}{\Gamma(\gamma)}\,J_{0t}^{1-\gamma}u_{1,n}(t)|_{t=0}t^{\gamma-1},$$

we have

$$u_{1,n}(t)=\frac{C_{1n}}{\Gamma(\gamma)}\,t^{\gamma-1}+J_{0t}^\alpha g_{1,n}(t)-\mu_{1,n}J_{0t}^\alpha u_{1,n}(t),\quad C_{1,n}=\text{const}.$$

Then, using the initial value condition (3.6), we represent the solution of the system (3.14) in the form

$$u_{1,n}(t)=\varphi_{1,n}\left[\frac{t^{\gamma-1}}{\Gamma(\gamma)}-\frac{\mu_{1,n}}{\Gamma(\gamma)}\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\mu_{1,n}(t-s)^\alpha\right)s^{\gamma-1}\,ds\right]+$$

$$+J_{0t}^\alpha g_{1,n}(t)-\mu_{1,n}\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\mu_{1,n}(t-s)^\alpha\right)J_{0t}^\alpha g_{1,n}(s)\,ds. \tag{3.16}$$

In representation (3.16) we take into account the following relations

$$\frac{t^{\gamma-1}}{\Gamma(\gamma)}-\frac{\mu_{1,n}}{\Gamma(\gamma)}\int\limits_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\mu_{1,n}(t-s)^{\alpha}\right)s^{\gamma-1}ds=t^{\gamma-1}E_{\alpha,\gamma}\left(-\mu_{1,n}t^{\alpha}\right),$$

$$\int\limits_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\mu_{1,n}(t-s)^{\alpha}\right)\,J_{0t}^{\alpha}g_{1,n}(s)ds=$$

$$=\frac{1}{\Gamma(\alpha)}\int\limits_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\mu_{1,n}(t-s)^{\alpha}\right)\,ds\int\limits_0^s(s-\theta)^{\alpha-1}g_{1,n}(\theta)\,d\theta=$$

$$=\frac{1}{\Gamma(\alpha)}\int\limits_0^t(t-s)^{\alpha-1}ds\int\limits_s^t(t-s)^{\alpha-1}(s-\theta)^{\alpha-1}E_{\alpha,\alpha}\left(-\mu_{1,n}(t-\theta)^{\alpha}\right)\,d\theta=$$

$$=\int\limits_0^t g_{1,n}(s)(t-s)^{2\alpha-1}E_{\alpha,2\alpha}\left(-\mu_{1,n}(t-s)^{\alpha}\right)ds.$$

Hence, we obtain

$$u_{1,n}(t)=\varphi_{1,n}t^{\gamma-1}E_{\alpha,\gamma}\left(-\mu_{1,n}t^{\alpha}\right)+\int\limits_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\mu_{1,n}(t-s)^{\alpha}\right)\,g_{1,n}(s)ds, \tag{3.17}$$

where

$$E_{\alpha,\gamma}(z)=\sum_{m=0}^{\infty}\frac{z^m}{\Gamma(\alpha m+\gamma)},\quad z,\alpha,\gamma\in \mathrm{C},\quad \mathrm{Re}\,(\alpha)>0$$

is Mittag-Leffler function.

In obtaining the equation (3.17) we took into account that the following representations are true:

$$E_{\alpha,\gamma}(z)=\frac{1}{\Gamma(\gamma)}+z\,E_{\alpha,\gamma+\alpha}(z),\quad \alpha>0,\quad \gamma>0,$$

$$\frac{1}{\Gamma(\alpha)}\int\limits_0^z(z-t)^{\alpha-1}E_{\alpha,\gamma}\left(k\,t^{\alpha}\right)\,t^{\gamma-1}dt=z^{\gamma+\alpha-1}E_{\alpha,\gamma+\alpha}\left(k\,z^{\alpha}\right),\quad \alpha>0,\quad \gamma>0.$$

By virtue of (3.15), the equation (3.17) we rewrite as

$$u_{1,n}(t)=\varphi_{1,n}t^{\gamma-1}E_{\alpha,\gamma}\left(-\mu_{1,n}t^{\alpha}\right)+$$

$$+\frac{1}{1+\lambda_{1,n}}\int\limits_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\mu_{1,n}(t-s)^{\alpha}\right)f_{1,n}(s,u)ds. \tag{3.18}$$

Instead of the equation (3.18) we consider the following CS of nonlinear integral equations (CSNIE)

$$t^{1-\gamma}u_{1,n}(t)=\varphi_{1,n}E_{\alpha,\gamma}\left(-\mu_{1,n}t^{\alpha}\right)+\frac{1}{1+\lambda_{1,n}}\int\limits_0^t K_{1,n}(t,s)\,f_{1,n}(s,u)ds, \tag{3.19}$$

where

$$K_{1,n}(t,s)=t^{1-\gamma}(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\mu_{1,n}(t-s)^{\alpha}\right),$$

$$\mu_{1,n}=\frac{\lambda_{1,n}}{1+\lambda_{1,n}},\quad \lambda_{1,n}=\left(\frac{2qn\pi}{p+q}\right)^2,\quad n=1\,,\,2\,,\,...$$

By similarly way, for the differential equation (3.4) with initial value condition (3.8) we obtain the following CSNIE

$$t^{1-\gamma}\tilde{u}_{2,m}(t) = \tilde{\varphi}_{2,m}E_{\alpha,\gamma}\left(-\mu_{2,m}t^{\alpha}\right) + \frac{1}{1+\lambda_{2,m}}\int\limits_{0}^{t} K_{2,m}(t,s)\tilde{f}_{2,m}(s,u)ds, \tag{3.20}$$

where

$$K_{2,m}(t,s) = t^{1-\gamma}(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\mu_{2,m}(t-s)^{\alpha}\right),$$

$$\mu_{2,m} = \frac{\lambda_{2,m}}{1+\lambda_{2,m}}, \quad \lambda_{2,m} = (2qm\pi)^2, \quad m = 1\,,\,2\,,\,...$$

Before to solve the countable system of differential equation (3.3) with initial value condition (3.7), we study the solvability of the CSNIE (3.19) and (3.20).

4. Unique solvability of CSNIE

In the set $\left\{\vec{\psi}(t) = \left(\psi_i(t)\right) : \psi_i(t) \in C[0,T],\ i = 1,\,2,\,...\right\}$ operations of addition of two elements and multiplication of an element by a scalar are defined coordinate-wise. This set is a linear vector space. We consider those elements of this vector space that satisfy the condition $\sum_{i=1}^{\infty}\left(\max\limits_{t\in[0,T]}|\,\psi_i(t)\,|\right)^2 < \infty$. This set we denote by $B_2[0,T]$ and is provided with a norm [16]

$$\left\|\,\vec{\psi}(t)\,\right\|_{B_2[0,T]} = \sqrt{\sum_{i=1}^{\infty}\left(\max_{t\in[0,T]}|\,\psi_i(t)\,|\right)^2} < \infty.$$

We use also coordinate Hilbert space ℓ_2 of number sequences $\left\{\zeta_i\right\}_{i=1}^{\infty}$ with a norm

$$\left\|\,\vec{\zeta}\,\right\|_{\ell_2} = \sqrt{\sum_{i=1}^{\infty}|\,\zeta_i\,|^2} < \infty$$

and the space $L_2[0,1]$ of square summable functions on an interval $[0,1]$ with a norm

$$\|\,\eta(x)\,\|_{L_2[0,1]} = \sqrt{\int\limits_0^1 |\,\eta(y)\,|^2\,dy} < \infty.$$

Smoothness condition S_1. Let in the domain $[0,1]$ the functions $\varphi_1(x) \in C^3[0,1]$ and $f_1(t,x,\cdot) \in C^{0,1}_{t,x}(\Omega\times\mathbb{R})$ have the piecewise continuous derivatives with respect to x up to the fourth and second order, respectively. Then, we integrate by parts the integrals

$$\varphi_{1,n} = \int\limits_0^1 \varphi_1(y)\,\omega_{1,n}(y)dy, \quad f_{1,n}(t,u) = \int\limits_0^1 f_1(t,y,\cdot)\,\omega_{1,n}(y)dy$$

fourth and second times on the variable x, respectively, and obtain the results

$$|\,\varphi_{1,n}\,| \le \left(\frac{p+q}{2q\pi}\right)^4 \frac{\left|\,\varphi_{1,n}^{(IV)}\,\right|}{n^4}, \quad |\,f_{1,n}(t,u)\,| \le \left(\frac{p+q}{2q\pi}\right)^2 \frac{|\,f''_{1,n}(t,u)\,|}{n^2}, \tag{4.1}$$

where

$$\varphi_{1,n}^{(IV)} = \int\limits_0^1 \frac{\partial^4\varphi_1(y)}{\partial\,y^4}\omega_{1,n}(y)dy, \quad f''_{1,n}(t) = \int\limits_0^1 \frac{\partial^2 f_1(t,y,\cdot)}{\partial\,y^2}\omega_{1,n}(y)dy.$$

Theorem 4.1. *Let the condition A and smoothness condition S_1 be fulfilled and*

1). $\int\limits_0^T \left\| \frac{G(s,x)}{s^{1-\gamma}} \right\|_{L_2[0,1]} ds \leq G_1 < \infty, \quad 0 < G_1 = \text{const}$;

2). $\max\limits_{0 \leq t \leq T} \|f_1(t,x,u)\|_{L_2[0,1]} \leq \delta_1, \ \ 0 < \delta_1 = \text{const}$;

3). $|\, f_1(t,x,u_1) - f_1(t,x,u_2)\,| \leq l_1(x)\,|\, u_1 - u_2\,|, \ \ 0 < l_1(x) \in L_2[0,1]$;

4). $\rho_1 = M_{1,4} G_1 < 1, \ \ M_{1,4} = M_{1,2} \sqrt{\sum\limits_{n=1}^{\infty *} \frac{1}{n^4}}\, \|\, l_1(x)\,\|_{L_2[0,1]}$.

Then CSNIE (3.19) *has a unique solution in the space $B_2[0,T]$.*

Proof. Using the principle of contracting mappings, we show that the operator, defined on the right-hand side of equation (3.19), has a unique fixed point.

The solution $\vec{u}_1(t) \in B_2[0,T]$ of the nonlinear integral equation (3.19) can be found by the following iteration process

$$\left\{ \begin{array}{l} t^{1-\gamma} u_{1,n}^0(t) = \varphi_{1,n} E_{\alpha,\gamma}\left(-\mu_{1,n} t^{\alpha}\right), \\ t^{1-\gamma} u_{1,n}^{\tau+1}(t) = t^{1-\gamma} u_{1,n}^0(t) + \frac{1}{1+\lambda_{1,n}} \int\limits_0^t K_{1,n}(t,s)\, f_{1,n}\left(s, u_{1,n}^{\tau}\right) ds, \quad \tau = 0,\, 1,\, 2,\, 3,\, \ldots, \end{array} \right. \tag{4.2}$$

where

$$f_{1,n}(t, u_{1,n}^{\tau}) = \int\limits_0^1 f_1\left(t, y, \int\limits_0^T \int\limits_0^1 G(s,z) \sum_{i=1}^{\infty *} u_{1,i}^{\tau}(s)\, \vartheta_{1,i}(z) dz ds\right) \omega_{1,n}(y) dy.$$

It is known that for all $\alpha \in (0,1)$, $\gamma \in \mathbb{R}$ and $\arg z = \pi$ there takes place the following estimate for Mittag-Leffler function [17]

$$|\, E_{\alpha,\gamma}(z)\,| \leq \frac{M_0}{1 + |z|},$$

where $0 < M_0 = \text{const}$ does not depend from z. In particularly, for all $0 < \alpha \leq \gamma \leq 1, \ 0 < \mu_{i,n} < 1, \ i = 1,2$, we have the estimate

$$|\, E_{\alpha,\gamma}\left(-\mu_{i,n} t^{\alpha}\right)| \leq M_0.$$

We estimate the zero approximation. By virtue of formulas in (4.1) and the fact that $\mu_{1,n} = \frac{\lambda_{1,n}}{1+\lambda_{1,n}} < 1, \ \ \lambda_{1,n} = \left(\frac{2qn\pi}{p+q}\right)^2$, applying the Cauchy-Shwartz inequality and Bessel inequality, from approximations (4.2) we have

$$\left\|\, t^{1-\gamma} \vec{u}_1^0(t)\,\right\|_{B_2[0,T]} \leq \sqrt{\sum_{n=1}^{\infty *} \max_{0 \leq t \leq T} \left|\, t^{1-\gamma} u_{1,n}^0(t)\,\right|^2} \leq \sum_{n=1}^{\infty *} \max_{0 \leq t \leq T} |\, \varphi_{1,n} E_{\alpha,\gamma}\left(-\mu_{1,n} t^{\alpha}\right)| \leq$$

$$\leq M_0 \sum_{n=1}^{\infty *} |\, \varphi_{1,n}\,| \leq M_0 \left(\frac{p+q}{2q\pi}\right)^4 \sum_{n=1}^{\infty *} \frac{1}{n^4} \left|\, \varphi_{1,n}^{(IV)}\,\right| \leq$$

$$\leq M_{1,1} \left\|\, \vec{\varphi}_1^{(IV)}\,\right\|_{\ell_2} \leq M_{1,1} \left\|\, \frac{\partial^4 \varphi_1(x)}{\partial\, x^4}\,\right\|_{L_2[0,1]} < \infty, \tag{4.3}$$

where

$$M_{1,1} = M_0 \left(\frac{p+q}{2q\pi}\right)^4 \sqrt{\sum_{n=1}^{\infty *} \frac{1}{n^8}}.$$

Due to the conditions of the Theorem 4.1, estimate (4.3) and applying the Cauchy-Shwartz inequality and Bessel's inequality, for the first difference $t^{1-\gamma}\left[u_{1,n}^1(t)-u_{1,n}^0(t)\right]$ we obtain

$$\left\| t^{1-\gamma}\left[\vec{u}_1^1(t)-\vec{u}_1^0(t)\right]\right\|_{B_2[0,T]} \le \sum_{n=1}^{\infty *}\frac{1}{\lambda_{1,n}}\max_{0\le t\le T}\int_0^t K_{1,n}(t,s)\left| f_{1,n}(t,u_{1,n}^0)\right| ds \le$$

$$\le \sum_{n=1}^{\infty *}\frac{1}{\lambda_{1,n}}\max_{0\le t\le T} t^{1-\gamma}\int_0^t (t-s)^{\alpha-1}\left| E_{\alpha,\alpha}\left(-\mu_{1,n}(t-s)^\alpha\right)\right| ds\times$$

$$\times\left|\int_0^1 f_1\left(t,y,\int_0^T\int_0^1 G(s,z)\sum_{i=1}^{\infty *}u_{1,i}^0(s)\,\vartheta_{1,i}(z)dzds\right)\omega_{1,n}(y)dy\right| \le$$

$$\le M_{1,2}\sum_{n=1}^{\infty *}\frac{1}{n^2}\max_{0\le t\le T}\left|\int_0^1 f_1\left(t,y,\int_0^T\int_0^1 G(s,z)\sum_{i=1}^{\infty *}u_{1,i}^0(s)\,\vartheta_{1,i}(z)dzds\right)\omega_{1,n}(y)dy\right| \le$$

$$\le M_{1,2}\sqrt{\sum_{n=1}^{\infty *}\frac{1}{n^4}}\left\|\int_0^1 f_1\left(t,y,\int_0^T\int_0^1 G(s,z)\sum_{i=1}^{\infty *}u_{1,i}^0(s)\,\vartheta_{1,i}(z)dzds\right)\omega_1(y)dy\right\|_{B_2[0,T]} \le$$

$$\le M_{1,2}\sqrt{\sum_{n=1}^{\infty *}\frac{1}{n^4}}\max_{0\le t\le T}\left\| f_1\left(t,y,\int_0^T\int_0^1 G(s,z)U_1^0(s,z)\,dzds\right)\right\|_{L_2[0,1]} \le$$

$$\le M_{1,2}\sqrt{\sum_{n=1}^{\infty *}\frac{1}{n^4}}\delta_1<\infty, \tag{4.4}$$

where

$$M_{1,2}=M_0\frac{T^{1+\alpha-\gamma}}{\alpha}\left(\frac{p+q}{2q\pi}\right)^2.$$

Now we consider the arbitrary difference $t^{1-\gamma}\left[u_{1,n}^{\tau+1}(t)-u_{1,n}^\tau(t)\right]$. By the same way as above, we obtain

$$\left\| t^{1-\gamma}\left[\vec{u}_1^{\tau+1}(t)-\vec{u}_1^\tau(t)\right]\right\|_{B_2[0,T]} \le \sum_{n=1}^{\infty *}\frac{1}{\lambda_{1,n}}\max_{0\le t\le T}\int_0^t K_{1,n}(t,s)\left| f_{1,n}(t,u_{1,n}^\tau)-f_{1,n}(t,u_{1,n}^{\tau-1})\right| ds \le$$

$$\le \sum_{n=1}^{\infty *}\frac{1}{\lambda_{1,n}}\max_{0\le t\le T} t^{1-\gamma}\int_0^t (t-s)^{\alpha-1}\left| E_{\alpha,\alpha}\left(-\mu_{1,n}(t-s)^\alpha\right)\right| ds\times$$

$$\times\left|\int_0^1 l_1(y)\int_0^T\int_0^1 G(t,z)\sum_{i=1}^{\infty *}\left| u_{1,i}^\tau(t)-u_{1,i}^{\tau-1}(t)\right| \vartheta_{1,i}(z)dzdt\,\omega_{1,n}(y)dy\right| \le$$

$$\le M_{1,2}\sum_{n=1}^{\infty *}\frac{1}{n^2}\left|\int_0^1 l_1(y)\omega_{1,n}(y)dy\right|\left|\int_0^T\int_0^1 G(t,z)\sum_{i=1}^{\infty *}\left| u_{1,i}^\tau(t)-u_{1,i}^{\tau-1}(t)\right| \vartheta_{1,i}(z)dzdt\right| \le$$

$$\le M_{1,3}\int_0^T\left|\sum_{i=1}^{\infty *}t^{1-\gamma}\left| u_{1,i}^\tau(t)-u_{1,i}^{\tau-1}(t)\right|\int_0^1\frac{G(t,z)}{t^{1-\gamma}}\vartheta_{1,i}(z)dz\right| dt \le$$

$$\leq M_{1,3} \left\| t^{1-\gamma} \left[\vec{u}_1^{\tau}(t) - \vec{u}_1^{\tau-1}(t) \right] \right\|_{B_2(T)} \int\limits_0^T \left\| \int\limits_0^1 \frac{G(t,z)}{t^{1-\gamma}} \vartheta_1(z) dz \right\|_{B_2[0,T]} dt \leq$$

$$\leq \rho_1 \cdot \left\| t^{1-\gamma} \left[\vec{u}_1^{\tau}(t) - \vec{u}_1^{\tau-1}(t) \right] \right\|_{B_2[0,T]}, \quad (4.5)$$

where

$$\rho_1 = M_{1,3} G_1, \;\; M_{1,3} = M_{1,2} \sqrt{\sum_{n=1}^{\infty *} \frac{1}{n^4}} \, \| \, l_1(x) \, \|_{L_2[0,1]} \, .$$

From estimates (4.3)-(4.5) it follows that the operator on the right-hand side of (3.19) is contracting and there is a fixed point [15]. So, existence and uniqueness of the solution $t^{1-\gamma}\vec{u}_1(t) \in B_2[0,T]$ to CSNIE (3.19) are proved. ☐

Smoothness condition $\tilde{S}_2$. Let in the domain $[0,1]$ the functions $\tilde{\varphi}_2(x) \in C^3[0,1]$ and $\tilde{f}_2(t,x,\cdot) \in C^{0,1}_{t,x}(\Omega \times R)$ have the piecewise continuous derivatives with respect to x up to the fourth and second order, respectively. Then, we integrate by parts

$$\tilde{\varphi}_{2,m} = \int\limits_0^1 \tilde{\varphi}_2(y) \, \omega_{2,m}(y) dy, \;\; \tilde{f}_{2,m}(t,u) = \int\limits_0^1 \tilde{f}_2(t,y,\cdot) \, \omega_{2,m}(y) \, dy$$

fourth and second times on the variable x, respectively, and obtain

$$| \, \tilde{\varphi}_{2,m} \, | \leq \left(\frac{p+q}{2q\pi} \right)^4 \frac{\left| \tilde{\varphi}_{2,m}^{(IV)} \right|}{m^4}, \quad \left| \tilde{f}_{2,m}(t,u) \right| \leq \left(\frac{p+q}{2q\pi} \right)^2 \frac{\left| \tilde{f''}_{2,m}(t,u) \right|}{m^2},$$

where

$$\tilde{\varphi}_{2,m}^{(IV)} = \int\limits_0^1 \frac{\partial^4 \tilde{\varphi}_2(y)}{\partial y^4} \omega_{2,m}(y) \, dy, \;\; \tilde{f''}_{2,m}(t) = \int\limits_0^1 \frac{\partial^2 \tilde{f}_2(t,y,\cdot)}{\partial y^2} \omega_{2,m}(y) \, dy.$$

Theorem 4.2. *Let the condition A and smoothness condition $\tilde{S}_2$ be fulfilled and*

1). $\int\limits_0^T \left\| \frac{G(t,x)}{t^{1-\gamma}} \right\|_{L_2[0,1]} dt \leq \tilde{G}_2 < \infty, \;\; 0 < \tilde{G}_2 = \text{const};$

2). $\max\limits_{0 \leq t \leq T} \left\| \tilde{f}_2(t,x,u) \right\|_{L_2[0,1]} \leq \tilde{\delta}_2, \;\; 0 < \tilde{\delta}_2 = \text{const};$

3). $\left| \tilde{f}_2(t,x,u_1) - \tilde{f}_2(t,x,u_2) \right| \leq \tilde{l}_2(x) \, | \, u_1 - u_2 \, |, \;\; 0 < \tilde{l}_2(x) \in L_2[0,1];$

4). $\tilde{\rho}_2 = M_{2,3} \tilde{G}_2 < 1, \;\; M_{2,3} = M_{2,2} \sqrt{\sum_{m=1}^{\infty} \frac{1}{m^4}} \left\| \tilde{l}_2(x) \right\|_{L_2[0,1]}, \;\; M_{2,2}, \, M_{2,3} = \text{const}.$

Then CSNIE (3.20) *has a unique solution in the space* $B_2[0,T]$.

The existence and uniqueness Theorem 4.2 for CS of nonlinear integral equations (3.20) is proved by similar way as in the case of Theorem 4.1.

The differential equation (3.3) consists two unknown functions. Therefore, the solution of the CSNIE (3.20) we denote by $\tilde{F}_{2,m}(t)$ and substitute it into equation (3.3)

$$D^{\alpha,\gamma} u_{2,m}(t) + \mu_{2,m} u_{2,m}(t) = \frac{1}{1+\lambda_{2,m}} f_{2,m}(t,u) - \frac{2\sqrt{\lambda_{2,m}}}{1+\lambda_{2,m}} \left(D^{\alpha,\gamma} \tilde{F}_{2,m}(t) + \tilde{F}_{2,m}(t) \right). \quad (4.6)$$

The equation (4.6) consists only one unknown function. So, we solve it with initial value condition (3.7). This problem is equivalent to the following Volterra integral equation

$$t^{1-\gamma} u_{2,m}(t) = \varphi_{2,m} E_{\alpha,\gamma} \left(-\mu_{2,m} t^{\alpha} \right) - \frac{2\sqrt{\lambda_{2,m}}}{1+\lambda_{2,m}} \int\limits_0^t K_{2,m}(t,s) \, \left(D^{\alpha,\gamma} \tilde{F}_{2,m}(s) + \tilde{F}_{2,m}(s) \right) ds +$$

$$+\frac{1}{1+\lambda_{2,m}}\int_{0}^{t}K_{2,m}(t,s)\,f_{2,m}(s,u)ds, \tag{4.7}$$

where

$$K_{2,m}(t,s)=t^{1-\gamma}(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\mu_{2,m}(t-s)^{\alpha}\right),$$
$$\mu_{2,m}=\frac{\lambda_{2,m}}{1+\lambda_{2,m}},\ \ \lambda_{2,m}=(2qm\pi)^{2},\ \ m=1,\,2,\,...$$

Smoothness condition S_2. Let in the domain $[0,1]$ the functions $\varphi_2(x)\in C^3[0,1]$, $f_2(t,x,\cdot)\in C^{0,1}_{t,x}(\Omega\times\mathbb{R})$ have the peace-wise continuous derivatives with respect to x up to the fourth order and second order, respectively. Then, by integrations by parts

$$\varphi_{2,m}=\int_{0}^{1}\varphi_2(y)\,\tilde{\omega}_{2,m}(y)dy,\quad f_{2,m}(t,u)=\int_{0}^{1}f_2(t,y,\cdot)\,\tilde{\omega}_{2,m}(y)dy$$

we obtain that there hold estimates

$$|\,\varphi_{2,m}\,|\le\left(\frac{1}{2q\pi}\right)^{4}\frac{\left|\varphi_{2,m}^{(IV)}\right|}{m^4}+4\left(\frac{1}{2q\pi}\right)^{5}\frac{\left|\tilde{\varphi}_{2,m}^{(IV)}\right|}{m^5},$$

$$|\,f_{2,m}(t,u)\,|\le\left(\frac{1}{2q\pi}\right)^{2}\frac{\left|\,f''_{2,m}(t,u)\,\right|}{m^2}+4\left(\frac{1}{2q\pi}\right)^{3}\frac{\left|\tilde{f}''{}_{2,m}(t,u)\right|}{m^3},$$

where

$$\varphi_{2,m}^{(IV)}=\int_{0}^{1}\frac{\partial^4\varphi_2(y)}{\partial\,y^4}\,\tilde{\omega}_{2,m}(y)\,dy,\quad f''_{2,m}(t,u)=\int_{0}^{1}\frac{\partial^2 f_2(t,y,\cdot)}{\partial\,y^2}\,\tilde{\omega}_{2,m}(y)\,dy.$$

Theorem 4.3. *Let the condition A and smoothness condition S_2 be fulfilled and*

1). $\int_0^T\left\|\frac{G(s,x)}{s^{1-\gamma}}\right\|_{L_2[0,1]}ds\le G_2<\infty,\ \ 0<G_2=\text{const};$

2). $\max\limits_{0\le t\le T}\|f_2(t,x,u)\|_{L_2[0,1]}\le\delta_2,\ \ 0<\delta_2=\text{const};$

3). $|\,f_2(t,x,u_1)-f_2(t,x,u_2)\,|\le l_2(x)\,|\,u_1-u_2\,|,\ \ 0<l_2(x)\in L_2[0,1];$

4). $\rho_3=M_{3,5}G_2<1,\ \ M_{3,5}=M_{3,3}\sqrt{\sum_{m=1}^{\infty}\frac{1}{m^4}}\,\|\,l_2(x)\,\|_{L_2[0,1]}.$

Then CSNIE (4.7) *has a unique solution in the space $B_2[0,T]$.*

Proof. We use the method of successive approximations:

$$\begin{cases} t^{1-\gamma}u^0_{2,m}(t)=\varphi_{2,m}E_{\alpha,\gamma}\left(-\mu_{2,m}t^{\alpha}\right)-\frac{2\sqrt{\lambda_{2,m}}}{1+\lambda_{2,m}}\int_0^t K_{2,m}(t,s)\,\left(D^{\alpha,\gamma}\tilde{F}_{2,m}(s)+\tilde{F}_{2,m}(s)\right)ds,\\ t^{1-\gamma}u^{\tau+1}_{2,m}(t)=t^{1-\gamma}u^0_{2,m}(t)+\frac{1}{1+\lambda_{2,m}}\int_0^t K_{2,m}(t,s)f_{2,m}\big(t,u^{\tau}_{2,m}\big)ds,\ \ \tau=0,\,1,\,2,\,3,\,\ldots,\end{cases}$$

where

$$f_{2,m}\big(t,u^{\tau}_{2,m}\big)=\int_0^1 f_2\left(t,y,\int_0^T\int_0^1 G(s,z)\sum_{j=1}^{\infty}u^{\tau}_{2,j}(s)\,\vartheta_{2,j}(z)dzds\right)\tilde{\omega}_{2,m}(y)dy.$$

By virtue of smoothness conditions S_2 and the fact that $\mu_{1,n}=\frac{\lambda_{1,n}}{1+\lambda_{1,n}}<1,\ \ \lambda_{1,n}=\left(\frac{2qn\pi}{p+q}\right)^2$, applying the Cauchy-Shwartz inequality and Bessel inequality, we have

$$\|\,t^{1-\gamma}\vec{u}^0_2(t)\,\|_{B_2[0,T]}\le M_{3,1}\left[\left\|\frac{\partial^4\,\varphi_2(x)}{\partial\,x^4}\right\|_{L_2[0,1]}+\left\|\frac{\partial^4\,\tilde{\varphi}_2(x)}{\partial\,x^4}\right\|_{L_2[0,1]}\right]+$$

$$+M_{3,2}\left[\left\| D^{\alpha,\gamma}\left[t^{1-\gamma}\vec{\tilde{F}}_2(t)\right]\right\|_{B_2[0,T]}+\left\| t^{1-\gamma}\vec{\tilde{F}}_2(t)\right\|_{B_2[0,T]}\right]<\infty, \tag{4.8}$$

where

$$M_{3,1}=M_0\max\left\{\left(\frac{1}{2q\pi}\right)^4\sqrt{\sum_{m=1}^{\infty}\frac{1}{m^8}};\,4\left(\frac{1}{2q\pi}\right)^5\sqrt{\sum_{m=1}^{\infty}\frac{1}{m^{10}}}\right\},$$

$$M_{3,2}=\frac{1}{2}M_0\frac{T^{1+\alpha-\gamma}}{\alpha}\left(\frac{1}{q\pi}\right)^2\sqrt{\sum_{m=1}^{\infty}\frac{1}{m^2}}.$$

Due to the conditions of the Theorem 4.3 and applying the Cauchy-Shwartz inequality and Bessel's inequality, for the first difference $t^{1-\gamma}\left[u_{2,m}^1(t)-u_{2,m}^0(t)\right]$ we obtain

$$\left\| t^{1-\gamma}\left[\vec{u}_2^1(t)-\vec{u}_2^0(t)\right]\right\|_{B_2[0,T]}\leq\sum_{m=1}^{\infty}\frac{1}{\lambda_{2,m}}\max_{0\leq t\leq T}\int_0^t K_{2,m}(t,s)\left| f_{2,m}(t,u_{2,m}^0)\right| ds\leq$$

$$\leq M_{3,3}\sum_{m=1}^{\infty}\frac{1}{m^2}\max_{0\leq t\leq T}\left|\int_0^1 f_2\left(t,y,\int_0^T\int_0^1 G(s,z)\sum_{j=1}^{\infty}u_{2,m}^0(s)\,\vartheta_{2,j}(z)dzds\right)\omega_{2,m}(y)dy\right|\leq$$

$$\leq M_{3,3}\sqrt{\sum_{m=1}^{\infty}\frac{1}{m^4}}\left\|\int_0^1 f_2\left(t,y,\int_0^T\int_0^1 G(s,z)\,U_2^0(s,z)dzds\right)\omega_{2,m}(y)dy\right\|_{B_2[0,T]}\leq M_{3,4}<\infty, \tag{4.9}$$

where

$$M_{3,4}=M_{3,3}\delta_2\sqrt{\sum_{m=1}^{\infty}\frac{1}{m^4}},\quad M_{3,3}=M_0\frac{T^{1+\alpha-\gamma}}{\alpha}\left(\frac{1}{2q\pi}\right)^2.$$

Now, considering the arbitrary difference $t^{1-\gamma}\left[u_{2,n}^{\tau+1}(t)-u_{2,n}^{\tau}(t)\right]$, we obtain

$$\left\| t^{1-\gamma}\left[\vec{u}_2^{\tau+1}(t)-\vec{u}_2^{\tau}(t)\right]\right\|_{B_2[0,T]}\leq\sum_{m=1}^{\infty}\frac{1}{\lambda_{2,m}}\max_{0\leq t\leq T}\int_0^t K_{2,m}(t,s)\left| f_{2,m}\left(t,u_{2,m}^{\tau}\right)-f_{2,m}\left(t,u_{2,m}^{\tau-1}\right)\right| ds\leq$$

$$\leq M_{3,3}\sum_{m=1}^{\infty}\frac{1}{m^2}\left|\int_0^1 l_2(y)\omega_{2,m}(y)dy\right|\cdot\left|\int_0^T\int_0^1 G(s,z)\sum_{j=1}^{\infty}\left| u_{2,j}^{\tau}(s)-u_{2,j}^{\tau-1}(s)\right|\,\vartheta_{2,j}(z)dzds\right|\leq$$

$$\leq M_{3,5}\left\| t^{1-\gamma}\left[\vec{u}_2^{\tau}(t)-\vec{u}_2^{\tau-1}(t)\right]\right\|_{B_2[0,T]}\int_0^T\left\|\int_0^1\frac{G(s,z)}{s^{1-\gamma}}\omega_2(z)dz\right\|_{B_2[0,T]}ds\leq$$

$$\leq\rho_3\cdot\left\| t^{1-\gamma}\left[\vec{u}_2^{\tau}(t)-\vec{u}_2^{\tau-1}(t)\right]\right\|_{B_2[0,T]}, \tag{4.10}$$

where

$$\rho_3=M_{3,5}G_2,\quad M_{3,5}=M_{3,3}\sqrt{\sum_{m=1}^{\infty}\frac{1}{m^4}}\left\| l_2(x)\right\|_{L_2[0,1]}.$$

From estimates (4.8)-(4.10) and last condition of the theorem it follows the existence and uniqueness of the solution $t^{1-\gamma}\vec{u}_2(t)\in B_2[0,T]$ to CSNIE (4.7). □

5. Mixed problem

Since the solution of the mixed problem (1.1)-(1.4) we look at (2.13), then for the function (2.14) from (3.13), (3.19), (3.20) and (4.7) we have

$$t^{1-\gamma}U(t,x)=\frac{\varphi_0}{\Gamma(\gamma)}\,\vartheta_0(x)+\int\limits_0^t K_0(t,s)\int\limits_0^1 f_0\left(s,y,\int\limits_0^T\int\limits_0^1 G(\theta,z)\,U_0(\theta,z)dzd\theta\right)\omega_0(y)dy\,ds+$$

$$+\sum_{n=1}^{\infty *}\vartheta_{1,n}(x)\Big[\varphi_{1,n}E_{\alpha,\gamma}\left(-\mu_{1,n}t^{\alpha}\right)+$$

$$+\frac{1}{1+\lambda_{1,n}}\int\limits_0^t K_{1,n}(t,s)\int\limits_0^1 f_1\left(s,y,\int\limits_0^T\int\limits_0^1 G(\theta,z)\,U_1(\theta,z)dzd\theta\right)\omega_1(y)dy\,ds\Bigg]+$$

$$+\sum_{m=1}^{\infty}\vartheta_{2,m}(x)\Bigg[\varphi_{2,m}E_{\alpha,\gamma}\left(-\mu_{2,m}t^{\alpha}\right)-\frac{2\sqrt{\lambda_{2,m}}}{1+\lambda_{2,m}}\int\limits_0^t K_{2,m}(t,s)\,\Big(D^{\alpha,\gamma}\tilde{F}_{2,m}(s)+\tilde{F}_{2,m}(s)\Big)\,ds+$$

$$+\frac{1}{1+\lambda_{2,m}}\int\limits_0^t K_{2,m}(t,s)\int\limits_0^1 f_2\left(s,y,\int\limits_0^T\int\limits_0^1 G(\theta,z)\,U_2(\theta,z)dzd\theta\right)\tilde{\omega}_2(y)dy\,ds\Bigg]+$$

$$+\sum_{m=1}^{\infty}\tilde{\vartheta}_{2,m}(x)\Big[\tilde{\varphi}_{2,m}E_{\alpha,\gamma}\left(-\mu_{2,m}t^{\alpha}\right)+$$

$$+\frac{1}{1+\lambda_{2,m}}\int\limits_0^t \tilde{K}_{2,m}(t,s)\int\limits_0^1 \tilde{f}_2\left(s,y,\int\limits_0^T\int\limits_0^1 G(\theta,z)\,U_2(\theta,z)dzd\theta\right)\omega_2(y)dy\,ds\Bigg]. \tag{5.1}$$

Theorem 5.1. *Let the conditions of Theorems 3.1-4.3 be satisfied. Then function* (5.1) *will be a unique solution to the mixed problem* (1.1)-(1.4) *and this solution belongs to the class* (1.5).

Proof. The existence and uniqueness of the solution of the mixed problem (1.1)-(1.4) follows from the validity of Theorems 3.1-4.3. This solution has the form of a Fourier series (5.1). Indeed, from the Theorems 3.1-4.3 we have $t^{1-\gamma}u_0(t)\in C_2[0,T]$ and $t^{1-\gamma}\vec{u}_1(t),\ t^{1-\gamma}\vec{u}_2(t),\ t^{1-\gamma}\vec{\tilde{u}}_2(t)\in B_2[0,T]$. To prove the convergence of the function (5.1), we use the calculations in the proofs of the Theorems 3.1-4.3:

$$\left|\,t^{1-\gamma}U(t,x)\,\right|\leq\frac{|\,\varphi_0\,|}{\Gamma(\gamma)}+\frac{2\delta_0}{1-x_0^2}\,\frac{T^{1+\alpha-\gamma}}{\alpha\,\Gamma(\gamma)}+$$

$$+M_{1,1}\left\|\,\frac{\partial^4\varphi_1(x)}{\partial\,x^4}\,\right\|_{L_2[0,1]}+M_{3,1}\left\|\,\frac{\partial^4\varphi_2(x)}{\partial\,x^4}\,\right\|_{L_2[0,1]}+(M_{2,1}+M_{3,1})\left\|\,\frac{\partial^4\tilde{\varphi}_2(x)}{\partial\,x^4}\,\right\|_{L_2[0,1]}+$$

$$+M_{3,2}\left[\left\|\,D^{\alpha,\gamma}\left[t^{1-\gamma}\vec{\tilde{F}}_2(t)\right]\right\|_{B_2[0,T]}+\left\|\,t^{1-\gamma}\vec{\tilde{F}}_2(t)\,\right\|_{B_2[0,T]}\right]+$$

$$+\delta_1M_{1,2}\sqrt{\sum_{n=1}^{\infty *}\frac{1}{n^4}}+\delta_2M_{3,3}\sqrt{\sum_{m=1}^{\infty}\frac{1}{m^4}}+\tilde{\delta}_2M_{2,2}\sqrt{\sum_{m=1}^{\infty}\frac{1}{m^4}}<\infty. \tag{5.2}$$

We will prove the belongness of the function (5.2) to the class (1.5). In this order we also consider the functions

$$t^{1-\gamma}D^{\alpha,\gamma}U(t,x),\ t^{1-\gamma}U_{xx}(t,x),\ t^{1-\gamma}D^{\alpha,\gamma}U_{xx}(t,x).$$

The proofs of convergence of these functions are similarly to the estimate (5.2). So, we show the convergence only one of them. It is not difficult to see that $\left| t^{1-\gamma} U_{xx}(t,x) \right| < \infty$. Indeed, for the series

$$t^{1-\gamma} U_{xx}(t,x) = -\sum_{n=1}^{\infty *} \lambda_{1,n} \vartheta_{1,n}(x) \Big[\varphi_{1,n} E_{\alpha,\gamma}\left(-\mu_{1,n} t^{\alpha}\right) +$$

$$+ \frac{1}{1+\lambda_{1,n}} \int_0^t K_{1,n}(t,s) \int_0^1 f_1 \left(s, y, \int_0^T \int_0^1 G(\theta,z)\, U_1(\theta,z) dz d\theta \right) \omega_1(y) dy\, ds \Bigg] -$$

$$- \sum_{m=1}^{\infty} \lambda_{2,m} \vartheta_{2,m}(x) \Bigg[\varphi_{2,m} E_{\alpha,\gamma}\left(-\mu_{2,m} t^{\alpha}\right) - \frac{2\sqrt{\lambda_{2,m}}}{1+\lambda_{2,m}} \int_0^t K_{2,m}(t,s) \, \left(D^{\alpha,\gamma} \tilde{F}_{2,m}(s) + \tilde{F}_{2,m}(s) \right) ds +$$

$$+ \frac{1}{1+\lambda_{2,m}} \int_0^t K_{2,m}(t,s) \int_0^1 f_2 \left(s, y, \int_0^T \int_0^1 G(\theta,z)\, U_2(\theta,z) dz d\theta \right) \tilde{\omega}_2(y) dy\, ds \Bigg] -$$

$$- \sum_{m=1}^{\infty} \left(2\sqrt{\lambda_{2,m}} \vartheta_{2,m}(x) + \lambda_{2,m} \tilde{\vartheta}_{2,m}(x) \right) \Big[\tilde{\varphi}_{2,m} E_{\alpha,\gamma}\left(-\mu_{2,m} t^{\alpha}\right) +$$

$$+ \frac{1}{1+\lambda_{2,m}} \int_0^t \tilde{K}_{2,m}(t,s) \int_0^1 \tilde{f}_2 \left(s, y, \int_0^T \int_0^1 G(\theta,z)\, U_2(\theta,z) dz d\theta \right) \omega_2(y) dy\, ds \Bigg], \tag{5.3}$$

we have

$$\left| t^{1-\gamma} U_{xx}(t,x) \right| \le M_0 \sum_{n=1}^{\infty *} \left| \lambda_{1,n} \varphi_{1,n} \right| + M_0 \sum_{m=1}^{\infty} \left| \lambda_{2,m} \varphi_{2,m} \right| + M_0 \sum_{m=1}^{\infty} \left(2\sqrt{\lambda_{2,m}} + \lambda_{2,m} \right) \left| \tilde{\varphi}_{2,m} \right| +$$

$$+ \sum_{n=1}^{\infty *} \frac{\lambda_{1,n}}{1+\lambda_{1,n}} \max_{0\le t\le T} \left| \int_0^1 f_1 \left(s, y, \int_0^T \int_0^1 G(\theta,z)\, U_1(\theta,z) dz d\theta \right) \omega_1(y) dy\, ds \right| \int_0^t K_{1,n}(t,s)\, ds +$$

$$+ \sum_{m=1}^{\infty} \frac{\lambda_{2,m}}{1+\lambda_{2,m}} \max_{0\le t\le T} \left| \int_0^1 f_2 \left(s, y, \int_0^T \int_0^1 G(\theta,z)\, U_2(\theta,z) dz d\theta \right) \tilde{\omega}_2(y) dy\, ds \right| \int_0^t K_{2,m}(t,s) ds +$$

$$+ \sum_{m=1}^{\infty} \frac{2\lambda_{2,m}}{1+\lambda_{2,m}} \int_0^t K_{2,m}(t,s) s^{\gamma-1} \left| D^{\alpha,\gamma} \left(\sqrt{\lambda_{2,m}} s^{1-\gamma} \tilde{F}_{2,m}(s) \right) + \sqrt{\lambda_{2,m}} s^{1-\gamma} \tilde{F}_{2,m}(s) \right| ds +$$

$$+ \sum_{m=1}^{\infty} \frac{2\sqrt{\lambda_{2,m}} + \lambda_{2,m}}{1+\lambda_{2,m}} \max_{0\le t\le T} \left| \int_0^1 \tilde{f}_2 \left(s, y, \int_0^T \int_0^1 G(\theta,z)\, U_2(\theta,z) dz d\theta \right) \tilde{\omega}_2(y) dy\, ds \right| \int_0^t K_{2,m}(t,s) ds.$$

Hence, by virtue of smoothness conditions, we obtain the following estimate

$$\left| t^{1-\gamma} U_{xx}(t,x) \right| \le M_0 \sum_{n=1}^{\infty *} \frac{1}{\lambda_{1,n}} \left| \int_0^1 \frac{\partial^4 \varphi_1(y)}{\partial y^4} \omega_{1,n}(y)\, dy \right| +$$

$$+ M_0 \sum_{m=1}^{\infty} \left[\frac{1}{\lambda_{2,m}} \left| \int_0^1 \frac{\partial^4 \varphi_2(y)}{\partial y^4} \omega_{2,m}(y)\, dy \right| + \frac{4}{\lambda_{2,m}\sqrt{\lambda_{2,m}}} \left| \int_0^1 \frac{\partial^4 \tilde{\varphi}_2(y)}{\partial y^4} \omega_{2,m}(y)\, dy \right| \right] +$$

$$+ M_0 \sum_{m=1}^{\infty} \left(\frac{2}{\sqrt{\lambda_{2,m}} \lambda_{2,m}} + \frac{1}{\lambda_{2,m}} \right) \left| \int_0^1 \frac{\partial^4 \tilde{\varphi}_2(y)}{\partial y^4} \omega_{2,m}(y)\, dy \right| +$$

$$+M_0\frac{T^{1+\alpha-\gamma}}{\alpha}\sum_{n=1}^{\infty*}\max_{0\le t\le T}\frac{1}{\lambda_{1,n}}\left|\int_0^1\frac{\partial^2 f_1(t,y,\cdot)}{\partial y^2}\,\omega_{1,n}(y)\,dy\right|+$$

$$+M_0\frac{T^{1+\alpha-\gamma}}{\alpha}\sum_{m=1}^{\infty}\max_{0\le t\le T}\frac{1}{\lambda_{2,m}}\left|\int_0^1\frac{\partial^2 f_2(t,y,\cdot)}{\partial y^2}\,\tilde{\omega}_{2,m}(y)\,dy\right|+$$

$$+M_0\frac{T^{1+\alpha-\gamma}}{\alpha}\sum_{m=1}^{\infty}\max_{0\le t\le T}\frac{4}{\lambda_{2,m}\sqrt{\lambda_{2,m}}}\left|\int_0^1\frac{\partial^2 \tilde{f}_2(t,y,\cdot)}{\partial y^2}\,\omega_{2,m}(y)\,dy\right|+$$

$$+2M_0\frac{\Gamma(\gamma)\,\Gamma(\alpha)}{\Gamma(\gamma+\alpha)}T^{\alpha}\sum_{m=1}^{\infty}\max_{0\le t\le T}\left|\,D^{\alpha,\gamma}\left(\sqrt{\lambda_{2,m}}t^{\gamma-1}\tilde{F}_{2,m}(t)\right)+\sqrt{\lambda_{2,m}}t^{\gamma-1}\tilde{F}_{2,m}(t)\right|+$$

$$+3M_0\frac{T^{1+\alpha-\gamma}}{\alpha}\sum_{m=1}^{\infty}\max_{0\le t\le T}\frac{1}{\lambda_{2,m}}\left|\int_0^1\frac{\partial^2 \tilde{f}_2(t,y,\cdot)}{\partial y^2}\,\omega_{2,m}(y)\,dy\right|. \tag{5.4}$$

By virtue of the results of the Theorem 3.1-4.3 and applying Cauchy-Shwartz inequality and Bessel's inequality, from (5.4) we obtain

$$\left|\,t^{1-\gamma}U_{xx}(t,x)\,\right|\le M_0\left\|\frac{\partial^4\varphi_1(x)}{\partial x^4}\right\|_{L_2[0,1]}\sqrt{\sum_{n=1}^{\infty*}\lambda_{1,n}^{-2}}+$$

$$+M_0\left[\left\|\frac{\partial^4\varphi_2(x)}{\partial x^4}\right\|_{L_2[0,1]}\sqrt{\sum_{m=1}^{\infty}\lambda_{2,m}^{-2}}+\left\|\frac{\partial^4\tilde{\varphi}_2(x)}{\partial x^4}\right\|_{L_2[0,1]}\sqrt{\sum_{m=1}^{\infty}\lambda_{2,m}^{-3}}\right]+$$

$$+M_0\left\|\frac{\partial^4\tilde{\varphi}_2(x)}{\partial x^4}\right\|_{L_2[0,1]}\left(2\sqrt{\sum_{m=1}^{\infty}\lambda_{2,m}^{-3}}+\sqrt{\sum_{m=1}^{\infty}\lambda_{2,m}^{-2}}\right)+$$

$$+M_0\frac{T^{1+\alpha-\gamma}}{\alpha}\max_{0\le t\le T}\left\|\frac{\partial^4 f_1(t,x,\cdot)}{\partial x^4}\right\|_{L_2[0,1]}\sqrt{\sum_{n=1}^{\infty*}\lambda_{2,n}^{-2}}+$$

$$+M_0\frac{T^{1+\alpha-\gamma}}{\alpha}\max_{0\le t\le T}\left\|\frac{\partial^4 f_2(t,x,\cdot)}{\partial x^4}\right\|_{L_2[0,1]}\sqrt{\sum_{m=1}^{\infty}\lambda_{2,m}^{-2}}+$$

$$+4M_0\frac{T^{1+\alpha-\gamma}}{\alpha}\max_{0\le t\le T}\left\|\frac{\partial^4 \tilde{f}_2(t,x,\cdot)}{\partial x^4}\right\|_{L_2[0,1]}\sqrt{\sum_{m=1}^{\infty}\lambda_{2,m}^{-3}}+$$

$$+2M_0\frac{\Gamma(\gamma)\,\Gamma(\alpha)}{\Gamma(\gamma+\alpha)}T^{\alpha}\sum_{m=1}^{\infty}\max_{0\le t\le T}\left|\,D^{\alpha,\gamma}\left(\sqrt{\lambda_{2,m}}t^{\gamma-1}\tilde{F}_{2,m}(t)\right)+\sqrt{\lambda_{2,m}}t^{\gamma-1}\tilde{F}_{2,m}(t)\right|+$$

$$+3M_0\frac{T^{1+\alpha-\gamma}}{\alpha}\max_{0\le t\le T}\left\|\frac{\partial^4 \tilde{f}_2(t,x,\cdot)}{\partial x^4}\right\|_{L_2[0,1]}\sqrt{\sum_{m=1}^{\infty}\lambda_{2,m}^{-2}}.$$

So, the series (5.3) is convergence. This completes the proof of the theorem. □

Example 5.2. At the end of the paper, we consider an example. In the domain $\Omega \equiv (0,T)\times(0,1)$ we consider the equation

$$\left(D^{\frac{1}{3},\frac{1}{2}} - D^{\frac{1}{3},\frac{1}{2}}\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x^2}\right)U(t,x) = t\left[x(x-1)\left(x-\frac{1}{2}\right)\right]^6 \int_0^T\int_0^1 syU(s,y)dyds$$

with mixed conditions

$$\lim_{t\to+0} J_{0t}^{\frac{1}{2}}U(t,x) = 5\left[x(x-1)\left(x-\frac{1}{2}\right)\right]^6,$$

$$U(t,0)=0,\quad U_x(t,1)=U_x\left(t,\frac{1}{2}\right),\quad 0\le t\le T.$$

The conditions

$$\frac{d^i\varphi(x)}{dx^i}\,|_{x=0}=0,\quad \frac{d^j\varphi(x)}{dx^j}\,|_{x=1}=\frac{d^j\varphi(x)}{dx^j}\Big|_{x=\frac{1}{2}},\quad i=0,2,4,6,\quad j=1,3,5,$$

$$\frac{d^i f(t,x,\cdot)}{dx^i}\,|_{x=0}=0,\quad \frac{d^j f(t,x,\cdot)}{dx^j}\,|_{x=1}=\frac{d^j f(t,x,\cdot)}{dx^j}\Big|_{x=\frac{1}{2}},\quad i=0,2,4,6,\quad j=1,3,5$$

are fulfilled.

We will check the conditions of the Theorem 3.1:

1). $\int_0^T\int_0^1 \frac{tx}{t^{1-\gamma}}xdxdt = \int_0^T\int_0^1 \sqrt{t}x^2dxdt = \frac{2T^{\frac{3}{2}}}{9} \le \frac{1}{4}T^{\frac{3}{2}} = G_0 < \infty$;

2). $\left\|t\left[x(x-1)\left(x-\frac{1}{2}\right)\right]^6\int_0^T\int_0^1 sy^2u_0^0(s)dyds\right\|_C \le \frac{5T^2}{3\sqrt{\pi}} = \delta_0$;

3). $\tilde{l}_0 = T\int_0^1 \left(x^3+1\right)dx = \frac{5}{4}T.$

For condition

$$\rho = \tilde{l}_0 G_0 \frac{2\delta_0}{1-x_0^2}\frac{T^{1+\alpha-\gamma}}{\alpha\,\Gamma(\gamma)} < 1$$

to be met, there must be

$$T < \left(\frac{\pi}{4}\right)^{\frac{3}{16}}.$$

The fulfillment of the conditions of other Theorems 4.1-4.3 is verified similarly.

Acknowledgements. This scientific work is made a report at the seventh international scientific conference "Mathematical Advances and Applications", which was held in Istanbul from May 8 to 11 at the Yildiz Technical University.

References

[1] V. M. Aleksandrov and E. V. Kovalenko, *Problems of Continuum Mechanics with Mixed Boundary Conditions*, Nauka, Moscow, 1986.

[2] R. R. Ashurov, B. J. Kadirkulov and M. A. Jalilov, *On a nonlocal problem for a mixed-type equation with a fractional order operator*, Lobachevskii Journal of Mathematics **45(7)** (2024), 3220–3232.

[3] R. Ashurov, B. Kadirkulov and M. Jalilov, *On an inverse problem of the Bitsadze–Samarskii type for a parabolic equation of fractional order*, Boletín de la Sociedad Matemática Mexicana **29(3)** (2023), 1-21.

[4] G. I. Barenblatt, Yu. P. Zheltov and I. N. Kochina, *On finitness conditions in the mechanics of continuous media in static problems of the theory of elasticity*, Prikl. Mat. Meh. **24(5)** (1960), 316-322.

[5] G. I. Barenblatt and Yu. P. Zheltov, *Fundamental equations of filtration of homogeneous liquids in fissured rocks*, Sov. Phys. Dokl. **132(5)** (1960), 522-525.

[6] N. K. Bari, *Biorthogonal systems and bases in Hilbert space*, Math. Uch. Zap. Moscow State Univ. **4** (1951), 69-107.

[7] M. Benchohra and J. E. Lazreg, *Existence and uniqueness results for nonlinear implicit fractional differential equations with boundary conditions*, Rom. J. Math. Comput. Sci. **4(1)** (2014), 60-72.

[8] V. A. Chernyatin, *Justification of the Fourier Method in a Mixed Problem for Partial Differential Equations*, Lomonosov University, Moscow, 1992.

[9] R. Gorenflo, A. A. Kilbas, F. Mainardi and S. V. Rogozin, *Mittag-Leffler Functions, Related Topics and Applications*, Springer, Berlin, Heidelberg, 2014.

[10] V. A. Ilyin, *On the solvability of mixed problems for hyperbolic and parabolic equations*, Uspekhi Mat. Nauk **15(2)** (1960), 97-154.

[11] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, North-Holland, 2006.

[12] D. Kumar and D. Baleanu, *Fractional calculus and its applications in physics*, Front. Phys. **7(6)** (2019), 1-2.

[13] A. A. Matchanova, B. J. Kadirkulov and T. K. Yuldashev, *Mixed problem for a linear Barenblatt-Zheltov-Kochina equation with fractional Hilfer operator*, Lobachevskii Journal of Mathematics **45(7)** (2024), 3333-3350.

[14] S. Patnaik , J. P. Hollkamp and F. Semperlotti, *Applications of variable-order fractional operators: a review*, Proceedings A, Royal Society, **A476** (2020), 20190498.

[15] A. V. Pskhu, *Fractional Partial Differential Equations*, Nauka, Moscow, 2005.

[16] T. K. Yuldashev, *Mixed value problem for nonlinear integro-differential equation with parabolic operator of higher power*, Comput. Math. Math. Phys. **52(1)** (2012), 105-116.

[17] T. K. Yuldashev and O. Kh. Abdullaev, *Unique solvability of a boundary value problem for a loaded fractional parabolic-hyperbolic equation with nonlinear terms*, Lobachevskii Journal of Mathematics **42(5)** (2021), 1113-1123.

[18] T. K. Yuldashev and T. A. Abduvahobov, *Periodic solutions for an impulsive system of fractional order integro-differential equations with maxima*, Lobachevskii Journal of Mathematics **44(10)** (2023), 4401-4409.

[19] T. K. Yuldashev and B. J. Kadirkulov, *Inverse problem for a partial differential equation with Gerasimov-Caputo-type operator and degeneration*, Fractal Fract. **5(2)** (2021), 1-13.

[20] T. K. Yuldashev and B. J. Kadirkulov, *On a boundary value problem for a mixed type fractional differential equations with parameters*, Proceedings of the Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan **47(1)** (2021), 112-123.

[21] T. K. Yuldashev, B. J. Kadirkulov and R. A. Bandaliyev, *On a mixed problem for Hilfer type fractional differential equation with degeneration*, Lobachevskii Journal of Mathematics **43(1)** (2022), 263-274.

Department of Higher Mathematics, Tashkent State Transport University, Temiryolchilar street, 1, Tashkent, Uzbekistan
Email address: tursun.k.yuldashev@gmail.com

Department of Mathematics and Physics, Alfraganus University, Yuqori Qoraqamish street, 2A, Tashkent, Uzbekistan
Email address: kadirkulovbj@gmail.com

V. I. Romanovskii Institute of Mathematics, Academy of Sciences of Uzbekistan, Tashkent, Uzbekistan
Email address: oygul87-87@mail.ru