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Homogeneity Tests for Lévy Processes and Applications

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Abstract

In this paper we will check the homogeneity/heterogeneity of Lévy processes using some non-parametric homogeneity tests. First we create two samples from two Lévy processes starting from the definition of the Lévy process, and next we test if the two samples have the same distribution.

Using the Lévy—Itô decomposition we will perform the homogeneity tests for given parts of the Lévi processes.

The study of the homogeneity of stock markets shocks is usefull because the eventualy homogeneity can produce a phenomenon analogue to the resonance that can be observed in mechanics. This resonance increase the idiosyncratic risk.

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1 Introduction

For studying of the jump processes, an essential role is played by the Poisson process, whose definition is given in the following (see [10, 2]).

Definition 1 *A Poisson process is a homogeneous Markov process X with the set of states $S = \mathbb{N}$ and the time set $T = [0, \infty)$ such that the transition probabilities are $P_t(n+k, n) = e^{-\lambda \cdot t} \cdot \frac{(\lambda \cdot t)^k}{k!}$ for any $t > 0$, $n \geq 0$ and $k \geq 0$, and the other transition probabilities are equal to zero.*

Definition 2 ([10, 2]) *Let N_t be a Poisson process having the intensity λ. The stochastic process given by $\widetilde{N}_t = N_t - \lambda \cdot t$ is called compensated (or centered) Poisson process.*

From given measures and from Poisson/compensated Poisson process we build Poisson/ compensated Poisson random measures (see [10, 2]).

Definition 3 *Let N_t be a Poisson process having the intensity λ, and the corresponding compensated Poisson process $\widetilde{N}_t$ in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider also the set $E \in \mathbb{R}^d$ and μ a given positive Radon measure μ on $(E, \mathcal{E})$.*

A Poisson random measure on E with the intensity measure μ is an integer valued random measure $M : \Omega \times \mathcal{E} \to \mathbb{N}$, $(\omega, A) \mapsto M(\omega, A)$ such that:

1. *For almost all $\omega \in \Omega$, $M(\omega, \cdot)$ is an integer-valued Radon measure on E: for any bounded measurable $A \subset E$, $M(A) < \infty$ is an integer-valued random variable.*

   item For each measurable set $A \subset E$, $M(\cdot, A) = M(A)$ is a Poisson random variable with parameter $\mu(A)$: $\forall k \in \mathbb{N}$, $P(M(A) = k) = e^{-\mu(A)} \cdot \frac{(\mu(A))^k}{k!}$.

2. *For any disjoint measurable sets $A_1, ..., A_n \in \mathcal{E}$, the random variables $M(A_1)$,...,$M(A_n)$ are independent*

The compensated Poisson random measure is $\widetilde{M}(A) = M(A) - \mu(A)$.

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Definition 4 ([10, 2]) *The jump measure of a Poisson process* $(N_t)_{t\geq 0}$ *is defined by* $J_N = \sum\limits_{n\geq 1} \delta_{(T_n,1)}$*:*

$$J_N([0,t]\times A) = \begin{cases} \sharp\{i\geq 1, T_i\in[0,t]\}, \ if \ 1\in A \\ 0, \ if \ 1\notin A \end{cases}.$$

Definition 5 ([10, 2]) *The function* $f:[0,T]\to\mathbb{R}^d$ *is called cadlag*[1] *if it is right-continuous with left limits: for each* $t\in[0,T]$ *the limits*

$f(t-) = \lim\limits_{s\nearrow t} f(s)$ *and* $f(t+) = \lim\limits_{s\searrow t} f(s)$

exist and $f(t) = f(t+)$*.*

We denote by $\Delta f(t) = f(t) - f(t-)$ the discontinuity (or the "jump") of f at t.

An analogue definition is for caglad functions, which are left-continuous with right limits. Considering now a stochastic process X_t, cadlag means that the jump occurs before, and caglad mans that the jump occurs after the given moment t. Because the jumps can be observed only after they have occured, in financial modeling there are prefered the cadlag stochastic processes (see [10, 2]).

Definition 6 ([10, 2]) *A cadlag stochastic process* $(X_t)_{t\geq 0}$ *on the probability space* $(\Omega,\mathcal{F},\mathbb{P})$ *with values in* $\mathbb{R}^d$ *such that* $X_0 = 0$ *is called Lévy process if it has the following properties*

1. *Independent increments: for any increasing sequence of time moments* $t_0, t_1,\ldots,t_n$*, the random variables* $X_{t_0}, X_{t_1}-X_{t_0},\ldots,X_{t_n}-X_{t_{n-1}}$ *are independent.*

2. *Stationary increments: the distribution law of* $X_{t+h}-X_t$ *does not depend on* t*.*

3. *Stochastic continuity:* $\forall\varepsilon>0, \lim\limits_{h\to 0} P(|X_{t+h}-X_t|\geq\varepsilon) = 0.$

Because it is proved (see [10, 2]) that the Poisson process is a particular case of Lévy process with piecewise constant sample paths, the third condition in the above definition does not imply that the sample paths are continuous.

[1]From French "continu à droite, limite à gauche"

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Definition 7 ([10, 2]) *A compound Poisson process with the intensity* λ *and the jump size* f *is a stochastic process*

$$X_t = \sum_{i=1}^{N_t} Y_i,$$

where N_t *is a Poisson process with the intensity* λ*, and the random variables* Y_i*,* $i \geq 1$ *are independent and they have the same distribution* f*.*

If the distribution f is such that $Y_i = 1$ with the probability 1, we have $X_t = N_t$. Therefore the Poisson process is a particular case of compound Poisson process. We have the following property of compound Poisson processes.

Proposition 1 ([10, 2]) *The stochastic process* $(X_t)_{t\geq 0}$ *is a compound Poisson process if and only if it is a Lévy process and its sample paths are piecewise constant functions.*

Proposition 2 ([10, 2]) *The characteristic function of a compound Poisson process* X_t *(i.e. the characteristic function of the random variable* X_t*) is*

$$E\left[e^{iu\cdot X_t}\right] = \exp\left\{ t\lambda \cdot \int_{\mathbb{R}^d} \left(e^{iu\cdot x}\right) f\left(dx\right) \right\},$$

where λ *is the jump intensity and* f *is the jump size distribution.*

Definition 8 ([10, 2]) *Let* $(X_t)_{t\geq 0}$ *be a Lévy process. The Lévy measure on* X *is defined by*

$$\nu\left(A\right) = E\left\{\sharp\left\{t \in [0,1] \middle| \, \Delta X_t \neq 0,\ \Delta X_t \in A\right\}\right\},\ A \in \mathcal{B}\left(\mathbb{R}^d\right).$$

Therefore, according to the above definition, $\nu\left(A\right)$ is the expected number of jumps whose sizes belong to A, per unit time.

Definition 9 ([10, 2]) *A Brownian motion is a stochastic process* $(B_t)_{t\geq 0}$ *with independent stationary increments, that follow a Gaussian distribution.*

Proposition 3 (The Lévy—Itô decomposition: see [10, 2]) *Let* $(X_t)_{t\geq 0}$ *be a Lévy process on* $\mathbb{R}^d$ *and* ν *its Lévy measure, given by Definition 8. Then*

1. ν *is a Radon measure on* $\mathbb{R}^d$ *such that*

   (a) $\int\limits_{|x|\leq 1} |x|^2 \nu(dx) < \infty.$

   (b) $\int\limits_{|x|\geq 1} \nu(dx) < \infty.$

2. *The jump measure of* X*, denoted by* J_X*, is a Poisson random measure on* $[0,\infty) \times \mathbb{R}^d$ *with the intensity measure* $\nu(dx)\, dt$*.*

3. *There exist a vector* $\gamma \in \mathbb{R}^d$ *and a d-dimensional Brownian motion* B_t *such that*

   (a) $X_t = t\gamma + B_t + X_t^1 + \lim\limits_{\varepsilon \searrow 0} \widetilde{X}_t^\varepsilon$*, where*

   (b) $X_t^1 = \int\limits_{|x|\geq 1, s\in[0,t]} x J_x(ds \times dx)$ *and*

   (c) $\widetilde{X}_t^\varepsilon = \int\limits_{\varepsilon\leq|x|\leq 1, s\in[0,t]} x J_x(ds \times dx) - \nu(dx)\, ds = \int\limits_{\varepsilon\leq|x|\leq 1, s\in[0,t]} x \widetilde{J}_x(ds \times dx).$

   The terms in the above decomposition are independent, and the convergence of the last term is almost sure and uniform in t *on* $[0,T]$*.*

From the Lévy—Itô decomposition we obtain the characteristic triplet (A, ν, γ), where A is the varianc-covariance matrix of the increments of B_t, ν is the Lévy measure, and γ is the constant vector from above. Computing the characteristic function for a Lévy process we obtain the following reprezentation, tacking into account the above Lévy—Itô decomposition.

Proposition 4 (The Lévy—Khinchin reprezentation: see [10, 2]) *Let* $(X_t)_{t\geq 0}$ *a Lévy process on* $\mathbb{R}^d$ *with the characteristic triplet* (A, ν, γ)*. Then the characteristic function is*

$$E\left[e^{iz\cdot X_t}\right] = e^{t\psi(z)}, z \in \mathbb{R}^d, \text{ and}$$

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$$\psi(z) = -\frac{1}{2} z \cdot Az + i\gamma \cdot z + \int\limits_{\mathbb{R}^d} \left(e^{iz \cdot x} - 1 - iz \cdot x 1_{|x| \leq 1}\right) \nu(dx).$$

If $d = 1$ we obtain

$$\psi(z) = -\frac{Az^2}{2} + i\gamma z + \int\limits_{-\infty}^{\infty} \left(e^{izx} - 1 - izx 1_{|x| \leq 1}\right) \nu(dx) \tag{1}$$

The moments of a Lévy process are computed from the Lévy—Khincin representation (see [10, 2]). We obtain

$$\begin{cases} E(X_t) = t\left(\gamma + \int\limits_{|x| \geq 1} x\nu(dx)\right) \\ Var(X_t) = t\left(\sigma^2 + \int\limits_{-\infty}^{\infty} x^2 \nu(dx)\right) \\ C_n(X_t) = t\left(\int\limits_{-\infty}^{\infty} x^n \nu(dx)\right) \ for\ n > 2 \end{cases}. \tag{2}$$

Consider now a discrete time series $X_1,\ X_2,...,\ X_n$. In the general case it can be decomposed in three parts (see [7, 15, 18]): the trend, the sesonal component and the stationary component. If ther is no sesonnal component, a method to remove the trend is the moving average. The moving average of order q is

$$\widehat{m}_t = \frac{\sum\limits_{j=-q}^{q} X_{t+j}}{2 \cdot q + 1}. \tag{3}$$

In [7] there are considered $X_t = X_1$ for $t < 1$, and $X_t = X_n$ for $t > n$, and in [15, 18] there are computed only the values for which $q < t \leq n - q$, hence all the terms in the above relation exist in the time series. A criterion to choose q used in [15] is the minimum variance of $X_t - \widehat{m}_t$.

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2 Testing the homogeneity of two Lévy processes and the decomposition Lévy—Itô

Consider two Lévy processes X_Δ, $X_{2\cdot\Delta}$,..., $X_{n\cdot\Delta}$, respectively Y_Δ, $Y_{2\cdot\Delta}$,..., $Y_{n\cdot\Delta}$, observed at the same time intervals, Δ. According to the definition of the Lévy processes, the random variables $X_{2\cdot\Delta} - X_\Delta$, $X_{3\cdot\Delta} - X_{2\cdot\Delta}$,..., $X_{n\cdot\Delta} - X_{(n-1)\cdot\Delta}$ are independent and identicaly distributed. The same thing we can say about $Y_{2\cdot\Delta} - Y_\Delta$, $Y_{3\cdot\Delta} - Y_{2\cdot\Delta}$,..., $Y_{n\cdot\Delta} - Y_{(n-1)\cdot\Delta}$. Therefore the problem of testing homogeneity of the Lévy processes can be reduced to test if two random variables have the same distribution.

The Kolmogorov—Smirnov goodness-of-fit test (see [9, 17]) tests the null hypothesis H_0: the random variable X has the cumulative distribution function F against the alternative hypothessis H_1: the random variable X has not the cumulative distribution function F, with a given first degree error ε. We have a sample of the size n from a population characterized by the random variable X. Denote by F^* the empirical cdf. The following theorem (see [9, 17]) is the theoretical basis of this test.

Theorem 5 *Let X be a random variable having the cdf F, and a sample $X_1, ..., X_n$ from a population characterized by the random variable X. Denote by $D = \max_x |F^*(x) - F(x)|$. Then for any $\lambda > 0$ we have*

$$\lim_{n\to\infty} P\left(D \leq \frac{\lambda}{\sqrt{n}}\right) = K(\lambda) = \sum_{k=-\infty}^{\infty} (-1)^k \cdot e^{-2k^2\lambda^2}.$$

We can notice that, due to the shape of the empirical cdf, the maximum in the above theorem is obtained in one of the sample values. The centils for D can be found in tables in any book of statistics. Denoting this centil by $D_n(1-\varepsilon)$ (n is the size of the sample), we accept the null hypothesis H_0 if $D < D_n(1-\varepsilon)$.

We can use this test for testing homogeneity if both cdfs are empirical, considering two samples of sizes m, respectively n. In this case D is the maximum distance between the empirical cdfs, and the relation from the theorem becomes (see [9, 17])

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$$\lim_{n\to\infty} P\left(D \le \lambda \cdot \sqrt{\frac{m+n}{m\cdot n}}\right) = K(\lambda) = \sum_{k=-\infty}^{\infty} (-1)^k \cdot e^{-2k^2\lambda^2}. \qquad (4)$$

Therefore we replace the value of n in the case of one sample by $n_1 = \frac{m\cdot n}{m+n}$ for the case of two samples. As a computational technique (in our $C++$ program) we estimate the above function K by sumation of k between -1000 and 1000, and we accept the homogeneity if $K\left(\sqrt{n_1}\cdot D\right) < 1-\varepsilon$. The problem can arise only in the case $n_1 < 35$ (this is not our case, as we will see in Section 3, because if $m = n$ we have $n_1 = \frac{n}{2}$, and in the considered application we have $n = 149$). For this case we need the table of centils, and we accept the null hypothesis H_0 if $D < D_n(1-\varepsilon)$ for n being the integer part of n_1 plus one, and we reject it if $D \ge D_n(1-\varepsilon)$ for n being the integer part of n_1. If D is between these values we need an interpolation technique to compute $D_n(1-\varepsilon)$ for $n = n_1$.

Another homogeneity test used in statistics is the Mann—Whitney—Wilcoxon test (known also as the ranks' sum test). We test the null hypotheses H_0: the two samples have the same distribution against the alternative hypothesis H_1: the two samples have not the same distribution. If the first sample has the size m, and the second one the size n, we order first the $m+n$ values in increasing order (see [9, 17]). If there are common values in the two samples we remove them, and m and n decrease according to the removed values.

Denoting by R_X and R_y the sum of the obtained ranks of the values from X, respectively from Y, we compute first

$$\begin{cases} W_X = m\cdot n + \frac{m(m+1)}{2} - R_X \\ W_Y = m\cdot n + \frac{n(n+1)}{2} - R_Y \\ W = \max(W_X, W_Y) \end{cases}. \qquad (5)$$

The expectation and the variance of W are

$$\begin{cases} E(W) = \frac{m\cdot n}{2} \\ Var(W) = \frac{m\cdot n\cdot(m+n+1)}{12} \end{cases}. \qquad (6)$$

If $m, n > 8$ the distribution of W can be approximated by a normal distribution (see [9, 17]). Therefore

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$$Z = \frac{W - E(W)}{\sqrt{Var(W)}} \sim N(0,1), \tag{7}$$

and we accept the null hypothesis of homogeneity with the first degree error ε if $Z < Z_{1-\varepsilon}$ ($Z_{1-\varepsilon}$ is the centil of level $1-\varepsilon$, or of the error ε for the standard normal distribution).

If $m = n$ (as in our case, where the values of the two Lévy processes are observed in the same moments of time) we can use the signed Wilcoxon ranks test. We order first each sample in increasing order, and we remove each pair (X_i, Y_i) with $X_i = Y_i$ (and n decrease by one for every such pair). Next we multiply the ranks with the signs of the corresponding differences between the values of the first sample and those of the second one (see [9, 17]): if $X_i - Y_i > 0$ we consider the signed rank i, and if $X_i - Y_i < 0$ we consider the signed rank $-i$.

Denoting by T_X the sum of positive ranks, and by T_Y the sum ov the negative ones, we obtain the expectations and the variances of these random variables in the case of homogeneity

$$\begin{cases} E(T_X) = E(-T_Y) = \frac{n(n+1)}{4} \\ Var(T_X) = Var(-T_Y) = \frac{n(n+1)(2 \cdot n+1)}{24} \end{cases}. \tag{8}$$

We consider now $T = T_X$ or $T = -T_Y$, and, in the same conditions as for the Mann—Whitney—Wilcoxon test ($n > 8$) we can approximate the distribution of T with a normal one. Therefore

$$Z = \frac{T - E(T)}{\sqrt{Var(T)}} \sim N(0,1), \tag{7'}$$

and we accept the null hypothesis of homogeneity with the first degree error ε if $Z < Z_{1-\varepsilon}$.

To implement the Lévy—Itô decomposition we take into account the appearance of σ^2 at the denominator of the skewness S in (2). Therefore, analogue to the moving average technique to extract the trend from a time series and the criterion used in [15] of minimum variance of the residues, we consider the same technique of moving average to extract the Brownian motion, but we use as criterion the maximum absolute value of the skewness.

3 Application

Example 1 *Consider the daily data from January 3 to August 1 on the Bucharest Stock Exchange indices BET-C and BET-FI*[2]. *We except from the above period, due to the lack of data, the Saturdays, Sundays and other legal/ religious hollidays.*

The data were downloaded from The Statistical Section of Bucharest Stock Exchange (see [23]).

The results of the homogeneity tests for $\varepsilon = 5\%$ in the case of the whole Lévy processes are in Table 1. We find by all the three tests that the two financial indices are homogeneous.

Table 1: The results in the case of whole BET-C and BET-FI.

Test	Involved statistics S	Centil/ involved *cdf* (S)
Kolmogorov—Smirnov	$D = 0.14865$	$K(\lambda) = 0.92401 < 0.95$
Mann—Whitney—Wilcoxon	$Z = 1.67353$	$\Phi(Z) = 0.9476 < 0.95$
Wilcoxon signed ranks	$Z = 1.69466$	$\Phi(Z) = 0.9499 < 0.95$

For the Lévy—Itô decomposition, we extract first the Brownian motion. The skewness with the maximum absolute value for the log-BET-C case is 0.23993 obtained for $q = 43$. The drift is $\gamma = 0.00013$, and the average amplitude of shocks is 0.00741. For the log-BET-FI case we obtain the skewness with the maximum absolute value 0.19678 obtained for $q = 2$. The drift is $\gamma = -0.00014$, and the average amplitude of shocks is 0.01049.

The results for shocks for $\varepsilon = 5\%$ are in Table 2. We find by all the three tests that the shocks in the two financial indices are homogeneous.

Table 2: The results in the case of shocks in BET-C, and in BET-FI.

Test	Involved statistics S	Centil/ involved *cdf* (S)
Kolmogorov—Smirnov	$D = 0.11486$	$K(\lambda) = 0.71703 < 0.95$
Mann—Whitney—Wilcoxon	$Z = 0.75595$	$\Phi(Z) = 0.781 < 0.95$
Wilcoxon signed ranks	$Z = 1.16454$	$\Phi(Z) = 0.9499 < 0.95$

We present in the following the graphics for BET-C and BET-FI when we compute

[2]BET is the main Bucharest Stock Exchange index, and it means Bucharest Exchange Trading. It contains the most liquide 10 companies.

BET-C means BET-Composite, and it contains all the listed companies, except the financial ones, i.e banks and societies for mass privatization—SIF=Societate de Investiţii Financiare=Society for Financial Investment (engl).

BET-FI means BET-FInancial, and it contains the financial institutions excepted by BET-C.

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the differences $X_n - X_1$ and $Y_n - Y_1$ for log-values (fig. 1), respectively when we compute the ratios $\frac{X_n}{X_1}$ and $\frac{Y_n}{Y_1}$ for the initial values (fig. 2).

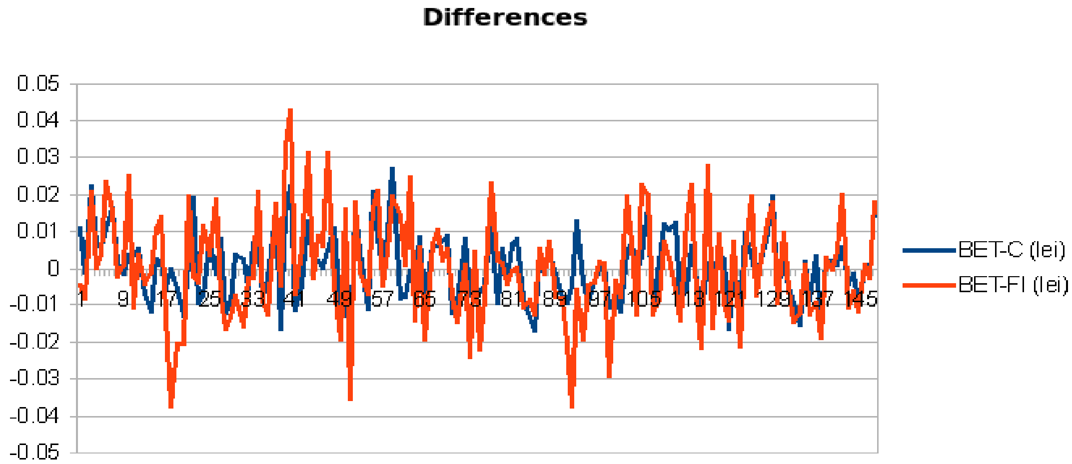

Fig. 1: BET-C and BET-FI in the case of the differences between logarithms.

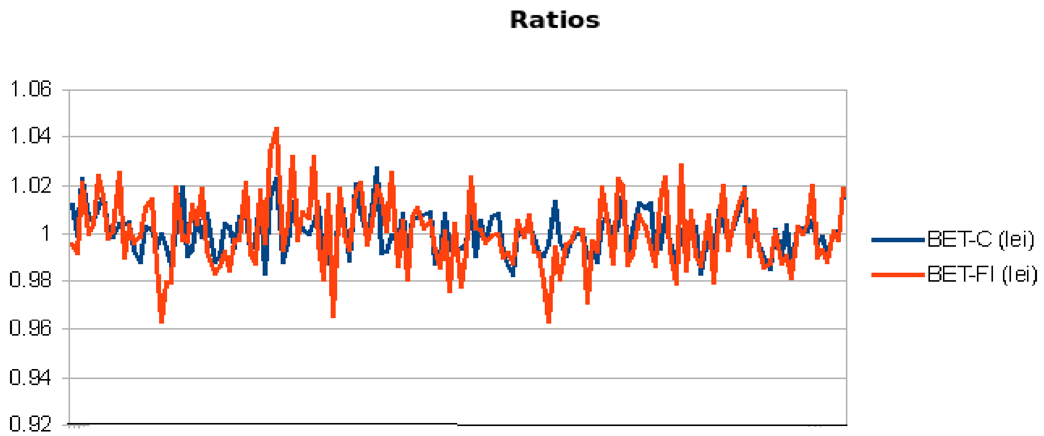

Fig. 2: BET-C and BET-FI in the case of the ratios between values.

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4 Conclusions

Because in our application we have considered logarithms for both Bucharest Stock Exchange indices, and the homogeneity test are nonparametric, we have replaced in our $C++$ program for these tests the differences between logarithms the ratio between initial data. Of course, for Lévy—Itô decomposition we have applied in the end of the corresponding $C++$ program the exponential to the parts of the Lévy processes. If we consider logarithms for no indices, we have to use obviously the differences. The problem is if we use only one logarithm, and in this case we have indeed to compute the corresponding logarithm data, and next to test the homogeneity.

In [10, 2] a parametric approach is provided. In [6] there are presented some non-parametric models, namely the normal variance-mean mixture models, the hyperbolic models and the NIG model. In the case of the normal variance-mean mixture models with self-decomposable mixing distribution there are two parts that we need to estimate: the parametric part (μ, Σ), for which standard parametric methods suffice, and the nonparametric part g for which we need non-parametric estimation techniques.

In our paper the techniques analogues to those for extracting the trend in the time series (and we mean the moving average method) is used for extracting the drift and the Brownian motion from the Lévy processes. An open problem is if we can use techniques analogues to other trend extracting methods, as the exponential smooth (see [15, 18, 7]), or elimination of the components with high frequences (see [7]).

All the tests used for testing the homogeneity are non-parametric ones. Of course, for the Brownian motions we can use some parametric tests, like the Tuckey test for equal expectations, or Hartley test for equal variances. But the goal of this paper is to provide some non-parametric techniques that can be applied for all parametric models as the Kou or the Merton model (see [10, 2]).

Open problems are to test the homogeneity for more then two models, and to test other proprieties for the distributions of $X_{(k+1)\Delta} - X_{k\Delta}$, as the independence (the Wald—Wolfowitz independence test: see [21, 9, 17]).

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