ON $\mathcal{Q}$-GENERALIZED FUNCTIONS OF POSITIVE REAL PART

A. A. YUSUF AND M. DARUS

Abstract. In this research work, the class of functions having a positive real part define by a generalized operator of Salagean differential and integral operators involving q-derivative is studied with the following properties includes inclusion, conditions for univalency, coefficient and Fekete-Szego inequalities. Our results are accompanied by their corollaries and implications, which also extend earlier results.

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1. Introduction

The concept of fractional calculus is a field of study that introduced the knowledge of special functions in th area of Mathematical analysis, originated in 1832 by Liouville and this has been studied extensively by researchers, (see [16], [17], [21], [24]).

The use of operator in the area geometric function theory was originated through the introduction of integral operator by Alexander in 1915, this has been utilized by several authors in {[1], [2], [3], [4], [10], [11], [12], [28], [30]} to introduced different classes of analytic and univalent functions of the form $f(0) = 0$ and $f'(0) = 1$ in the unit disk $\{\mathbf{\Omega} = |\eta| < 1\}$ such as $\Re e \eta f^{'}(\eta)/f(\eta) > \zeta$ referred to as a starlike function of order ζ, $Ref'(\eta) > \zeta$ which is the class of bounded turning of order ζ and $\Re e(\eta f')'/f'(\eta) > \zeta$ which is the class of convex functions of order ζ, such that $\eta \in \mathbf{\Omega}$ (see [23], [25], [26]).

Several classes of these analytic and univalent function has been proved to satisfy the condition of the functions having a positive real parts $\mathbf{p}(\eta)$ define as $\mathbf{p}(\eta) = 1 + s_1\eta + s_2\eta^2 + \cdots$ in the unit disk $\mathbf{\Omega}$, such that $\Re e\mathbf{p}(\eta) > 0$ and by $\mathbf{P}_\zeta$ if $\Re e\mathbf{p}(\eta) > \zeta$ for some real number $0 \leq \zeta < 1$, extensively studied in [9].

The differential operator $\mathcal{D}^n$ such that $n \in \mathbb{N} \cup \{0\}$, which is the Salagean differential operator defined as $\mathcal{D}^n f(\eta) = \mathcal{D}(\mathcal{D}^{n-1} f(\eta)) = \eta[\mathcal{D}^{n-1} f(\eta)]$ with $\mathcal{D}^0 f(\eta) = f(\eta)$, introduced in [27] and the integral operator of one parameter denoted as $\mathcal{I}^\sigma$ define as $\mathcal{I}^\sigma f(\eta) = \frac{2^\sigma}{\eta\Gamma\sigma} \int_0^\eta (log\frac{\eta}{t})^{\sigma-1} f(t)dt$ introduced by Jung-Kim-Srivastava in [7, 8] and this has been used severally by different authors to generalize classes of analytic functions.

Let $f \in A$ be the class of analytic functions of the form

$$f(\eta) = \eta + \sum_{k=2}^{\infty} e_k \eta^k.$$

Defining the Salagean operator on $f \in A$, we have

$$\mathcal{D}^n f(\eta) = \eta + \sum_{k=2}^{\infty} e_k \eta^k.$$

Defining one-parameter Jung-Kim-Srivastava integral operator on $f \in A$, we have

$$\mathcal{I}^\sigma f(\eta) = \frac{2^\sigma}{\eta\Gamma(\sigma)} \int_0^\eta (\log \frac{\eta}{t})^{\sigma-1} f(t)dt = \eta + \sum_{k=2}^{\infty} \left(\frac{2}{1+k}\right)^\sigma \eta^k.$$

Taking the inverse of the integral operator $\mathcal{I}_\sigma$, then

$$\mathcal{I}_\sigma f(\eta) = \frac{2^{-\sigma}}{\eta\Gamma(-\sigma)} \int_0^\eta \left(\log \frac{\eta}{t}\right)^{-(\sigma-1)} f(t)dt = \eta + \sum_{k=2}^{\infty} \left(\frac{k+1}{2}\right)^\sigma \eta^k.$$

We have

$$\mathcal{D}^n(\mathcal{I}_\sigma f(\eta)) = \eta + \sum_{k=2}^{\infty} \left(\frac{k+1}{2}\right)^\sigma e_k \eta^k = \mathcal{I}_\sigma(D^n f(\eta)^\alpha) = \mathcal{D}^n_\sigma f(\eta).$$

Varying the parameters, we have $\mathcal{D}^0_\sigma f(\eta) = I_\sigma f(\eta)$, $\mathcal{D}^n_0 f(\eta) = D^n f(\eta)$, so that $\mathcal{D}^0_0 f(\eta) = f(\eta)$.

The concepts of q-calculus was introduced by Jackson [17], with the authors (see [14], [15] [19], [20], [29]) developed on this concept as a tool of defining different classes of analytic functions in the area of Geometric function Theory (GFT).

Definition 1.1. The *q-differentiation* of function $f(0) = 0$ and $f'(0) = 1$, for $q \in (0, 1)$ is defined by

$$\mathbf{D}_q f(0) = f'(0), \mathbf{D}_q f(\eta) = \frac{f(\eta) - f(q\eta)}{\eta(1-q)} (\eta \neq 0), \mathbf{D}^2_q f(\eta) = \mathbf{D}_q(\mathbf{D}_q f(\eta)), \tag{1.1}$$

where

$$\mathbf{D}_q f(\eta) = 1 + \sum_{k=2}^{\infty} [k]_q a_k \eta^{k-1}, z\mathbf{D}^2_q f(\eta) = \sum_{k=2}^{\infty} [k-1]_q [k]_q a_k \eta^{k-1} \tag{1.2}$$

such that $[k]_q = \frac{1-q^k}{1-q}$ and $q \to 1$, $[k]_q = k$.

Recent works on q-calculus has shown different approach on the use of differential and integral operator and its applications in the area of geometry functions theory.

Adoptions of the q-differential operator $\mathbf{D}_q$ on the generalized operator $\mathcal{D}^n_\sigma$, we have $\mathbf{D}_q(\mathcal{D}^n_\sigma f(\eta)) = \mathbf{D}^{1,n}_{q,\sigma}$, as a new operator to introduce a new class of analytic function denoted as $\mathcal{J}^n_{q,\sigma}(\theta, \zeta)$, define as follows:

Definition 1.2. The function $f \in A$, is said to be in the class of *q-generalized functions* denoted as $\mathcal{J}^n_{q,\sigma}(\theta, \zeta)$, if it satisfies the geometric condition

$$\Re e \left\{ \mathbf{D}^{1,n}_{q,\sigma} f(\eta) + \frac{1+e^{i\theta}}{2} \eta \mathbf{D}^{2,n}_{q,\sigma} f(\eta) \right\} > \zeta, \tag{1.3}$$

where $\zeta \in [0, 1)$, $\theta \in (-\pi, \pi]$ and $\{n, \sigma \in 0 \cup \mathbb{N}\}$.

With the same parameters, the following classes can be deduced from (1.3).

i. For $n = 0$, we have

$$\Re e\left\{\mathbf{D}_q^1(\mathcal{I}_\sigma f(\eta)) + \frac{1+e^{i\theta}}{2}\eta\mathbf{D}_q^2(\mathcal{I}_\sigma f(\eta))\right\} > \zeta.$$

ii. For $\sigma = 0$, we have

$$\Re e\left\{\mathbf{D}_q^1(\mathcal{D}^n f(\eta)) + \frac{1+e^{i\theta}}{2}\eta\mathbf{D}_q^2(\mathcal{D}^n f(\eta))\right\} > \zeta.$$

iii. For $n = 0$, $\sigma = 0$, we have

$$\Re e\left\{\mathbf{D}_q^1 f(\eta) + \frac{1+e^{i\theta}}{2}\eta\mathbf{D}_q^2 f(\eta)\right\} > \zeta.$$

iv For $n = 0$, $\sigma = 0$ and $\theta = \pi$, we have

$$\Re e\left\{\mathbf{D}_q^1 f(\eta)\right\} > \zeta,$$

which is the class of q-bounded turning denoted as $\mathcal{R}_q(\zeta)$ [16].

v. For $n = 0$, $\sigma = 0$, $q \to 1$, we have

$$\Re e\left\{f'(\eta) + \frac{1+e^{i\theta}}{2}\eta f''(\eta)\right\} > \zeta.$$

vi. For $n = 0$, $\sigma = 0$, $q \to 1$ and $\theta = 0$, we have

$$\Re e\left\{f'(\eta) + \eta f''(\eta)\right\} > \zeta.$$

vii. For $n = 0$, $\sigma = 0$, $q \to 1$ and $\theta = \pi$, we have

$$\Re e f'(\eta) > \zeta,$$

which is the class of bounded turning of order ζ denoted as $\mathcal{R}(\zeta)$.

2. Preliminary Lemmas

Lemma 2.1. [8] *Let $p(\eta)$ be holomorphic in Ω with $p(0) = 1$. Suppose that*

$$Re\left(1 + \frac{\eta\mathbf{D}_q^1 p(\eta)}{p(\eta)}\right) > \frac{3\zeta - 1}{2\zeta}.$$

Then

$$Rep(\eta) > 2^{1-\frac{1}{\zeta}}, \frac{1}{2} \le \zeta < 1, \eta \in \Omega. \tag{2.1}$$

and the constant $2^{1-\frac{1}{\zeta}}$ is the best possible.

Lemma 2.2. [9] *Let $p \in \mathcal{P}$. where $p(\eta) = 1 + s_1\eta + s_2\eta^2 + \cdots$, then*

$$|s_k| \le 2, k = 1, 2, 3, \cdots. \tag{2.2}$$

Lemma 2.3. [6]

Let $p \in \mathcal{P}$, then for any real or complex number μ, we have sharp inequalities

$$\left|s_2 - \mu\frac{s_1^2}{2}\right| \le 2\max\{1, |1-\mu|\}. \tag{2.3}$$

Lemma 2.4. [6] *Let $u = u_1 + u_2 i$, $v = v_1 + v_2 i$ and $\Phi(u, v)$ a complex valued function satisfying*
(i) $\Phi(u, v)$ is continuous in a domain Ω of $\mathbb{C}^2$.
(ii) $(1, 0) \in \Omega$ and $Re\Phi(1, 0) > 0$.
(iii) $Re\Phi(\zeta + (1-\zeta)u_2 i, v_1) \le \zeta$ when $(\zeta + (1-\zeta)u_2 i, v_1) \in \mathbf{\Omega}$ and $2v_1 \le -(1-\zeta)(1+u_2^2))$ for $0 \le \zeta < 1$.
If $p \in P$ such that $(p(\eta), \eta p^{'}(\eta)) \in \Omega$ and $Re(p(\eta), \eta p^{'}(\eta)) > \zeta$ for $\eta \in \mathbf{\Omega}$. Then $Rep(\eta) > \zeta$ in $\mathbf{\Omega}$.

3. Main Results

Theorem 3.1. *The function* $f \in \mathcal{J}^n_{q,\sigma}(\theta,\zeta)$, *if*

$$\mathcal{J}^n_{q,\sigma}(\theta,\zeta) \subset \mathcal{R}_q(\zeta),$$

for $\zeta \in [0,1)$, $\theta \in (-\pi,\pi]$, $\{n,\sigma \in 0 \cup \mathbb{N}\}$, $\eta \in \mathbf{\Omega}$.

Proof. Let

$$p(\eta) = \mathbf{D}^{1,n}_{q,\sigma} f(\eta),$$

so that

$$\mathbf{D}^{\mathbf{1}}_{\mathbf{q}} p(\eta) = \mathbf{D}^{2,n}_{q,\sigma} f(\eta),$$

with $r = (1+e^{i\theta})/2$.
By simple calculation

$$\Re e(p(\eta) + \mathbf{r}\eta\mathbf{D}^{\mathbf{2}}_{\mathbf{q}}\mathbf{p}(\eta) > \zeta. \tag{3.1}$$

From Lemma (2.4) and $p(\eta)$,

$$\Phi(u,v) = u + \mathbf{r}\mathbf{v}$$

on the domain $\Omega \subset \mathbb{C}^2$. Then $\Phi(u,v)$ satisfies the condition (1) of the Lemma (2.4).
Also for $(0,1) \in \Omega$, $\Phi(1,0) = 1$, implies that $\Re e(\Phi(1,0)) > 0$ and

$$\Phi(\zeta + (1-\zeta)u_2 i, v_1 = \zeta + \frac{1+cos\zeta}{2}v_1 + ((1-\zeta)u_2 + \frac{\sin\zeta}{2}v_1)i.$$

Thus

$$\Re e(\Phi(\zeta + (1-\zeta)u_2 i, v_1)) = \zeta + \frac{1+\cos\zeta}{2}v_1 \leq \zeta.$$

For

$$v_1 \leq -\frac{1}{2}(1-\zeta)(1+u_2^2).$$

Since $\Phi(u,v)$ satisfies all the conditions of Lemma (2.4).
Then

$$\Re e(p(\eta)) = \Re e(\mathbf{D}^{1,n}_{q,\sigma} f(\eta)) > \zeta,$$

and the proof is completed. □

Corollary 3.2. *The class* $\mathcal{J}^n_{q,\sigma}(\theta,\zeta)$ *consists of univalent functions.*

Corollary 3.3. *For* $q \to 1$, *the class* $\mathcal{J}^n_{q,\sigma}(\theta,\zeta) \subset \mathcal{R}(\zeta)$.

Theorem 3.4. *For* $\zeta \in [1/2,1)$, $\theta \in (-\pi,\pi]$, $\{n,\sigma \in 0 \cup \{\mathbb{N}\}\}$. *If* $f \in \mathcal{J}^n_{q,\sigma}(\theta,\zeta)$ *satisfies the condition,*

$$\Re e\frac{\mathbf{D}^{1,n}_{q,\sigma}(\mathbf{D}^{1,n}_{q,\sigma} f(\eta) + \mathbf{r}\eta\mathbf{D}^{\mathbf{2},\mathbf{n}}_{\mathbf{q},\sigma}\mathbf{f}(\eta))}{\mathbf{D}^{1,n}_{q,\sigma} f(\eta) + \mathbf{r}\eta\mathbf{D}^{\mathbf{2},\mathbf{n}}_{\mathbf{q},\sigma}\mathbf{f}(\eta)} > \frac{\zeta-1}{2\zeta}, \tag{3.2}$$

then

$$\mathbf{D}^{1,n}_{q,\sigma} f(\eta) + \mathbf{r}\eta\mathbf{D}^{\mathbf{2},\mathbf{n}}_{\mathbf{q},\sigma}\mathbf{f}(\eta) > \mathbf{2}^{\frac{\zeta-\mathbf{1}}{\zeta}}.$$

Proof. By logarithmic differentiation of $p(\eta) = \mathbf{D}^{1,n}_{q,\sigma} f(\eta) + \mathbf{r}\eta\mathbf{D}^{\mathbf{2},\mathbf{n}}_{\mathbf{q},\sigma}\mathbf{f}(\eta)$. Then

$$\frac{\eta\mathbf{D}_q p(\eta)}{p(\eta)} + 1 = \frac{\mathbf{D}^{1,n}_{q,\sigma}(\mathbf{D}^{1,n}_{q,\sigma} f(\eta) + \mathbf{r}\eta\mathbf{D}^{\mathbf{2},\mathbf{n}}_{\mathbf{q},\sigma}\mathbf{f}(\eta))}{\mathbf{D}^{1,n}_{q,\sigma} f(\eta) + \mathbf{r}\eta\mathbf{D}^{\mathbf{2},\mathbf{n}}_{\mathbf{q},\sigma}\mathbf{f}(\eta)} + 1.$$

From Lemma (2.1),

$$\Re e\left(\frac{\eta\mathbf{D}^1_q p(\eta)}{p(\eta)} + 1\right) = \Re e\left(\frac{\mathbf{D}^{1,n}_{q,\sigma}(\mathbf{D}^{1,n}_{q,\sigma} f(\eta) + \mathbf{r}\eta\mathbf{D}^{\mathbf{2},\mathbf{n}}_{\mathbf{q},\sigma}\mathbf{f}(\eta))}{\mathbf{D}^{1,n}_{q,\sigma} f(\eta) + \mathbf{r}\eta\mathbf{D}^{\mathbf{2},\mathbf{n}}_{\mathbf{q},\sigma}\mathbf{f}(\eta)} + 1\right) > \frac{3\zeta-1}{2\zeta}.$$

So that

$$\Re e\left(\frac{\mathbf{D}_{q,\sigma}^{1,n}(\mathbf{D}_{q,\sigma}^{1,n}f(\eta)+\mathbf{r}\eta\mathbf{D}_{\mathbf{q},\sigma}^{\mathbf{2,n}}\mathbf{f}(\eta))}{\mathbf{D}_{q,\sigma}^{1,n}f(\eta)+\mathbf{r}\eta\mathbf{D}_{\mathbf{q},\sigma}^{\mathbf{2,n}}\mathbf{f}(\eta)}\right)>\frac{\zeta-1}{2\zeta},$$

and the proof is completed. □

Corollary 3.5. *If the inequality (3.2) is satisfied by the function $f(\eta)$, then $f(\eta)\in\mathcal{J}_{q,\sigma}^{n}(\theta,2^{(1-\zeta)/\zeta}$.*

Corollary 3.6. *If $n=0$,*

$$\Re e\left\{\frac{\mathbf{D}_{q,\sigma}^{1}(\mathbf{D}_{q,\sigma}f(\eta)+\mathbf{r}\eta\mathbf{D}_{\mathbf{q},\sigma}^{\mathbf{2}}\mathbf{f}(\eta))}{\mathbf{D}_{q,\sigma}^{1}f(\eta)+\mathbf{r}\eta\mathbf{D}_{\mathbf{q},\sigma}^{\mathbf{2}}\mathbf{f}(\eta)}\right\}>\frac{\zeta-1}{2\zeta}.$$

Then

$$\mathbf{D}_{q,\sigma}^{1}f(\eta)+\mathbf{r}\eta\mathbf{D}_{\mathbf{q},\sigma}^{\mathbf{2}}\mathbf{f}(\eta)>\mathbf{2}^{\frac{\zeta-\mathbf{1}}{\zeta}}.$$

Corollary 3.7. *If $\sigma=0$,*

$$\Re e\left\{\frac{\mathbf{D}_{q}^{1,n}(\mathbf{D}_{q}^{n}f(\eta)+\mathbf{r}\eta\mathbf{D}_{\mathbf{q}}^{\mathbf{2,n}}\mathbf{f}(\eta))}{\mathbf{D}_{q}^{1,n}f(\eta)+\mathbf{r}\eta\mathbf{D}_{\mathbf{q}}^{\mathbf{2,n}}\mathbf{f}(\eta)}\right\}>\frac{\zeta-1}{2\zeta}.$$

Then

$$\mathbf{D}_{q}^{1,n}f(\eta)+\mathbf{r}\eta\mathbf{D}_{\mathbf{q}}^{\mathbf{2,n}}\mathbf{f}(\eta)>\mathbf{2}^{\frac{\zeta-\mathbf{1}}{\zeta}}.$$

Corollary 3.8. *If $n=0$, $\sigma=0$,*

$$\Re e\left\{\frac{\mathbf{D}_{q}^{1}(\mathbf{D}_{q}^{1}f(\eta)+\mathbf{r}\eta\mathbf{D}_{\mathbf{q}}^{\mathbf{2}}\mathbf{f}(\eta))}{\mathbf{D}_{q}^{1}f(\eta)+\mathbf{r}\eta\mathbf{D}_{\mathbf{q}}^{\mathbf{2}}\mathbf{f}(\eta)}\right\}>\frac{\zeta-1}{2\zeta}.$$

Then

$$\mathbf{D}_{q}^{1}f(\eta)+\mathbf{r}\eta\mathbf{D}_{\mathbf{q}}^{\mathbf{2}}\mathbf{f}(\eta)>\mathbf{2}^{\frac{\zeta-\mathbf{1}}{\zeta}}.$$

Corollary 3.9. *If $\theta=0$, $\zeta=1/2$,*

$$\Re e\left\{\frac{\mathbf{D}_{q,\sigma}^{1,n}(\mathbf{D}_{q,\sigma}^{1,n}f(\eta)+\eta\mathbf{D}_{q,\sigma}^{2,n}f(\eta))}{\mathbf{D}_{q,\sigma}^{1,n}f(\eta)+\eta\mathbf{D}_{q,\sigma}^{2,n}f(\eta)}\right\}>-\frac{1}{2}.$$

Then

$$\mathbf{D}_{q,\sigma}^{1,n}f(\eta)+\eta\mathbf{D}_{q,\sigma}^{2,n}f(\eta)>\frac{1}{2}.$$

Corollary 3.10. *If $\theta=\pi$, $\zeta=1/2$,*

$$\Re e\left\{\frac{\mathbf{D}_{q,\sigma}^{1,n}(\mathbf{D}_{q,\sigma}^{1,n}f(\eta)}{\mathbf{D}_{q,\sigma}^{n}f(\eta)}\right\}>-\frac{1}{2}.$$

Then

$$\Re e(\mathbf{D}_{q,\sigma}^{n}f(\eta))>\frac{1}{2}.$$

Corollary 3.11. *If $n=0$, $\sigma=0$, $\theta=0$, $\zeta=1/2$,*

$$\Re e\left\{\frac{\mathbf{D}_{q}^{1}(\mathbf{D}_{q}^{1}f(\eta)+\eta\mathbf{D}_{q}^{2}f(\eta))}{\mathbf{D}_{q}^{1}f(\eta)+\eta\mathbf{D}_{q}^{2,n}f(\eta)}\right\}>-\frac{1}{2}.$$

Then

$$\Re e(\mathbf{D}_{q}^{1}f(\eta)+\eta\mathbf{D}_{q}^{2}f(\eta))>\frac{1}{2}.$$

Corollary 3.12. *If* $n = 0,\ \sigma = 0,\ \theta = \pi,\ \zeta = 1/2,$

$$\Re e\left\{\frac{\mathbf{D}_q^1(\mathbf{D}_q^1 f(\eta)}{\mathbf{D}_q^1 f(\eta)}\right\} > -\frac{1}{2}.$$

Then

$$\Re e(\mathbf{D}_q^1 f(\eta)) > \frac{1}{2}.$$

Corollary 3.13. *If* $n = 0,\ \sigma = 0,\ \theta = 0,\ \zeta = 1/2,\ q \to 1,$

$$\Re e\left\{\frac{2\eta f'' f(\eta) + \eta^2 f'''(\eta))}{f'(\eta) + \eta f''(\eta)}\right\} > -\frac{1}{2}.$$

Then

$$\Re e(f'(\eta) + \eta f''(\eta)) > \frac{1}{2}.$$

Corollary 3.14. *If* $n = 0,\ \sigma = 0,\ \theta = \pi,\ \zeta = 1/2,\ q \to 1,$

$$\Re e\left\{\frac{\eta f'' f(\eta)}{f'(\eta)}\right\} > -\frac{1}{2}.$$

Then

$$\Re e(f'(\eta)) > \frac{1}{2}.$$

Theorem 3.15. *If* $f \in \mathcal{J}_{q,\sigma}^n(\theta, \zeta)$. *Then*

$$|e_2| \le \frac{4(1-\beta)}{|\frac{3}{2}|^\sigma |2|^n [2]_q |2 + (1+e^{i\theta})[1]_q|}, \tag{3.3}$$

$$|e_3| \le \frac{4(1-\beta)}{|2|^\sigma 3^n [3]_q |2 + (1+e^{i\theta})[2]_q|}, \tag{3.4}$$

$$|e_4| \le \frac{4(1-\beta)}{|\frac{5}{2}|^\sigma 4^n [4]_q |2 + (1+e^{i\theta})[3]_q|}, \tag{3.5}$$

$$|e_5| \le \frac{4(1-\beta)}{|3|^\sigma 5^n [5]_q |2 + (1+e^{i\theta})[4]_q|}. \tag{3.6}$$

Proof. Since $f \in \mathcal{J}_{q,\sigma}^n(\theta, \zeta)$, then

$$\begin{aligned}\mathbf{D}_{q,\sigma}^{1,n} f(\eta) &= 1 + \sum_{k=2}^{\infty}\left(\frac{k+1}{2}\right)^\sigma k^n [\ k]_q e_k \eta^{k-1} \\ &= 1 + \left(\frac{3}{2}\right)^\sigma 2^n [\ 2]_q e_2 z + 2^\sigma 3^n [\ 3]_q e_3 \eta^2 + \left(\frac{5}{2}\right)^\sigma 4^n [\ 4]_q e_4 \eta^3 \\ &+ 3^\sigma 5^n [\ 5]_q e_5 z^4 + \left(\frac{7}{2}\right)^\sigma 6^n [\ 6]_q e_5 \eta^5 + \cdots\end{aligned} \tag{3.7}$$

$$\begin{aligned}z\mathbf{D}_{q,\sigma}^{2,n} f(\eta) &= \sum_{k=2}^{\infty}\left(\frac{k+1}{2}\right)^\sigma k^n [\ k]_q [\ k-1]_q e_k \eta^{k-1} \\ &= \left(\frac{3}{2}\right)^\sigma 2^n [\ 2]_q [\ 1]_q e_2 \eta + 2^\sigma 3^n [\ 3]_q [\ 2]_q e_3 \eta^2 \\ &+ \left(\frac{5}{2}\right)^\sigma 4^n [\ 4]_q [\ 3]_q e_4 \eta^3 + 3^\sigma 5^n [\ 5]_q [\ 4]_q e_5 z^4 + \left(\frac{7}{2}\right)^\sigma 6^n [\ 6]_q [\ 5]_q e_5 \eta^5 + \cdots\end{aligned} \tag{3.8}$$

$$p_\beta(\eta) = 1 + (1-\beta)s_1\eta + (1-\beta)s_2\eta^2 + (1-\beta)s_3\eta^3 + (1-\beta)s_4\eta^4 + (1-\beta)s_5\eta^5 + \cdots \tag{3.9}$$

$$\begin{aligned}\mathbf{D}^{1;n}_{q,\sigma}f(\eta) + \frac{1+e^{i\theta}}{2}z\mathbf{D}^{2,n}_{q,\sigma}f(\eta) = 1 &+ \left[\left(\frac{3}{2}\right)^\sigma 2^n[\ 2]_q + \frac{1+e^{i\theta}}{2}\left(\frac{3}{2}\right)^\sigma 2^n[\ 2]_q[\ 1]_q\right]e_2\eta \\ &+ \left[2^\sigma 3^n[\ 3]_q + \frac{1+e^{i\theta}}{2}2^\sigma 3^n[\ 3]_q[\ 2]_q\right]e_3\eta^2 \\ &+ \left[\left(\frac{5}{2}\right)^\sigma 4^n[\ 4]_q + \frac{1+e^{i\theta}}{2}\left(\frac{5}{2}\right)^\sigma 4^n[\ 4]_q[\ 3]_q\right]e_4\eta^3 \\ &+ \left[3^\sigma 5^n[\ 5]_q + \frac{1+e^{i\theta}}{2}3^\sigma 5^n[\ 5]_q[\ 4]_q\right]e_5\eta^4 + \cdots \end{aligned} \tag{3.10}$$

Equating equation (3.8) and (3.9), and by Lemma 2.2, we have the inequalities (3.2), (3.3), (3.4) and (3.5).

□

Corollary 3.16. *Varying the value of the parameters n, σ, θ, ζ, the coefficients inequalities $|e_2|$, $|e_3|$, $|e_4|$ and $|e_5|$ of different classes of univalent function can be deduced, especially for the class of bounded turning.*

Theorem 3.17. *If $f \in \mathcal{J}^n_{q,\sigma}(\theta,\zeta)$. Then*

$$|e_3 - \mu e_2^2| \leq \frac{2(1-\beta)}{2^\sigma 3^n[3]_q[2+(1+e^{i\theta})[2]_q]} max\{1, |1-\mathcal{M}|\}. \tag{3.11}$$

where

$$\mathcal{M} = \frac{2(1-\beta)}{2^\sigma 3^{-2\sigma}2^{2n}([2])_q^2/[3]_q|2+(1+e^{i\theta})([1])_q^2/[2]_q|}. \tag{3.12}$$

Proof. Since $f \in \mathcal{J}^n_{q,\sigma}(\theta,\zeta)$. Then

$$e_2 = \frac{2(1-\beta)s_1}{(\frac{3}{2})^\sigma 2^n[2]_q[2+(1+e^{i\theta})[1]_q]} \tag{3.13}$$

$$e_3 = \frac{2(1-\beta)s_2}{2^\sigma 3^n[3]_q[2+(1+e^{i\theta})[2]_q]}. \tag{3.14}$$

So that

$$|e_3 - \mu e_2^2| = \left|\frac{2(1-\beta)s_2}{(\frac{3}{2})^\sigma 2^n[2]_q[2+(1+e^{i\theta})[1]_q]} - \mu\left\{\frac{2(1-\beta)s_1}{2^\sigma 3^n[3]_q[2+(1+e^{i\theta})[2]_q]}\right\}^2\right| \tag{3.15}$$

From Lemma (2.3) and by simple calculation, we obtained the inequality (3.10). □

Corollary 3.18. *Varying the value of the parameters n, σ, θ, ζ, the Fekete-Szego functional $|e_3 - \mu e_2^2|$ of different classes of univalent function can be deduced, especially for the class of bounded turning.*

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Department of Mathematics, College of Physical Sciences, Federal University of Agriculture, Abeokuta
Email address: yusufaa@funaab.edu.ng

Department of Mathematics, Faculty of Science, Universiti Kebangsaan Malaysia
Email address: maslina@ukm.edu.my