PARALLEL ITERATION METHODS WITH APPLICATION TO VARIATIONAL INEQUALITY PROBLEMS

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Abstract. In this paper, we define new iteration methods for altering points and generalized altering points of Lipschitzian mappings. We proved the convergence results and data dependency of this new iteration methods under suitable assumptions. We also give an application for solution of nonlinear variational inequalities.

Mathematics Subject Classification (2010): 47J05, 47H09, 65K05, 65K10.
Key words: Altering points, parallel iteration methods, variational inequality.

Article history:
Received: August 9, 2024
Received in revised form: October 8, 2024
Accepted: October 10, 2024

1. Introduction and Preliminaries

Let K be a nonempty subset of a normed space B and $A : K \to B$ an operator.
i. A is said to be L-Lipschitzian if there exists a constant $L > 0$, such that

$$\|Ap - Aq\| \leq L\|p - q\| \text{ for all } p, q \in K.$$

ii. A is said to be ω-inverse strongly monotone (ω-ism) if there exist a constant $\omega > 0$ such that

$$\langle Ap - Aq, p - q\rangle \geq \omega\|p - q\|^2 \text{ for all } p, q \in K.$$

Let K be a nonempty closed convex subset of B. We use P_K to denote the projection from B onto K; namely, for $p \in B$, $P_K p$ is the unique point in K with the property:

$$\|p - P_K p\| = \inf \{\|p - q\| : q \in K\}.$$

The projection operator $P_K : B \to B$ is nonexpansive mapping.
Now let's remember the definition of alternating point again.

Definition 1.1. ([2]). Let B be a metric space, K_1 and K_2 be nonempty subsets of B. We say $p \in K_1$ and $q \in K_2$ are *altering points* of mappings $A_1 : K_1 \to K_2$ and $A_2 : K_2 \to K_1$ if

$$\begin{cases} A_1(p) = q \\ A_2(q) = p \end{cases} \tag{1.1}$$

Sahu [2] proved some convergence results for Lipschitz continuous mappings that have altering points using Picard, Mann, and S-iteration processs. He also introduced the parallel-S iteration process to reach the altering points of nonlinear mappings as follows:

$$\begin{cases} p_{f+1} = A_2\left[(1-\alpha_f)\, q_f + \alpha_f A_1 p_f\right] \\ q_{f+1} = A_1\left[(1-\alpha_f)\, p_f + \alpha_f A_2 q_f\right] \end{cases} \tag{1.2}$$

where $(p_1, q_1) \in K_1 \times K_2$ and $\{\alpha_f\}$ is a real sequence in $[0, 1]$.

Sahu et al. [1] proposed a parallel Mann iteration process as follows:

$$\left\{\begin{array}{l} p_{f+1}=(1-\alpha_f)\,p_f+\alpha_f A_2 q_f \\ q_{f+1}=(1-\alpha_f)\,q_f+\alpha_f A_1 p_f \end{array}\right. \tag{1.3}$$

where $(p_1,q_1)\in K_1\times K_2$ and $\{\alpha_f\}$ is a real sequence in $[0,1]$.

The authors in [1] compared the convergence results between the iteration processes (1.2) and (1.3). They showed that convergence speed of the iteration process (1.2) is better than the iteration process (1.3). They also gave a numerical example for it. After, Sintunavarat and Pitea [3] iteration process defined two parallel fixed point iteration processs as follows:

$$\begin{array}{ll} p_{f+1}=(1-\alpha_f)\,A_2 z_f+\alpha_f A_2 w_f & q_{f+1}=(1-\alpha_f)\,A_1 u_f+\alpha_f A_1 v_f \\ z_f=(1-\beta_f)\,q_f+\beta_f w_f & u_f=(1-\beta_f)\,p_f+\beta_f v_f \\ w_f=(1-\gamma_f)\,q_f+\gamma_f A_1 p_f & v_f=(1-\gamma_f)\,p_f+\gamma_f A_2 q_f \end{array} \tag{1.4}$$

Taking $\gamma_f=1$ for all $f\in\mathbb{N}$ in iteration process (1.4), it reduces the following iteration process:

$$\begin{array}{ll} p_{f+1}=(1-\alpha_f)\,A_2 z_f+\alpha_f A_2 w_f & q_{f+1}=(1-\alpha_f)\,A_1 u_f+\alpha_f A_1 v_f \\ z_f=(1-\beta_f)\,q_f+\beta_f w_f & u_f=(1-\beta_f)\,p_f+\beta_f v_f \\ w_f=A_1 p_f & v_f=A_2 q_f \end{array} \tag{1.5}$$

where $\{\alpha_f\},\{\beta_f\}$ and $\{\gamma_f\}$ are real sequences in $[0,1]$.

Sintunavarat and Pitea [3] iteration process showed that iteration process (1.5) has a better convergence speed than iteration process (1.2) using a numerical example under suitable conditions. Moreover, they analyzed the data dependency result of this iteration process.

Now, we will give some known results:

Lemma 1.2. *([4]) Let B be a metric space. For a given $z\in B, p\in K$ satisfies the inequality*

$$\langle p-z,q-p\rangle\geq 0,\forall q\in K$$

if and only if

$$p=P_K[z]$$

where P_K is the projection of B onto K. In addition, the projection operator P_K is nonexpansive and satisfies $\langle p-q,P_Kp-P_Kq\rangle\geq\|P_Kp-P_Kq\|^2$, for all $p,q\in B$.

Lemma 1.3. *([5]) Let K_1 and K_2 be nonempty closed convex subsets of a normed space B. Let $A_1:K_1\to B$ and $A_2:K_2\to B$ be nonlinear operators and let δ and θ be positive real numbers. Define $Q=I-\delta A_1$ and $W=I-\theta A_2$. Then the following are equivalent:*

i) p^* and q^* are altering points of mappings $P_{K_2}U$ and $P_{K_1}V$.

ii) $(p^*,q^*)\in K_1\times K_2$ is a solution of the following system of variational inequalities:

Find $(p^*,q^*)\in K_1\times K_2$ such that

$$\left\{\begin{array}{ll} \langle q^*-Q\,(p^*)\,,p-q^*\rangle\geq 0 & \textit{for all } p\in K_2 \\ \langle p^*-Q\,(q^*)\,,p-p^*\rangle\geq 0 & \textit{for all } p\in K_1. \end{array}\right.$$

Definition 1.4. ([4]) Let B be a metric space and $A,S:B\to B$ be two operators. S is called an *approximate operator* of A for all $p\in B$ and a fixed $\varepsilon>0$ if $\|Ap-Sp\|\leq\varepsilon$.

Lemma 1.5. *([4]) Let $\{\gamma_f\}$ be a real sequence and there exists $f_0\in\mathbb{N}$ such that, for all $f\geq f_0$ satisfying the following condition:*

$$\gamma_{f+1}\leq(1-\sigma_f)\,\gamma_f+\sigma_f\rho_f,$$

where $\sigma_f\in(0,1)$ such that $\sum_{f=1}^{\infty}\sigma_f=\infty$. Then, the following inequality holds:

$$0\leq\lim_{f\to\infty}\sup\gamma_f\leq\lim_{f\to\infty}\sup\rho_f.$$

2. Main Results

Now, we introduce the following iteration process for the altering points of the Lipschitz mappings:

Let K_1 and K_2 be nonempty convex subsets of a normed space B. Also, let $A_1 : K_1 \to K_2$ and $A_2 : K_2 \to K_1$ be two mappings. For $(p_1, q_1) \in K_1 \times K_2$, our iteration method is as follows:

$$\left\{\begin{array}{cc} p_{f+1} = A_2 z_f & q_{f+1} = A_1 u_f \\ z_f = A_1 A_2 w_f & u_f = A_2 A_1 v_f \\ w_f = A_1[(1-\alpha_f)A_2 q_f + \alpha_f A_2 A_1 p_f] & v_f = A_2[(1-\alpha_f)A_1 p_f + \alpha_f A_1 A_2 q_f] \end{array}\right. \tag{2.1}$$

where $\{\alpha_f\}$ is a real sequence in $[0, 1]$.

The state of the above iteration given in $K_1 \times K_2$ in K_1 is as follows. We will also use this to prove the following theorem in a simpler way.

$$\left\{\begin{array}{c} p_{f+1} = A z_f \\ z_f = A^2 w_f \\ w_f = A[(1-\alpha_f)A p_f + \alpha_f A^2 p_f] \end{array}\right.$$

Theorem 2.1. *Assume that K_1 and K_2 are nonempty closed convex subsets of a Banach space B. We also suppose that $A_1 : K_1 \to K_2$ and $A_2 : K_2 \to K_1$ be two Lipschitz mappings with constants L_1 and L_2 such that $L_1 L_2 < 1$. Then,*

i. There exists a unique point $(p, q) \in K_1 \times K_2$ such that p and q are altering points of mappings A_1 and A_2.

ii. For arbitrary $p_1 \in K_1$, the sequence $\{(p_f, q_f)\} \in K_1 \times K_2$ generated by (2.1) converges to (p, q).

Proof. If $A_1 : K_1 \to K_2$ and $A_2 : K_2 \to K_1$ are two Lipschitz continuous mappings with Lipschitz constants L_1 and L_2 such that $L_1 L_2 < 1$, we know that the mapping $A := A_2 A_1 : K_1 \to K_1$ is contraction. If A is contraction, we know that A^2 is also contraction. Therefore, there exists a unique point $(p, q) \in K_1 \times K_2$ such that p and q are altering points of mappings A_1 and A_2. From (2.1) and Definition 1.1, we have

$$\begin{aligned} \|p_{f+1} - p\| &= \|A z_f - p\| \\ &= \|A_2 A_1 z_f - A_2 q\| \\ &\leq L_2 \|A_1 z_f - q\| \\ &= L_2 \|A_1 z_f - A_1 p\| \\ &\leq L_2 L_1 \|z_f - p\| \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} \|z_f - p\| &= \|A^2 w_f - p\| \\ &= \|(A_2 A_1)^2 w_f - (A_2 A_1) p\| \\ &\leq L_1^2 L_2^2 \|w_f - p\|. \end{aligned} \tag{2.3}$$

From hypothesis, we know that $L_1L_2 < 1$. Using (2.3) and $L_1L_2 < 1$, we get

$$\begin{aligned}
\|w_f - p\| &= \|A[(1-\alpha_f)Ap_f + \alpha_f A^2 p_f] - p\| \\
&= \|A_2A_1[(1-\alpha_f)Ap_f + \alpha_f A^2 p_f] - A_2 q\| \\
&\leq L_2\|A_1[(1-\alpha_f)Ap_f + \alpha_f A^2 p_f] - q\| \\
&= L_2\|A_1[(1-\alpha_f)Ap_f + \alpha_f A^2 p_f] - A_1 p\| \\
&= L_2L_1[(1-\alpha_f)\|Ap_f - p\| + \alpha_f\|A^2p_f - p\|] \\
&= L_2L_1[(1-\alpha_f)\|A_2A_1p_f - A_2q\| + \alpha_f\|(A_2A_1)^2p_f - (A_2A_1)p\|] \\
&\leq L_2L_1[(1-\alpha_f)L_2\|A_1p_f - q\| + \alpha_f L_2L_1\|(A_2A_1)p_f - p\|] \\
&= L_2L_1[(1-\alpha_f)L_2\|A_1p_f - A_1p\| + \alpha_f L_2L_1\|(A_2A_1)p_f - (A_2A_1)p\|] \\
&\leq L_2L_1[(1-\alpha_f)L_2L_1\|p_f - p\| + \alpha_f L_2^2L_1^2\|p_f - p\|] \\
&= L_2^2L_1^2[(1-\alpha_f + \alpha_f L_2L_1)\|p_f - p\|] \\
&= L_2^2L_1^2(1-(1-L_2L_1)\alpha_f)\|p_f - p\|.
\end{aligned} \tag{2.4}$$

If (2.2), (2.3) and (2.4) are combined, we obtain

$$\begin{aligned}
\|p_{f+1} - p\| &\leq L_2L_1L_2^2L_1^2L_2^2L_1^2[1-(1-L_2L_1)\alpha_f]\|p_f - p\| \\
&= L_2^5L_1^5[1-(1-L_2L_1)\alpha_f]\|p_f - p\| \\
&\leq (L_2L_1)^5\|p_f - p\|
\end{aligned}$$

which implies that

$$\|p_{f+1} - p\| \leq (L_2L_1)^{5f}\|p_1 - p\|. \tag{2.5}$$

If we take limit on both sides of (2.5) and using $L_1L_2 < 1$, we have

$$\lim_{f\to\infty} \|p_f - p\| = 0.$$

Also, since A_1 is a continuous mapping, we have $q_f = A_1p_f \to A_1p = q$. Thus, we obtain that $(p_f, q_f) \to (p, q)$. □

Now, we will show that the convergence of the parallel iteration method (2.1) to the unique altering points of the Lipschitz mappings. We also give a data dependence result for the parallel iteration method (2.1).

Theorem 2.2. *Assume that K_1 and K_2 are nonempty closed convex subsets of a Banach space B. We also suppose that $A_1 : K_1 \to K_2$ and $A_2 : K_2 \to K_1$ be two Lipschitz mappings with constants L_1 and L_2 such that $L_1 + L_2 < 1$. Then, the sequence $\{(p_f, q_f)\}$ in $K_1 \times K_2$ generated by (2.1) converges strongly to a unique point (p, q) in $K_1 \times K_2$ so that p and q are altering points of mappings A_1 and A_2.*

Proof. From Theorem 2.1, we know that there exists a unique point (p, q) in $K_1 \times K_2$ so that p and q are altering points of mappings A_1 and A_2. From the method (2.1) and Definition 1.1, we have

$$\begin{aligned}
\|p_{f+1} - p\| &= \| A_2z_f - A_2q \| \\
&\leq L_2 \| z_f - q \|
\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}
\|z_f - q\| &= \|A_1A_2w_f - q\| \\
&= \|A_1A_2w_f - A_1p\| \\
&\leq L_1\|A_2w_f - p\| \\
&= L_1\|A_2w_f - A_2q\| \\
&\leq L_1L_2\|w_f - q\|.
\end{aligned} \tag{2.7}$$

Using (2.7), we have

$$\begin{aligned}
\|w_f - q\| &= \|A_1[(1-\alpha_f)A_2q_f + \alpha_f A_2A_1p_f] - q\| \\
&= \|A_1[(1-\alpha_f)A_2q_f + \alpha_f A_2A_1p_f] - A_1p\| \\
&\leq L_1\,\|(1-\alpha_f)A_2q_f + \alpha_f A_2A_1p_f - p\| \\
&= L_1\,\|(1-\alpha_f)A_2q_f + \alpha_f A_2A_1p_f - A_2q\| \\
&\leq L_1\,[(1-\alpha_f)\,\|A_2q_f - A_2q\| + \alpha_f\,\|A_2A_1p_f - A_2q\|] \\
&\leq L_1\,[(1-\alpha_f)L_2\,\|q_f - q\| + \alpha_f L_2\,\|A_1p_f - q\|] \\
&\leq L_1\,[(1-\alpha_f)L_2\,\|q_f - q\| + \alpha_f L_2L_1\,\|p_f - p\|] \\
&\leq L_1L_2(1-\alpha_f)\|q_f - q\| + \alpha_f L_1^2L_2\|p_f - p\|.
\end{aligned} \tag{2.8}$$

Combine (2.6), (2.7) and (2.8), we obtain

$$\begin{aligned}
\|p_{f+1} - p\| &\leq L_2[L_1L_2\|w_f - q\|] \\
&\leq L_1L_2^2[L_1L_2(1-\alpha_f)\|q_f - q\| + \alpha_f L_1^2L_2\|p_f - p\|] \\
&\leq L_1[\|q_f - q\| + \|p_f - p\|].
\end{aligned} \tag{2.9}$$

We can also obtain the following inequality can be obtained using the similar processes in (2.6)–(2.9)

$$\|q_{f+1} - q\| \leq L_2[\|q_f - q\| + \|p_f - p\|]. \tag{2.10}$$

If we add (2.9) and (2.10) by side, we obtain

$$\|p_{f+1} - p\| + \|q_{f+1} - q\| \leq \mu[\|p_f - p\| + \|q_f - q\|] \tag{2.11}$$

where $\mu = L_1 + L_2 < 1$. Now, we define the norm $\|\cdot\|_*$on $B \times B$ by $\|(p,q)\|_* = \|p\| + \|q\|$ for all $(p,q) \in B \times B$. We know that $(B \times B, \|\cdot\|_*)$ is a Banach space. Using (2.11), we have

$$\|(p_{f+1}, q_{f+1}) - (p,q)\|_* \leq \mu\|(p_f, q_f) - (p,q)\|_* \tag{2.12}$$

by induction, we get

$$\|\ (p_{f+1}, q_{f+1}) - (p,q)\ \|_* \leq \mu^f\ \|\ (p_1, q_1) - (p,q)\|_*. \tag{2.13}$$

Taking the limit on both sides of above inequality, we have

$$\lim_{f\to\infty} \|(p_{f+1}, q_{f+1}) - (p,q)\|_* = 0 \tag{2.14}$$

which implies that

$$\lim_{f\to\infty} \|p_f - p\| = \lim_{f\to\infty} \|q_f - q\| = 0.$$

Therefore, $\{p_f\}$ and $\{q_f\}$ converge to p and q, respectively. □

Now, we discuss the data dependency concept of iteration method (2.1) for Lipschitz mappings:

Theorem 2.3. *Assume that K_1 and K_2 are nonempty closed convex subsets of a Banach space B. We also suppose that $A_1 : K_1 \to K_2$ and $A_2 : K_2 \to K_1$ be two Lipschitz mappings with constants L_1 and L_2 such that $L_1 + L_2 < 1$. Let S_1, S_2 be approximate operators of A_1 and A_2, respectively. Let $\{p_f\}$ and $\{q_f\}$ be iterative sequences generated by (2.1) and define iterative sequences $\{a_f\}$ and $\{b_f\}$ as follows:*

$$\begin{cases}
a_{f+1} = S_2k_f & b_{f+1} = S_1h_f \\
k_f = S_1S_2d_f & h_f = S_2S_1k_f \\
d_f = S_1[(1-\alpha_f)S_2b_f + \alpha_f S_2S_1a_f] & k_f = S_2[(1-\alpha_f)S_1a_f + \alpha_f S_1S_2b_f]
\end{cases} \tag{2.15}$$

where $\{\alpha_f\}$ and $\{\beta_f\}$ are real sequences in $[0,1]$. In addition, we suppose that there exist nonnegative constants ε_1 and ε_2 such that $\|A_1\vartheta - S_1\vartheta\| \leq \varepsilon_1$ and $\|A_2\sigma - S_2\sigma\| \leq \varepsilon_2$ for all $\vartheta \in K_1$ and $\sigma \in K_2$. If

$(p,q) \in K_1 \times K_2$, which are altering points of mappings A_1 and A_2, and $(a,b) \in K_1 \times K_2$, which are altering points of mappings S_1 and S_2, such that $(a_f, b_f) \to (a,b)$ as $f \to \infty$, then we have

$$\|(p,q)-(a,b)\|_* = \|p-a\| + \|q-b\| \leq \frac{L_2\varepsilon_1 + L_1\varepsilon_2 + 2\varepsilon_1 + 2\varepsilon_2}{1-L_1-L_2}.$$

Proof. From iteration methods (2.1) and (2.15), we obtain

$$\begin{aligned} \|p_{f+1}-a_{f+1}\| &\leq \|A_2z_f - S_2k_f\| \\ &\leq \|A_2z_f - A_2k_f\| + \|A_2k_f - S_2k_f\| \\ &\leq L_2\|z_f - k_f\| + \varepsilon_2 \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} \|z_f - k_f\| &\leq L_1\|A_2w_f - S_2d_f\| \\ &\leq L_1\|A_2w_f - A_2d_f\| + \|A_2d_f - S_2d_f\| \\ &\leq L_1L_2\|w_f - d_f\| + \varepsilon_2. \end{aligned} \tag{2.17}$$

Using above inequality (2.17), we have

$$\begin{aligned} \|w_f - d_f\| &\leq A_1[(1-\alpha_f)A_2q_f + \alpha_f[A_2A_1p_f] \\ &\quad -S_1[(1-\alpha_f)S_2b_f + \alpha_fS_2S_1a_f] \\ &\leq L_1\{(1-\alpha_f)[A_2q_f - S_2b_f] + \alpha_f[A_2A_1p_f - S_2S_1a_f]\} \\ &\leq L_1\{(1-\alpha_f)\|A_2q_f - A_2b_f\| + \|A_2b_f - S_2b_f\| \\ &\quad +\alpha_fL_2[\|A_1p_f - A_1a_f\| + \|A_1a_f - S_1a_f\|]\} \\ &\leq L_1[L_2(1-\alpha)\|q_f - b_f\| + \varepsilon_2 + \alpha_fL_2L_1\|p_f - a_f\| + \varepsilon_1]. \end{aligned} \tag{2.18}$$

If we combine (2.16), (2.17) and (2.18), we get

$$\begin{aligned} \|p_{f+1}-a_{f+1}\| &\leq L_2(L_1L_2\|w_f - d_f\| + \varepsilon_2) + \varepsilon_2 \\ &\leq L_2^2L_1[L_1L_2(1-\alpha)\|q_f - b_f\| + \varepsilon_2 \\ &\quad +\alpha_fL_2L_1\|p_f - a_f\| + \varepsilon_1] + \varepsilon_2 + \varepsilon_2 \\ &\leq L_2[\|q_f - b_f\| + \|p_f - a_f\|] + L_2\varepsilon_1 + 2\varepsilon_2. \end{aligned} \tag{2.19}$$

Using similar operations, we obtain the following inequality

$$\|q_{f+1} - b_{f+1}\| \leq L_1[\|p_f - a_f\| + \|q_f - b_f\|] + L_1\varepsilon_2 + 2\varepsilon_1 \tag{2.20}$$

From (2.19) and (2.20), we get that the following inequality:

$$\begin{aligned} &\|p_{f+1}-a_{f+1}\| + \|q_{f+1}-b_{f+1}\| \\ \leq\ &(L_1+L_2)\left[\|p_f - a_f\| + \|q_f - b_f\|\right] + L_2\varepsilon_1 + L_1\varepsilon_2 + 2\varepsilon_1 + 2\varepsilon_2 \end{aligned} \tag{2.21}$$

There exists a real number $L \in (0,1)$ such that $1 - L = L_1 + L_2 < 1$. Hence, we have

$$\begin{aligned} \|p_{f+1}-a_{f+1}\| + \|q_{f+1}-b_{f+1}\| &\leq (1-L)[\|p_f - a_f\| + \|q_f - b_f\|] \\ &\quad + \frac{L[L_2\varepsilon_1 + L_1\varepsilon_2 + 2\varepsilon_1 + 2\varepsilon_2]}{L}. \end{aligned} \tag{2.22}$$

Denote that

$$\begin{cases} \gamma_f = \|p_f - a_f\| + \|q_f - b_f\| \\ \sigma_f = L \in (0,1) \\ \rho_f = \frac{L_2\varepsilon_1 + L_1\varepsilon_2 + 2\varepsilon_1 + 2\varepsilon_2}{L} \end{cases}$$

It is now easy to check that (2.22) satisfies all the requirements of Lemma 1.5. Hence, it follows by its conclusion that

$$0 \leq \lim_{f\to\infty} \sup[\|p_f - a_f\| + \|q_f - b_f\|] \leq \lim_{f\to\infty} \sup \frac{L_2\varepsilon_1 + L_1\varepsilon_2 + 2\varepsilon_1 + 2\varepsilon_2}{L} \tag{2.23}$$

Since, $(a_f, b_f) \to (a, b)$ as $f \to \infty$, then we obtain

$$\|p - q\| + \|a - b\| \leq \frac{L_2\varepsilon_1 + L_1\varepsilon_2 + 2\varepsilon_1 + 2\varepsilon_2}{L}. \tag{2.24}$$

□

Next, we will introduce the following iteration process for the altering points of the three Lipschitz mappings:

Now, firstly we will give some definition and theorem in order to prove our main results.

Theorem 2.4. *([2]) Let K_1 and K_2 be nonempty closed subsets of a complete metric space B and let $A_1 : K_1 \to K_2$ and $A_2 : K_2 \to K_1$ be two Lipschitz continuous mappings with Lipschitz constants L_1 and L_2 such that $L_1L_2 < 1$. Then we have the following:*

(a) There exists a unique point $(p, q) \in K_1 \times K_2$ such that p^* and q^* are altering points of mappings A_1 and A_2.

(b) For arbitrary $p_0 \in K_1$, a sequence $\{(p_f, q_f)\}$ in $K_1 \times K_2$ generated by

$$\begin{cases} q_f = A_1 p_f \\ p_{f+1} = A_2 q_f \end{cases}$$

converges to (p^*, q^*).

Lemma 2.5. *([2]) Let K_1 and K_2 be two nonempty closed subset of a Banach space B. Let $\{S_f\}$ be a sequence of mappings from K_1 into K_2 such that $\{S_f\}$ is nearly nonexpansive with sequence $\{a_f\}$ in $[0, \infty)$, i.e.,*

$$||S_f p - S_f q|| \leq ||p - q|| + a_f \text{ for all } p\ , q \in K_1 \text{ and } f \in \mathbb{N}.$$

Let A_1 be a mapping from K_1 into K_2 defined by $A_1 z = \lim_{f\to\infty} S_f z$ for all $z \in K_1$. Then A_1 is nonexpansive.

Lemma 2.6. *([4])Let $\{a_f\}$ satisfy the following inequality:*

$$a_{f+1} \leq \omega a_f + \sigma_f,$$

where $a_f \geq 0$, $\sigma_f \geq 0$ with $\lim_{f\to\infty} \sigma_f = 0$, and $0 \leq \omega < 1$. Then $a_f \to 0$ as $f \to \infty$.

Assume that K_1 , K_2 and K_3 are nonempty closed convex subsets of a Banach space B. Also, suppose that $A_1 : K_1 \to K_2$, $A_2 : K_2 \to K_3$ and $A_3 : K_3 \to K_1$ be three mappings. For an arbitrary $(p_0, q_0, z_0) \in K_1 \times K_2 \times K_3$, a iteration process is defined by

$$\begin{cases} p_{f+1} = A_3A_2A_1w_f \\ w_f = (1 - \alpha_f - \beta_f)A_3z_f + \alpha_f A_3A_2q_f + \beta_f A_3A_2A_1p_f \end{cases}$$
$$\begin{cases} q_{f+1} = A_1A_3A_2u_f \\ u_f = (1 - \alpha_f - \beta_f)A_1p_f + \alpha_f A_1A_3z_f + \beta_f A_1A_3A_2q_f \end{cases} \tag{2.25}$$
$$\begin{cases} z_{f+1} = A_2A_1A_3a_f \\ a_f = (1 - \alpha_f - \beta_f)A_2q_f + \alpha_f A_2A_1p_f + \beta_f A_2A_1A_3z_f. \end{cases}$$

where $\{\alpha_f\}$ and $\{\beta_f\}$ are real sequences in $[0, 1]$.

Theorem 2.7. *Let K_1, K_2 and K_3 be three nonempty closed convex subsets of a Banach space B. Let $A_1 : K_1 \to K_2$, $A_2 : K_2 \to K_3$ and $A_3 : K_3 \to K_1$ be three Lipschitz mappings with constants $L_1 \leq 1$, $L_2 \leq 1$ and $L_3 \leq 1$ such that $L_1L_2L_3 < 1$. Then the sequence $\{(p_f, q_f, z_f)\}$ in $K_1 \times K_2 \times K_3$ generated by a iteration method (2.25) converges strongly to a unique point $(p^*, q^*, z^*) \in K_1 \times K_2 \times K_3$ such that p^*, q^* and z^* are altering points of mappings A_1, A_2 and A_3.*

Proof. From Theorem 2.4, we know that there exist a unique point $(p^*, q^*, z^*) \in K_1 \times K_2 \times K_3$ such that p^*, q^* and z^* are altering points of mappings A_1, A_2 and A_3.

$$\mu := \max\left\{ \begin{array}{c} L_1^2L_2^2L_3^2\beta_f + L_1^2L_2L_3(1-\alpha_f-\beta_f) + \alpha_f L_1^2L_2^2L_3, L_1^2L_2^2L_3^2\beta_f \\ +L_1L_2L_3^2(1-\alpha_f-\beta_f) + \alpha_f L_1L_2^2L_3^2, L_1^2L_2^2L_3^2\beta_f \\ +L_1L_2^2L_3(1-\alpha_f-\beta_f) + \alpha_f L_1^2L_2L_3^2 \end{array} \right\}.$$

Using (2.25), we obtain

$$\begin{aligned}
\|p_{f+1} - p^*\| &= \|A_3A_2A_1[(1-\alpha_f-\beta_f)A_3z_f + \alpha_f A_3A_2q_f + \beta_f A_3A_2A_1p_f] - p^*\| \\
&\leq \|A_3A_2A_1[(1-\alpha_f-\beta_f)A_3z_f + \alpha_f A_3A_2q_f + \beta_f A_3A_2A_1p_f] - A_3z^*\| \\
&\leq L_3\|A_2A_1[(1-\alpha_f-\beta_f)A_3z_f + \alpha_f A_3A_2q_f + \beta_f A_3A_2A_1p_f] - z^*\| \\
&\leq L_3\|A_2A_1[(1-\alpha_f-\beta_f)A_3z_f + \alpha_f A_3A_2q_f + \beta_f A_3A_2A_1p_f] - A_2q^*\| \\
&\leq L_3L_2\|A_1[(1-\alpha_f-\beta_f)A_3z_f + \alpha_f A_3A_2q_f + \beta_f A_3A_2A_1p_f] - q^*\| \\
&\leq L_3L_2\|A_1[(1-\alpha_f-\beta_f)A_3z_f + \alpha_f A_3A_2q_f + \beta_f A_3A_2A_1p_f] - A_1p^*\| \\
&\leq L_3L_2L_1\|[(1-\alpha_f-\beta_f)A_3z_f + \alpha_f A_3A_2q_f + \beta_f A_3A_2A_1p_f] - p^*\| \\
&\leq L_3L_2L_1\|(1-\alpha_f-\beta_f)A_3z_f - p^*\| + \|\alpha_f A_3A_2q_f - p^*\| \\
&\quad +\|\beta_f A_3A_2A_1p_f - p^*\| \\
&\leq L_3L_2L_1\|(1-\alpha_f-\beta_f)A_3z_f - A_3z^*\| + \|\alpha_f A_3A_2q_f - A_3z^*\| \\
&\quad +\|\beta_f A_3A_2A_1p_f - A_3z^*\| \\
&\leq L_3L_2L_1[(1-\alpha_f-\beta_f)L_3\|z_f - z^*\| + \alpha_f L_3\|A_2q_f - z^*\| \\
&\quad +\beta_f L_3\|A_2A_1p_f - z^*\|] \\
&\leq L_3L_2L_1[(1-\alpha_f-\beta_f)L_3\|z_f - z^*\| + \alpha_f L_3\|A_2q_f - A_2q^*\| \\
&\quad +\beta_f L_3\|A_2A_1p_f - A_2q^*\|] \\
&\leq L_3L_2L_1[(1-\alpha_f-\beta_f)L_3\|z_f - z^*\| + \alpha_f L_3L_2\|q_f - q^*\| \\
&\quad +\beta_f L_3L_2\|A_1p_f - q^*\|] \\
&\leq L_3L_2L_1[(1-\alpha_f-\beta_f)L_3\|z_f - z^*\| + \alpha_f L_3L_2\|q_f - q^*\| \\
&\quad +\beta_f L_3L_2\|A_1p_f - A_1p^*\|] \\
&\leq L_3L_2L_1[(1-\alpha_f-\beta_f)L_3\|z_f - z^*\| + \alpha_f L_3L_2\|q_f - q^*\| \\
&\quad +\beta_f L_3L_2L_1\|p_f - p^*\|].
\end{aligned}$$

□

Theorem 2.8. *Proof.* We also have

$$\begin{aligned}
\|q_{f+1} - q^*\| &\leq L_3L_2L_1[(1-\alpha_f-\beta_f)L_1\|p_f - p^*\| \\
&\quad +\alpha_f L_1L_3\|z_f - z^*\| + \beta_f L_3L_2L_1\|q_f - q^*\|]
\end{aligned}$$

and

$$\begin{aligned}
\|z_{f+1} - z^*\| &\leq L_3L_2L_1[(1-\alpha_f-\beta_f)L_1\|p_f - p^*\| \\
&\quad +\alpha_f L_1L_3\|z_f - z^*\| + \beta_f L_3L_2L_1\|q_f - q^*\|].
\end{aligned}$$

If we add above inequalities, we obtain the following inequality

$$\begin{aligned} (2.26) \quad & \|p_{f+1}-p^*\|+\|q_{f+1}-q^*\|+\|z_{f+1}-z^*\| \\ \leq\ & L_3L_2L_1[(1-\alpha_f-\beta_f)L_3\|z_f-z^*\|+\alpha_f L_3L_2\|q_f-q^*\| \\ & +\beta_f L_3L_2L_1\|p_f-p^*\|+(1-\alpha_f-\beta_f)L_1\|p_f-p^*\|+\alpha_f L_1L_3\|z_f-z^*\| \\ & +\beta_f L_3L_2L_1\|q_f-q^*\|+(1-\alpha_f-\beta_f)L_1\|p_f-p^*\|+\alpha_f L_1L_3\|z_f-z^*\| \\ & +\beta_f L_3L_2L_1\|q_f-q^*\|] \\ \leq\ & \mu\|p_f-p^*\|+\mu\|q_f-q^*\|+\mu\|z_f-z^*\| \\ =\ & \mu(\|p_f-p^*\|+\|q_f-q^*\|+\|z_f-z^*\|). \end{aligned}$$

Now, let be define the norm $\|.\|_1$ on $B\times B\times B$ by $\|(p,q,z)\|_1=\|p\|+\|q\|+\|z\|$ for all $(p,q,z)\in B\times B\times B$. We know that $(B\times B\times B,\|.\|_1)$ is a Banach space. From (2.26), we have

$$\|(p_{f+1},q_{f+1},z_{f+1})-(p^*,q^*,z^*)\|_1\leq\mu\|(p_f,q_f,z_f)-(p^*,q^*,z^*)\|_1$$

noticing that $\mu\in(0,1)$, it fallows that $\lim_{f\to\infty}\|(p_f,q_f,z_f)-(p^*,q^*,z^*)\|_1=0$. Thus, we obtain $\lim_{f\to\infty}\|p_f-p^*\|=\lim_{f\to\infty}\|q_f-q^*\|=\lim_{f\to\infty}\|z_f-z^*\|=0$. Therefore $\{p_f\},\{q_f\}$ and $\{z_f\}$ converge to p^*, q^* and z^*, respectively. □

Theorem 2.9. *Let K_1, K_2 and K_3 be three nonempty closed convex subsets of a Banach space B. Let $A_1:K_1\to K_2$, $A_2:K_2\to K_3$ and $A_3:K_3\to K_1$ be three Lipschitz mappings with constants $L_1\leq 1$, $L_2\leq 1$ and $L_3\leq 1$ such that $L_1+L_2+L_3<1$. Then the sequence $\{(p_f,q_f,z_f)\}$ in $K_1\times K_2\times K_3$ generated by a iteration method (2.25) converges strongly to a unique point $(p^*,q^*,z^*)\in K_1\times K_2\times K_3$ such that p^*, q^* and z^* are altering points of mappings A_1, A_2 and A_3.*

Proof. From the proof of Theorem 2.7, we know that

$$\begin{aligned} \|p_{f+1}-p^*\| \leq\ & L_3L_2L_1[(1-\alpha_f-\beta_f)L_3\|z_f-z^*\|+\alpha_f L_3L_2\|q_f-q^*\| \\ & +\beta_f L_3L_2L_1\|p_f-p^*\|] \end{aligned}$$

which implies that

$$\|p_{f+1}-p^*\|\leq L_3\left[\|z_f-z^*\|+\|q_f-q^*\|+\|p_f-p^*\|\right]. \tag{2.27}$$

If we use again the proof of Theorem 2.7, we write

$$\|q_{f+1}-q^*\|\leq L_2\left[\|z_f-z^*\|+\|q_f-q^*\|+\|p_f-p^*\|\right] \tag{2.28}$$

and

$$\|z_{f+1}-z^*\|\leq L_1\left[\|z_f-z^*\|+\|q_f-q^*\|+\|p_f-p^*\|\right]. \tag{2.29}$$

From (2.27), (2.28) and (2.29), we get

$$\|p_{f+1}-p^*\|+\|q_{f+1}-q^*\|+\|z_{f+1}-z^*\|\leq(L_1+L_2+L_3)\left[\|z_f-z^*\|+\|q_f-q^*\|+\|p_f-p^*\|\right].$$

Let be define the norm $\|.\|_1$ on $B\times B\times B$ by $\|(p,q,z)\|_1=\|p\|+\|q\|+\|z\|$ for all $(p,q,z)\in B\times B\times B$. We know that $(B\times B\times B,\|.\|_1)$ is a Banach space. Then

$$\|(p_{f+1},q_{f+1},z_{f+1})-(p^*,q^*,z^*)\|_1\leq(L_1+L_2+L_3)\,\|(p_f,q_f,z_f)-(p^*,q^*,z^*)\|_1$$

in which $L_1+L_2+L_3<1$. From induction principle, we have

$$\|(p_{f+1},q_{f+1},z_{f+1})-(p^*,q^*,z^*)\|_1\leq(L_1+L_2+L_3)^f\,\|(p_1,q_1,z_1)-(p^*,q^*,z^*)\|_1. \tag{2.30}$$

If we take the limit on both sides of (2.30), we get

$$\lim_{f\to\infty}\|(p_{f+1},q_{f+1},z_{f+1})-(p^*,q^*,z^*)\|_1=0.$$

Then the sequence $\{(p_f,q_f,z_f)\}$ converges to (p^*,q^*,z^*). □

Theorem 2.10. *Let K_1, K_2 and K_3 be three nonempty closed convex subsets of a Banach space B. Let $\{S_f\}$ be a sequence of mappings from K_1 into K_2 such that $\{S_f\}$ is nearly nonexpansive with sequence $\{a_f\}$ and let A_1 be a nonexpansive mapping from K_1 into K_2 defined by $A_1 z = \lim_{f\to\infty} S_f z$ for all $z \in K_1$. Let $\{R_f\}$ be a sequence of mappings from K_2 into K_3 such that $\{R_f\}$ is nearly nonexpansive with sequence $\{b_f\}$ and let A_2 be a nonexpansive mapping from K_2 into K_3 defined by $A_2 w = \lim_{f\to\infty} R_f w$ for all $w \in K_2$. $A_3 : K_3 \to K_1$ be a contraction with Lipschitz constant L. Then we have the following:*

(a) There exists a unique element $(p^*, q^*, z^*) \in K_1 \times K_2 \times K_3$ such that p^*, q^* and z^* are altering points of mappings A_1 , A_2 and A_3.

(b) For arbitrary $(p^*, q^*, z^*) \in K_1 \times K_2 \times K_3$, a sequence $\{(p_f, q_f, z_f)\}$ in $K_1 \times K_2 \times K_3$ generated by

$$
\begin{aligned}
p_{f+1} &= A_3 z_f \\
z_f &= R_f q_f \\
q_f &= S_f p_f
\end{aligned}
\tag{2.31}
$$

converges strongly to (p^*, q^*, z^*).

Proof. (a) From Lemma 2.5, we have that $A_1 : K_1 \to K_2$ is nonexpansive. From the hypothesis, we know that $A_2 : K_2 \to K_1$ is a contraction. Hence, from Theorem 2.4 (a), there exists a unique point $(p, q) \in K_1 \times K_2$ such that p^* and q^* are altering points of mappings A_1 and A_2. □

Theorem 2.11. *Proof.* (b) Using 2.31, we obtain

$$
\begin{aligned}
\|q_f - q^*\| &= \|S_f(p_f) - A_1(p^*)\| \\
&\leq \|S_f(p_f) - S_f(p^*)\| + \|S_f(p^*) - A_1(p^*)\| \\
&\leq \|p_f - p^*\| + \|S_f(p^*) - A_1(p^*)\| + a_f,
\end{aligned}
\tag{2.32}
$$

$$
\begin{aligned}
\|z_f - z^*\| &= \|R_f(q_f) - A_2 q^*\| \\
&\leq \|R_f(q_f) - R_f(q^*)\| + \|R_f(q^*) - A_2 q^*\| \\
&\leq \|q_f - q^*\| + \|R_f(q^*) - A_2 q^*\| + b_f
\end{aligned}
\tag{2.33}
$$

and

$$
\begin{aligned}
\|p_{f+1} - p^*\| &= \|A_3 z_f - A_3 z^*\| \\
&\leq L\|z_f - z^*\|.
\end{aligned}
$$

If we combine the above inequalities, we have

$$
\begin{aligned}
\|p_{f+1} - p^*\| &\leq L[\|q_f - q^*\| + \|R_f(q^*) - A_2 q^*\| + b_f] \\
&\leq L[\|p_f - p^*\| + \|S_f(p^*) - A_1(p^*)\| + a_f \\
&\quad + \|R_f(q^*) - A_2\,(q^*)\,\| + b_f]
\end{aligned}
$$

Since $||S_f(p^*) - A_1(p^*)|| + a_f \to 0$ and $\|R_f(q^*) - A_2\,(q^*)\,\| + b_f \to 0$ as $f \to \infty$, it follows from Lemma 2.6 that $\lim_{f\to\infty} ||p_f - p^*|| = 0$. If we take limit in the inequality (2.32) and we use these limits $||S_f(p^*) - A_1(p^*)|| + a_f \to 0$ and $|p_f - p^*|| \to 0$ as $f \to \infty$, we obtain $\lim_{f\to\infty} q_f = q^*$. Similarly, we also obtain that $\lim_{f\to\infty} z_f = z^*$ from (2.33). Therefore the sequence $\{(p_f, q_f, z_f)\}$ converges strongly to (p^*, q^*, z^*). □

3. Application

In this section, we will present an application for solution of nonlinear variational inequalities under suitable conditions by rewriting iteration process (2.1) with the help of certain mappings as under:

Let K_1 and K_2 be nonempty closed convex subsets of B and let $F_{K_2} : B \to K_2$ and $F_{K_1} : B \to K_1$ be nonlinear operators and let δ and θ be positive real numbers. Define $Q = I - \delta A_1$ and $W = I - \theta A_2$. Then the following are equivalent:

Let $A_1 : K_1 \to B$ and $A_2 : K_2 \to B$ be nonlinear operators and let $\delta, \theta \in (0, \infty)$. We consider the following altering problem [5]. find element $(p^*, q^*) \in K_1 \times K_2$ such that

$$\begin{cases} F_{K_2}(I - \delta A_1)(p^*) = q^* \\ F_{K_1}(I - \theta A_2)(q^*) = p^* \end{cases} \tag{3.1}$$

The operators F_{K_2} and F_{K_1} play a key role in the mathematical modeling (3.1). If $F_{K_2} = P_{K_2}$ and $F_{K_1} = P_{K_1}$ then, from Lemma 1.3, the system (3.1) is equivalent to the following general system of nonlinear variational inequalities in p:

Find $(p^*, q^*) \in K_1 \times K_2$ such that

$$\begin{cases} P_{K_2}(I - \delta A_1)(p^*) = q^*, \\ P_{K_1}(I - \theta A_2)(q^*) = p^*, \end{cases} \quad \text{i.e.,} \tag{3.2}$$

$$\begin{cases} \langle \delta A_1(p^*) + q^* - p^*, p - q^* \rangle \geq 0 & \text{for all } p \in K_2 \\ \langle \theta A_2(q^*) + p^* - q^*, p - p^* \rangle \geq 0 & \text{for all } p \in K_1. \end{cases}$$

In view of Theorem 2.1, the solution of systems (3.2) can be computed by the parallel iteration process (2.1) under suitable conditions. In this direction, we deal with the computation of nonlinear variational inequalities (3.2) using the parallel iteration method (2.1).

Theorem 3.1. *Let K_1 and K_2 be nonempty closed convex subsets of p. Let $A_1 : K_1 \to B$ be δ_{A_1}-inverse strongle monotone and let and $A_2 : K_2 \to B$ be δ_{A_2}-inverse strongle monotone operators. Suppose that $\delta \in \left(0, \frac{2}{\delta_{A_1}}\right)$ and $\theta \in \left(0, \frac{2}{\delta_{A_2}}\right)$ such that Alt($P_{K_2}(I - \delta A_1), P_{K_1}(I - \theta A_2)$)$\neq \emptyset$. For arbitrary $(p_1, q_1) \in K_1 \times K_2$, let $\{(p_f, q_f)\}$ be a sequence in $K_1 \times K_2$ defined by the parallel iteration process (2.1):*

$$\begin{cases} p_{f+1} = P_{K_1}(I - \theta A_2) z_f \\ z_f = P_{K_2}(I - \delta A_1)[P_{K_1}(I - \theta A_2) w_f] \\ w_f = P_{K_2}(I - \delta A_1) \begin{bmatrix} (1 - \alpha_f) P_{K_1}(I - \theta A_2) q_f \\ + \alpha_f P_{K_1}(I - \theta A_2)[P_{K_2}(I - \delta A_1) p_f] \end{bmatrix} \end{cases}$$

$$\begin{cases} q_{f+1} = P_{K_2}(I - \delta A_1) u_f \\ z_f = P_{K_1}(I - \theta A_2)[P_{K_2}(I - \delta A_1) v_f] \\ v_f = P_{K_1}(I - \theta A_2) \begin{bmatrix} (1 - \alpha_f) P_{K_2}(I - \delta A_1) p_f \\ + \alpha_f P_{K_2}(I - \delta A_1)[P_{K_1}(I - \theta A_2) q_f] \end{bmatrix} \end{cases}$$

where $\{\alpha_f\}$ is a sequence in $(0, 1)$ satisfying the condition $\sum_{f=1}^{\infty} \alpha_f (1 - \alpha_f) = \infty$. Then $\{(p_f, q_f)\}$ converges weakly to an element $(p^*, q^*) \in K_1 \times K_2$ which solves the nonlinear variational inequalities (3.2).

Proof. We know that $I - \delta A_1$ and $I - \theta A_2$ are nonexpansive for $\delta \in \left(0, \frac{2}{\delta_{A_1}}\right)$ and $\theta \in \left(0, \frac{2}{\delta_{A_2}}\right)$. Therefore, the proof of this theorem follows from Theorem 2.1. □

4. Conclusions

In conclusion, we introduced new iteration methods for altering points and generalized altering points of Lipschitzian mappings. We proved the convergence of this new iteration methods under suitable assumptions. We also showed that this iteration method is data dependent. Finally, we gave an application for solution of nonlinear variational inequalities under suitable conditions.

References

[1] D.R. Sahu, S.M. Kang and A. Kumar, *Convergence analysis of parallel S-iteration process for system of generalized variational inequalities*, Journal of Function Spaces **2017(1)** (2017), 5847096.

[2] D.R. Sahu, *Altering points and applications*, Nonlinear Studies **21(2)** (2014), 349-365.

[3] W. Sintunavarat and A. Pitea, *On a new iteration scheme for numerical reckoning fixed points of Berinde mappings with convergence analysis*, J. Nonlinear Sci. Appl. **9(5)** (2016), 2553-2562.
[4] S.M. Soltuz, T. Grosan and A. Pitea, *Data dependence for Ishikawa iteration when dealing with contractive like operators*, Fixed Point Theory and Applications **2008** (2008), 1-7.
[5] X. Zhao, D.R. Sahu and C.F. Wen, *Iterative methods for system of variational inclusions involving accretive operators and applications*, Fixed Point Theory **19(2)** (2018), 801-822.

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