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DEPTH AND STANLEY DEPTH OF THE EDGE IDEALS OF SOME m-LINE GRAPHS AND m-CYCLIC GRAPHS WITH A COMMON VERTEX

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ABSTRACT. We give some precise formulas for the depth of the quotient rings of the edge ideals associated to a graph consisting, either of the union of some line graphs $L_{3r_1}, \ldots, L_{3r_{k_1}}, L_{3s_1+1}, \ldots, L_{3s_{k_2}+1}$ and $L_{3t_1+2}, \ldots, L_{3t_{k_3}+2}$ or of the union of cycle graphs $C_{3r_1}, \ldots, C_{3r_{k_1}}, C_{3s_1+1}, \ldots, C_{3s_{k_2}+1}$ and $C_{3t_1+2}, \ldots, C_{3t_{k_3}+2}$, with a common vertex. We also give some tight bounds for their Stanley depths.

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1. INTRODUCTION

Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field K and M a finitely generated $\mathbb{Z}^n$-graded S-module. For a homogeneous element $u \in M$ and a subset $Z \subseteq \{x_1, \ldots, x_n\}$, $uK[Z]$ denotes the K-subspace of M generated by all the homogeneous elements of the form uv, where v is a monomial in $K[Z]$. The $\mathbb{Z}^n$-graded K-subspace $uK[Z]$ is said to be a Stanley space of dimension $|Z|$ if it is a free $K[Z]$-module, where, as usual, $|Z|$ denotes the number of elements of Z. A Stanley decomposition of M is a decomposition of M as a finite direct sum of $\mathbb{Z}^n$-graded K-vector spaces

$$\mathcal{D} : M = \bigoplus_{i=1}^{r} u_i K[Z_i]$$

where each $u_i K[Z_i]$ is a Stanley space of M. The number

$$\operatorname{sdepth}(\mathcal{D}) = \min\{|Z_i| : i = 1, \ldots, r\}$$

is called the Stanley depth of decomposition $\mathcal{D}$ and the quantity

$$\operatorname{sdepth}(M) := \max\{\operatorname{sdepth}(\mathcal{D}) \mid \mathcal{D} \text{ is a Stanley decomposition of } M\}$$

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ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2015, VOLUME 5, ISSUE 2, p.118-129

is called the Stanley depth of M. Stanley [10] conjectured that

$$\operatorname{sdepth}(M) \geq \operatorname{depth}(M)$$

for all $\mathbb{Z}^n$-graded S-modules M. This conjecture proves to be false, in general, for $M = S/I$ and $M = J/I$, where $I \subset J \subset S$ are monomial ideals, see [4].

Herzog, Vlădoiu and Zheng [6] introduced a method to compute the Stanley depth of a factor of two monomial ideals which was later developed into an effective algorithm by Rinaldo [9] implemented in CoCoA [3]. However, it is difficult to compute this invariant, even in some very particular cases. For instance in [1] Biró et al. proved that $\operatorname{sdepth}(\mathfrak{m}) = \lceil \frac{n}{2} \rceil$ where $\mathfrak{m} = (x_1, \ldots, x_n)$ is the graded maximal ideal of S and $\lceil \frac{n}{2} \rceil$ denote the smallest integer $\geq \frac{n}{2}$. For a friendly introduction on Stanley depth we refer the reader to [5].

Let I_n and J_n be the edge ideals associated to the line, respectively, cycle graph of length n. Morey [7] proved that $\operatorname{depth}(S/I_n) = \lceil \frac{n}{3} \rceil$. Replacing depth by stanley depth, Ştefan [11] showed that $\operatorname{sdepth}(S/I_n) = \lceil \frac{n}{3} \rceil$. In [2], Cimpoeaş proved that $\operatorname{depth}(S/J_n) = \lceil \frac{n-1}{3} \rceil$ and $\operatorname{sdepth}(S/J_n) = \lceil \frac{n-1}{3} \rceil$ for $n \equiv 0 \pmod 3$ and $n \equiv 2 \pmod 3$. He also proved that $\lceil \frac{n-1}{3} \rceil \leq \operatorname{sdepth}(S/J_n) \leq \lceil \frac{n}{3} \rceil$ for $n \equiv 1 \pmod 3$. Let I and J be the edge ideals associated to the graph consisting of the union of line graphs $L_{3r_1}, \ldots, L_{3r_{k_1}}$, $L_{3s_1+1}, \ldots, L_{3s_{k_2}+1}$ and $L_{3t_1+2}, \ldots, L_{3t_{k_3}+2}$ with a common vertex, respectively, the graph consisting of the union of cycle graphs $C_{3r_1}, \ldots, C_{3r_{k_1}}$, $C_{3s_1+1}, \ldots, C_{3s_{k_2}+1}$ and $C_{3t_1+2}, \ldots, C_{3t_{k_3}+2}$ with a common vertex, then using similar techniques, we prove that

(1) $\operatorname{sdepth}(\frac{S}{I}) \geq \operatorname{depth}(\frac{S}{I}) = \begin{cases} \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i, & \text{if } k_3 = 0, \\ \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i + 1, & \text{otherwise;} \end{cases}$

(2) $\operatorname{sdepth}(\frac{S}{I}) \leq \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i + 1.$

In the fourth section, we prove that

(1) $\operatorname{sdepth}(\frac{S}{J}) \geq \operatorname{depth}(\frac{S}{J}) = \begin{cases} \sum_{i=1}^{k_2} s_i, & \text{if } k_1 = k_3 = 0, \\ \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i - k_1 + 1, & \text{otherwise;} \end{cases}$

(2) $\operatorname{sdepth}(\frac{S}{J}) \leq \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i - k_1 + 1;$

(3) $\operatorname{sdepth}(\frac{J}{I}) \geq \operatorname{depth}(\frac{J}{I}) = \begin{cases} \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_3} t_i - k_1 + 2, & \text{if } k_2 = 0, \\ \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i - k_1 + 1, & \text{otherwise.} \end{cases}$

2. Preliminaries

We first recall some definitions about graphs and their edge ideals in order to make this paper self-contained. However, for more details on the notions, we refer the reader to [13, 14].

Definition 2.1. Let $G_i = (V(G_i), E(G_i))$ be some graphs with vertex set $V(G_i)$ and edge set $E(G_i)$ for $1 \leq i \leq k$. The union of graphs $G_1, G_2, \ldots, G_k$, written as $G_1 \cup G_2 \cup \cdots \cup G_k$, is the graph with vertex set $\bigcup_{i=1}^{k} V(G_i)$ and edge set $\bigcup_{i=1}^{k} E(G_i)$.

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2015, VOLUME 5, ISSUE 2, p.118-129

Definition 2.2. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{x_1, \ldots, x_n\}$ and edge set $E(G)$. Suppose that $x_1, \ldots, x_n$ are variables over the field K. The edge ideal of graph G in the polynomial ring $S = K[x_1, \ldots, x_n]$ is the squarefree monomial ideal

$$I(G) = (\{x_i x_j \mid \{x_i, x_j\} \in E(G)\}).$$

For the sake of simplicity, we will use the same notation $x_i x_j$ for the monomial and for the corresponding edge of graph G.

Definition 2.3. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{x_1, \ldots, x_m\}$ and edge set $E(G)$. Then G is called a line graph of length m, denoted by L_m, if its edge set $E = \{x_i x_{i+1} \mid 1 \leq i \leq m-1\}$. Similarly, if $m \geq 3$, then G is called a cycle graph of length m, denoted by C_m, if its edge set $E = \{x_i x_{i+1} \mid 1 \leq i \leq m-1\} \cup \{x_m x_1\}$.

We recall the well known Depth Lemma, see for instance [13, Lemma 1.3.9] or [12, Lemma 3.1.4].

Lemma 2.4. *(Depth Lemma) Let $0 \to L \to M \to N \to 0$ be a short exact sequence of modules over a local ring S, or a Noetherian graded ring with S_0 local, then*

(i) $depth(M) \geq min\{depth(L), depth(N)\}$;
(ii) $depth(L) \geq min\{depth(M), depth(N) + 1\}$;
(iii) $depth(N) \geq min\{depth(L) - 1, depth(M)\}$.

The most of the statements of the Depth Lemma are wrong if we replace depth by Stanley depth. Rauf [8] proved the analog of Lemma 2.4 (i) for Stanley depth.

Lemma 2.5. *Let $0 \to L \to M \to N \to 0$ be a short exact sequence of finitely generated $\mathbb{Z}^n$-graded S-modules. Then*

$$sdepth(M) \geq min\{sdepth(L), sdepth(N)\}.$$

Using Depth Lemma, Morey in [7] proved the following result.

Lemma 2.6. *Let L_m be a line graph of length m and $I(L_m)$ its edge ideal. Then $depth(S/I(L_m)) = \lceil \frac{m}{3} \rceil$.*

Replacing depth by Stanley depth, Ştefan in [11] showed the following result.

Lemma 2.7. *Let L_m be a line graph of length m and $I(L_m)$ its edge ideal. Then $sdepth(S/I(L_m)) = \lceil \frac{m}{3} \rceil$.*

3. Depth and Stanley depth of the edge ideals of some m-line graphs with a common vertex

In this section, we will give some formulas for depth and Stanley depth of the quotient rings of the edge ideals of some m-line graphs with a common vertex. We assume that G is the m-line graph formed by joining m line graphs $L_{3r_1}, \ldots, L_{3r_{k_1}}, L_{3s_1+1}, \ldots, L_{3s_{k_2}+1}, L_{3t_1+2}, \ldots, L_{3t_{k_3}+2}$ at a common vertex, where $k_1 + k_2 + k_3 = m$ and $k_i \geq 0$ for $i = 1, 2, 3$. We adopt the following notation to edges of graph G:
$E(L_{3r_i,i}) = \{x_1 x_{2,i}, x_{2,i} x_{3,i}, \ldots, x_{3r_i-1,i} x_{3r_i,i}\}$ for any $1 \leq i \leq k_1$ and $r_1 \leq \cdots \leq r_{k_1}$,
$E(L_{3s_i+1,i}) = \{x_1 y_{2,i}, y_{2,i} y_{3,i}, \ldots, y_{3s_i,i} y_{3s_i+1,i}\}$ for any $1 \leq i \leq k_2$ and $s_1 \leq \cdots \leq s_{k_2}$,
$E(L_{3t_i+2,i}) = \{x_1 z_{2,i}, z_{2,i} z_{3,i}, \ldots, z_{3t_i+1,i} z_{3t_i+2,i}\}$ for all $1 \leq i \leq k_3$ and $t_1 \leq \cdots \leq t_{k_3}$.

Set K be any field, $S = K[x_1, x_{2,1}, \ldots, x_{3r_1,1}, \ldots, x_{2,k_1}, \ldots, x_{3r_{k_1},k_1}, y_{2,1}, \ldots, y_{3s_1+1,1}, \ldots, y_{2,k_2}, \ldots, y_{3s_{k_2}+1,k_2}, z_{2,1}, \ldots, z_{3t_1+2,1}, \ldots, z_{2,k_3}, \ldots, z_{3t_{k_3}+2,k_3}]$ the polynomial ring. The edge ideal of graph G is $I = (x_1 x_{2,1}, x_{2,1} x_{3,1}, \ldots, x_{3r_1-1,1} x_{3r_1,1}, \ldots, x_1 x_{2,k_1}, x_{2,k_1} x_{3,k_1}, \ldots, x_{3r_{k_1}-1,k_1} x_{3r_{k_1},k_1}, x_1 y_{2,1}, y_{2,1} y_{3,1}, \ldots, y_{3s_1,1} y_{3s_1+1,1}, \ldots, x_1 y_{2,k_2}, y_{2,k_2} y_{3,k_2}, \ldots, y_{3s_{k_2},k_2} y_{3s_{k_2}+1,k_2}, x_1 z_{2,1}, z_{2,1} z_{3,1}, \ldots, z_{3t_1+1,1} z_{3t_1+2,1}, \ldots, x_1 z_{2,k_3}, \ldots, z_{3t_{k_3}+1,k_3} z_{3t_{k_3}+2,k_3})$.

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2015, VOLUME 5, ISSUE 2, p.118-129

Example 3.1. The following graph G is the union of 5 line graphs L_3, L_4, L_5, L_6 and L_7 with a common vertex x_1.

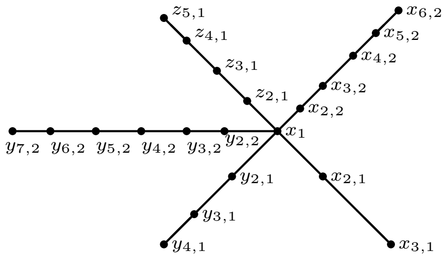

Figure 1

The edge ideal of graph G is $I = (x_1x_{2,1}, x_{2,1}x_{3,1}, x_1x_{2,2}, x_{2,2}x_{3,2}, x_{3,2}x_{4,2}, x_{4,2}x_{5,2}, x_{5,2}x_{6,2}, x_1y_{2,1}, y_{2,1}y_{3,1}, y_{3,1}y_{4,1}, x_1y_{2,2}, y_{2,2}y_{3,2}, y_{3,2}y_{4,2}, y_{4,2}y_{5,2}, y_{5,2}y_{6,2}, y_{6,2}y_{7,2}, x_1z_{2,1}, z_{2,1}z_{3,1}, z_{3,1}z_{4,1}, z_{4,1}z_{5,1})$.

We need the following lemma (See [8, Theorem 3.1]).

Lemma 3.2. *Let $I \subset S_1 = K[x_1, \ldots, x_m]$, $J \subset S_2 = K[y_1, \ldots, y_n]$ be monomial ideals and $S = K[x_1, \ldots, x_m, y_1, \ldots, y_n]$. Then*

$$\mathrm{sdepth}\,(S/(IS, JS) \geq \mathrm{sdepth}\,(S_1/IS_1) + \mathrm{sdepth}\,(S_2/JS_2).$$

Now, we prove the main results of this section. We adopt the following convention: whenever, in a sum, the index runs from 1 to 0, the sum has to be taken equal to zero.

Theorem 3.3. *Let G be a graph consisting of the union of line graphs $L_{3r_1}, \ldots, L_{3r_{k_1}}, L_{3s_1+1}, \ldots, L_{3s_{k_2}+1}$ and $L_{3t_1+2}, \ldots, L_{3t_{k_3}+2}$ with a common vertex x_1, where $k_i \geq 0$ for $i = 1, 2, 3$. Let I be its edge ideal. Then*

$$\mathrm{sdepth}\,(\frac{S}{I}) \geq \mathrm{depth}\,(\frac{S}{I}) = \begin{cases} \sum\limits_{i=1}^{k_1} r_i + \sum\limits_{i=1}^{k_2} s_i, & \text{if } k_3 = 0, \\ \sum\limits_{i=1}^{k_1} r_i + \sum\limits_{i=1}^{k_2} s_i + \sum\limits_{i=1}^{k_3} t_i + 1, & \text{otherwise.} \end{cases}$$

In particular, S/I satisfies the Stanley conjecture.

Proof. Note that $(I : x_1) = (x_{2,1}, \ldots, x_{2,k_1}, y_{2,1}, \ldots, y_{2,k_2}, z_{2,1}, \ldots, z_{2,k_3}, x_{3,1}x_{4,1}, \ldots, x_{3r_1-1,1}x_{3r_1,1}, \ldots, x_{3,k_1}x_{4,k_1}, \ldots, x_{3r_{k_1}-1,k_1}x_{3r_{k_1},k_1}, y_{3,1}y_{4,1}, \ldots, y_{3s_1,1}y_{3s_1+1,1}, \ldots, y_{3,k_2}y_{4,k_2}, \ldots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2}, z_{3,1}z_{4,1}, \ldots, z_{3t_1+1,1}z_{3t_1+2,1}, \ldots, z_{3,k_3}z_{4,k_3}, \ldots, z_{3t_{k_3}+1,k_3}z_{3t_{k_3}+2,k_3})$ and $(I, x_1) = (x_{2,1}x_{3,1}, \ldots, x_{3r_1-1,1}x_{3r_1,1}, \ldots, x_{2,k_1}x_{3,k_1}, \ldots, x_{3r_{k_1}-1,k_1}x_{3r_{k_1},k_1}, y_{2,1}y_{3,1}, \ldots, y_{3s_1,1}y_{3s_1+1,1}, \ldots, y_{2,k_2}y_{3,k_2}, \ldots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2}, z_{2,1}z_{3,1}, \ldots, z_{3t_1+1,1}z_{3t_1+2,1}, \ldots, z_{2,k_3}z_{3,k_3}, \ldots, z_{3t_{k_3}+1,k_3}z_{3t_{k_3}+2,k_3}, x_1)$, thus we get that

$$\begin{aligned} \frac{S}{(I:x_1)} \quad &\cong \quad \frac{K[x_{3,1}, \ldots, x_{3r_1,1}]}{(x_{3,1}x_{4,1}, \ldots, x_{3r_1-1,1}x_{3r_1,1})} \otimes_K \cdots \otimes_K \frac{K[x_{3,k_1}, \ldots, x_{3r_{k_1},k_1}]}{(x_{3,k_1}x_{4,k_1}, \ldots, x_{3r_{k_1}-1,k_1}x_{3r_{k_1},k_1})} \\ &\otimes_K \frac{K[y_{3,1}, \ldots, y_{3s_1+1,1}]}{(y_{3,1}y_{4,1}, \ldots, y_{3s_1,1}y_{3s_1+1,1})} \otimes_K \cdots \otimes_K \frac{K[y_{3,k_2}, \ldots, y_{3s_{k_2}+1,k_2}]}{(y_{3,k_2}y_{4,k_2}, \ldots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})} \\ &\otimes_K \frac{K[z_{3,1}, \ldots, z_{3t_1+2,1}]}{(z_{3,1}z_{4,1}, \ldots, z_{3t_1+1,1}z_{3t_1+2,1})} \otimes_K \cdots \otimes_K \frac{K[z_{3,k_3}, \ldots, z_{3t_{k_3}+2,k_3}]}{(z_{3,k_3}z_{4,k_3}, \ldots, z_{3t_{k_3}+1,k_3}z_{3t_{k_3}+2,k_3})} \otimes_K K[x_1], \end{aligned}$$

and

$$\frac{S}{(I,x_1)} \cong \frac{K[x_{2,1},\dots,x_{3r_1,1}]}{(x_{2,1}x_{3,1},\dots,x_{3r_1-1,1}x_{3r_1,1})}\otimes_K\cdots\otimes_K \frac{K[x_{2,k_1},\dots,x_{3r_{k1},k_1}]}{(x_{2,k_1}x_{3,k_1},\dots,x_{3r_{k1}-1,k_1}x_{3r_{k1},k_1})}$$
$$\otimes_K \frac{K[y_{2,1},\dots,y_{3s_1+1,1}]}{(y_{2,1}y_{3,1},\dots,y_{3s_1,1}y_{3s_1+1,1})}\otimes_K\cdots\otimes_K \frac{K[y_{2,k_2},\dots,y_{3s_{k_2}+1,k_2}]}{(y_{2,k_2}y_{3,k_2},\dots,y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})}$$
$$\otimes_K \frac{K[z_{2,1},\dots,z_{3t_1+2,1}]}{(z_{2,1}z_{3,1},\dots,z_{3t_1+1,1}z_{3t_1+2,1})}\otimes_K\cdots\otimes_K \frac{K[z_{2,k_3},\dots,z_{3t_{k_3}+2,k_3}]}{(z_{2,k_3}z_{3,k_3},\dots,z_{3t_{k_3}+1,k_3}z_{3t_{k_3}+2,k_3})}.$$

Therefore, by Lemmas 2.6, 2.7, 3.2 and [13, Proposition 2.2.20, Theorem 2.2.21], we obtain that $\operatorname{sdepth}(\frac{S}{I:x_1}) \geq \operatorname{depth}(\frac{S}{I:x_1}) = \sum_{i=1}^{k_1}\lceil\frac{3r_i-2}{3}\rceil + \sum_{i=1}^{k_2}\lceil\frac{3s_i-1}{3}\rceil + \sum_{i=1}^{k_3}\lceil\frac{3t_i}{3}\rceil + 1 = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i + 1$, and $\operatorname{sdepth}(\frac{S}{(I,x_1)}) \geq \operatorname{depth}(\frac{S}{I,x_1}) = \sum_{i=1}^{k_1}\lceil\frac{3r_i-1}{3}\rceil + \sum_{i=1}^{k_2}\lceil\frac{3s_i}{3}\rceil + \sum_{i=1}^{k_3}\lceil\frac{3t_i+1}{3}\rceil = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i + k_3$.

Using Lemma 2.5 on the short exact sequence

$$(1) \qquad 0 \longrightarrow S/(I:x_1) \xrightarrow{\cdot x_1} S/I \longrightarrow S/(I,x_1) \longrightarrow 0,$$

we conclude that $\operatorname{sdepth}(\frac{S}{I}) \geq \begin{cases} \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i, & \text{if } k_3 = 0;\\ \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i + 1, & \text{otherwise.}\end{cases}$

If $k_3 \neq 0$, then $\operatorname{depth}(\frac{S}{(I,x_1)}) \geq \operatorname{depth}(\frac{S}{I:x_1})$. Using Lemma 2.4 on the short exact sequence (1), it follows that $\operatorname{depth}(\frac{S}{(I,x_1)}) = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i + 1$.

Assume that $k_3 = 0$. We claim that we have the S-module isomorphism

$$\frac{(I:x_1)}{I} \cong \bigoplus_{i=1}^{k_1} x_{2,i}\Big(\frac{K[x_{3,1},\dots,x_{3r_1,1}]}{(x_{3,1}x_{4,1},\dots,x_{3r_1-1,1}x_{3r_1,1})}\otimes_K\cdots\otimes_K\frac{K[x_{3,i-1},\dots,x_{3r_{i-1},i-1}]}{(x_{3,i-1}x_{4,i-1},\dots,x_{3r_{i-1}-1,i-1}x_{3r_{i-1},i-1})}$$
$$\otimes_K\frac{K[x_{4,i},\dots,x_{3r_i,i}]}{(x_{4,i}x_{5,i},\dots,x_{3r_i-1,i}x_{3r_i,i})}\otimes_K\frac{K[x_{2,i+1},\dots,x_{3r_{i+1},i+1}]}{(x_{2,i+1}x_{3,i+1},\dots,x_{3r_{i+1}-1,i+1}x_{3r_{i+1},i+1})}\otimes_K\cdots$$
$$\otimes_K\frac{K[x_{2,k_1},\dots,x_{3r_{k_1},k_1}]}{(x_{2,k_1}x_{3,k_1},\dots,x_{3r_{k_1}-1,k_1}x_{3r_{k_1},k_1})}\Big)\otimes_K\frac{K[y_{2,1},\dots,y_{3s_1+1,1}]}{(y_{2,1}y_{3,1},\dots,y_{3s_1,1}y_{3s_1+1,1})}\otimes_K\cdots$$
$$\otimes_K\frac{K[y_{2,k_2},\dots,y_{3s_{k_2}+1,k_2}]}{(y_{2,k_2}y_{3,k_2},\dots,y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})}\Big)[x_{2,i}]\oplus\Big(\bigoplus_{i=1}^{k_2} y_{2,i}\Big(\frac{K[x_{3,1},\dots,x_{3r_1,1}]}{(x_{3,1}x_{4,1},\dots,x_{3r_1-1,1}x_{3r_1,1})}\otimes_K\cdots$$
$$\otimes_K\frac{K[x_{3,k_1},\dots,x_{3r_{k_1},k_1}]}{(x_{3,k_1}x_{4,k_1},\dots,x_{3r_{k_1}-1,k_1}x_{3r_{k_1},k_1})}\otimes_K\frac{K[y_{3,1},\dots,y_{3s_1+1,1}]}{(y_{3,1}y_{4,1},\dots,y_{3s_1,1}y_{3s_1+1,1})}\otimes_K\cdots$$
$$\otimes_K\frac{K[y_{3,i-1},\dots,y_{3s_{i-1}+1,i-1}]}{(y_{3,i-1}y_{4,i-1},\dots,y_{3s_{i-1},i-1}y_{3s_{i-1}+1,i-1})}\otimes_K\frac{K[y_{4,i},\dots,y_{3s_i+1,i}]}{(y_{4,i}y_{5,i},\dots,y_{3s_i,i}y_{3s_i+1,i})}\otimes_K\cdots$$
$$\otimes_K\frac{K[y_{2,i+1},\dots,y_{3s_{i+1}+1,i+1}]}{(y_{2,i+1}y_{3,i+1},\dots,y_{3s_{i+1},i+1}y_{3s_{i+1}+1,i+1})}\otimes_K\cdots\otimes_K\frac{K[y_{2,k_2},\dots,y_{3s_{k_2}+1,k_2}]}{(y_{2,k_2}y_{3,k_2},\dots,y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})}\Big)[y_{2,i}]\Big),$$

where $x_{i,0} = y_{j,0} = 0$ for $3 \leq i \leq k_1$, $3 \leq j \leq k_2$. Indeed, if $u \in (I:x_1)$ is a monomial such that $u \notin I$, then $x_{2,i}|u$ or $y_{2,j}|u$ for some $1 \leq i \leq k_1$ or $1 \leq j \leq k_2$.

If $x_{2,1}|u$, then we can write u as $u = x_{2,1}^{\alpha}v$ with $\alpha \geq 1$ and $x_{2,1} \nmid v$. Since $u \notin I$, we have that $v \in K[x_{4,1},\dots,x_{3r_1,1},x_{2,2},\dots,x_{3r_2,2},\dots,x_{2,k_1},\dots,x_{3r_{k_1},k_1},y_{2,1},\dots,y_{3s_1+1,1},\dots,y_{2,k_2},\dots,y_{3s_{k_2}+1,k_2}]$ and

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2015, VOLUME 5, ISSUE 2, p.118-129

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2015, VOLUME 5, ISSUE 2, p.118-129

$v \notin (x_{4,1}x_{5,1}, \dots, x_{3r_1-1,1}x_{3r_1,1}, x_{2,2}x_{3,2}, \dots, x_{3r_2-1,2}x_{3r_2,2}, \dots, x_{2,k_1}x_{3,k_1}, \dots, x_{3r_{k_1}-1,k_1}x_{3r_{k_1},k_1}, y_{2,1}y_{3,1}, \dots, y_{3s_1,1}y_{3s_1+1,1}, \dots, y_{2,k_2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})$. Similarly, if $x_{2,2}|u$ and $x_{2,1} \nmid u$, then $u = x_{2,2}^{\alpha}v$ with $\alpha \geq 1$ and $v \in K[x_{3,1}, \dots, x_{3r_1,1}, x_{4,2}, \dots, x_{3r_2,2}, x_{2,3}, \dots, x_{3r_3,3}, \dots, x_{2,k_1}, \dots, x_{3r_{k_1},k_1}, y_{2,1}, \dots, y_{3s_1+1,1}, \dots, y_{2,k_2}, \dots, y_{3s_{k_2}+1,k_2}]$ and $v \notin (x_{3,1}x_{4,1}, \dots, x_{3r_1-1,1}x_{3r_1,1}, x_{4,2}x_{5,2}, \dots, x_{3r_2-1,2}x_{3r_2,2}, x_{2,3}x_{3,3}, \dots, x_{3r_3-1,3}x_{3r_3,3}, \dots, x_{2,k_1}x_{3,k_1}, \dots, x_{3r_{k_1}-1,k_1}x_{3r_{k_1},k_1}, y_{2,1}y_{3,1}, \dots, y_{3s_1,1}y_{3s_1+1,1}, \dots, y_{2,k_2}y_{3,k_2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})$. Other cases can be shown in a similar way as the above.

Therefore, by [13, Proposition 2.2.20, Theorem 2.2.21] and Lemma 2.6, it follows that $\operatorname{depth}(\frac{I:x_1}{I}) = \sum_{i=1}^{k_1-1} \lceil \frac{3r_i-2}{3} \rceil + \lceil \frac{3r_{k_1}-3}{3} \rceil + \sum_{i=1}^{k_2-1} \lceil \frac{3s_i-1}{3} \rceil + \lceil \frac{3s_{k_2}-2}{3} \rceil + 1 = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i$.

Now, using Lemma 2.4 on the short exact sequence

$$0 \longrightarrow (I : x_1)/I \longrightarrow S/I \longrightarrow S/(I : x_1) \longrightarrow 0, \qquad (2)$$

this completes the proof. $\square$

Assume $n = \sum_{i=1}^{k_1} 3r_i + \sum_{i=1}^{k_2} (3s_i+1) + \sum_{i=1}^{k_3} (3t_i+2) - (k_1+k_2+k_3) + 1$. We identify S/I with the $\mathbb{Z}^n$-graded K-subvector space I^c of S which is generated by all monomials $u \in S \setminus I$. Set the set

$$P = \{a \in \mathbb{N}^n : x^a \in I^c \text{ and } x^a | x_1 \prod_{\substack{2 \leq i \leq 3r_j, \\ 1 \leq j \leq k_1}} x_{i,j} \prod_{\substack{2 \leq i \leq 3s_j+1, \\ 1 \leq j \leq k_2}} y_{i,j} \prod_{\substack{2 \leq i \leq 3t_j+2 \\ 1 \leq j \leq k_3}} z_{i,j}\},$$

where $a = (a(1), a(2,1), \dots, a(3r_1,1), \dots, a(2,k_1), \dots, a(3r_{k_1},k_1), b(2,1), \dots, b(3s_1+1,1), \dots, b(2,k_2), \dots, b(3s_{k_2}+1,k_2), c(2,1), \dots, c(3t_1+2,1), \dots, c(2,k_3), \dots, c(3t_{k_3}+2,k_3)) \in \mathbb{N}^n$ and

$$x^a = x_1^{a(1)} \prod_{\substack{2 \leq i \leq 3r_j, \\ 1 \leq j \leq k_1}} x_{i,j}^{a(i,j)} \prod_{\substack{2 \leq i \leq 3s_j+1, \\ 1 \leq j \leq k_2}} y_{i,j}^{b(i,j)} \prod_{\substack{2 \leq i \leq 3t_j+2 \\ 1 \leq j \leq k_3}} z_{i,j}^{c(i,j)}.$$

Consider the natural partial order on $\mathbb{N}^n$ which is given by componentwise comparison, i.e. if $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$, then $a \geq b$ if and only if $a_i \geq b_i$ for all $i = 1, \dots, n$. With respect to the partial order induced on P, it becomes a poset where $a \geq a'$ if and only if $x^{a'}|x^a$.

Let $\mathcal{P} : P = \bigcup_{i=1}^{r} [F_i, G_i]$ be a partition of P, We denote $\operatorname{sdepth}(\mathcal{P}) = \min\{|G_i| : 1 \leq i \leq r\}$. Also, we define the Stanley depth of P, to be the number

$$\operatorname{sdepth}(P) = \max\{\operatorname{sdepth}(\mathcal{P}) \mid \mathcal{P} \text{ is a partition of } P\}.$$

Herzog, Vlădoiu and Zheng proved in [6] that $\operatorname{sdepth}(\frac{S}{I}) = \operatorname{sdepth}(P)$. Now, for $d \in \mathbb{N}$ and $\sigma \in P$, we denote

$$\mathcal{P}_d := \{a \in P : |a| = d\} \quad \text{and} \quad \mathcal{P}_{d,\sigma} := \{a \in \mathcal{P}_d : x^{\sigma} | x^a\},$$

where for $a = (a(1), a(2,1), \dots, a(3r_1,1), \dots, a(2,k_1), \dots, a(3r_{k_1},k_1), b(2,1), \dots, b(3s_1+1,1), \dots, b(2,k_2), \dots, b(3s_{k_2}+1,k_2), c(2,1), \dots, c(3t_1+2,1), \dots, c(2,k_3), \dots, c(3t_{k_3}+2,k_3)) \in \mathbb{N}^n$, $|a| := a(1) + \sum_{j=1}^{k_1} \sum_{i=2}^{3r_j} a(i,j) + \sum_{j=1}^{k_2} \sum_{i=2}^{3s_j+1} b(i,j) + \sum_{j=1}^{k_3} \sum_{i=2}^{3t_j+2} c(i,j)$.

With these notations, we are able to prove the following result.

Theorem 3.4. *Let G be a graph as in Theorem 3.3 and I be its edge ideal. Then*

$$sdepth\,(\frac{S}{I}) \le \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i + 1.$$

Proof. Firstly, we claim: if $\sigma \in \mathcal{P}$ such that $\mathcal{P}_{d,\sigma} = \emptyset$, then sdepth $(\mathcal{P}) < d$.

Indeed, let $\mathcal{P} : P = \bigcup_{i=1}^{r} [F_i, G_i]$ be a partition of P with sdepth $(\mathcal{P}) =$ sdepth (P). Since $\sigma \in P$, it follows that $\sigma \in [F_i, G_i]$ for some i. If $|G_i| \ge d$, then it follows that $\mathcal{P}_{d,\sigma} \ne \emptyset$, since there exist some subsets in the interval $[F_i, G_i]$ of cardinality d which contain σ, a contradiction! Thus, $|G_i| < d$ and therefore sdepth $(\mathcal{P}) < d$.

We set $d = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i + 1$ and $\sigma = (a(1), a(2,1), \ldots, a(3r_1, 1), \ldots, a(2, k_1), \ldots, a(3r_{k_1}, k_1),$ $b(2,1), \ldots, b(3s_1{+}1, 1), \ldots, b(2, k_2), \ldots, b(3s_{k_2}{+}1, k_2), c(2,1), \ldots, c(3t_1{+}2, 1), \ldots, c(2, k_3), \ldots, c(3t_{k_3}{+}2, k_3))$ $\in \mathbb{N}^n$, where $a(1) = 1$, for any $1 \le i \le k_1$, $1 \le j \le r_i - 1$, $a(l,i) = \begin{cases} 1 : l = 3j+1 \text{ } or \text{ } l = 3r_i \\ 0 : otherwise \end{cases}$, for any $1 \le i \le k_2$, $1 \le j \le s_i$, $b(l,i) = \begin{cases} 1 : l = 3j+1 \\ 0 : otherwise \end{cases}$, and for any $1 \le i \le k_3$, $1 \le j \le t_i$, $c(l,i) = \begin{cases} 1 : l = 3j+1 \\ 0 : otherwise \end{cases}$. We obtain that $\mathcal{P}_{d+1,\sigma} = \emptyset$. Indeed, if monomial

$$u = x_1 \prod_{i=1}^{k_1} (x_{3r_i,i} \prod_{j=1}^{r_i - 1} x_{3j+1,i}) \prod_{i=1}^{k_2} (\prod_{j=1}^{s_i} y_{3j+1,i}) \prod_{i=1}^{k_3} (\prod_{j=1}^{t_i} z_{3j+1,i}),$$

one can easily see that if $a = (a(1), a(2,1), \ldots, a(3r_1, 1), \ldots, a(2, k_1), \ldots, a(3r_{k_1}, k_1), b(2,1), \ldots,$ $b(3s_1 + 1, 1), \ldots, b(2, k_2), \ldots, b(3s_{k_2}{+}1, k_2), c(2,1), \ldots, c(3t_1 + 2, 1), \ldots, c(2, k_3), \ldots, c(3t_{k_3}{+}\,2, k_3)) \in \mathbb{N}^n$ such that $a(l,i) \ne 0$ for some $l \notin \{3j+1, 3r_i \mid 1 \le j \le r_i - 1, 1 \le i \le k_1\}$ or $b(l,i) \ne 0$ for some $l \notin \{3j+1 \mid 1 \le j \le s_i, 1 \le i \le k_2\}$ or $c(l,i) \ne 0$ for some $l \notin \{3j+1 \mid 1 \le j \le t_i, 1 \le i \le k_3\}$, then $u \cdot x^a \in I$. Therefore, by previous remark, sdepth $(\frac{S}{I}) =$ sdepth $(\mathcal{P}) \le d = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i + 1$, as required. ☐

4. Depth and Stanley depth of the edge ideals of some m-cyclic graphs with a common vertex

In this section, we will give some formulas for depth and Stanley depth of the quotient rings of the edge ideals of some m-cyclic graphs with a common vertex. We assume that G is the m-cyclic graph formed by joining m cycles $C_{3r_1}, \ldots, C_{3r_{k_1}}, C_{3s_1+1}, \ldots, C_{3s_{k_2}+1}, C_{3t_1+2}, \ldots, C_{3t_{k_3}+2}$ at a common vertex, where $k_1 + k_2 + k_3 = m$ and $k_i \ge 0$ for $i = 1, 2, 3$. We adopt the following notation to edges of graph G:
$E(C_{3r_i,i}) = \{x_1x_{2,i}, x_{2,i}x_{3,i}, \ldots, x_{3r_i,i}x_1\}$ for any $1 \le i \le k_1$ and $r_1 \le r_2 \le \cdots \le r_{k_1}$,
$E(C_{3s_i+1,i}) = \{x_1y_{2,i}, y_{2,i}y_{3,i}, \ldots, y_{3s_i+1,i}x_1\}$ for any $1 \le i \le k_2$ and $s_1 \le \cdots \le s_{k_2}$,
$E(C_{3t_i+2,i}) = \{x_1z_{2,i}, z_{2,i}z_{3,i}, \ldots, z_{3t_i+2,i}x_1\}$ for all $1 \le i \le k_3$ and $t_1 \le t_2 \le \cdots \le t_{k_3}$.
Let K be any field, $S = K[x_1, x_{2,1}, \ldots, x_{3r_1,1}, \ldots, x_{2,k_1}, \ldots, x_{3r_{k_1},k_1}, y_{2,1}, \ldots, y_{3s_1+1,1}, \ldots, y_{2,k_2}, \ldots, y_{3s_{k_2}+1,k_2},$ $z_{2,1}, \ldots, z_{3t_1+2,1}, \ldots, z_{2,k_3}, \ldots, z_{3t_{k_3}+2,k_3}]$ the polynomial ring. Then the edge ideal of graph G is $J =$ $(x_1x_{2,1}, x_{2,1}x_{3,1}, \ldots, x_{3r_1,1}x_1, \ldots, x_1x_{2,k_1}, x_{2,k_1}x_{3,k_1}, \ldots, x_{3r_{k_1},k_1}x_1, x_1y_{2,1}, y_{2,1}y_{3,1}, \ldots, y_{3s_1+1,1}x_1, \ldots,$ $x_1y_{2,k_2}, y_{2,k_2}y_{3,k_2}, \ldots, y_{3s_{k_2}+1,k_2}x_1, x_1z_{2,1}, z_{2,1}z_{3,1}, \ldots, z_{3t_1+2,1}x_1, \ldots, x_1z_{2,k_3}, z_{2,k_3}z_{3,k_3}, \ldots, z_{3t_{k_3}+2,k_3}x_1)$.

Example 4.1. The following graph G is the union of 5 circle graphs C_3, C_4, C_5, C_6 and C_7 with a common vertex x_1.

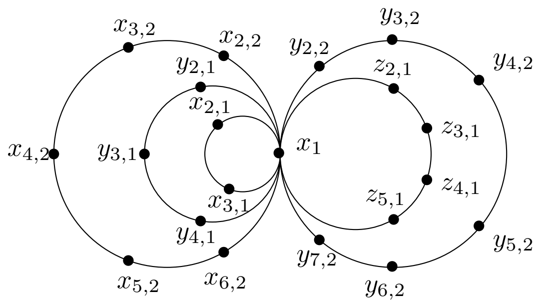

Figure 2

The edge ideal of graph G is $J = (x_1x_{2,1}, x_{2,1}x_{3,1}, x_{3,1}x_1, x_1x_{2,2}, x_{2,2}x_{3,2}, x_{3,2}x_{4,2}, x_{4,2}x_{5,2}, x_{5,2}x_{6,2}, x_{6,2}x_1, x_1y_{2,1}, y_{2,1}y_{3,1}, y_{3,1}y_{4,1}, y_{4,1}x_1, x_1y_{2,2}, y_{2,2}y_{3,2}, y_{3,2}y_{4,2}, y_{4,2}y_{5,2}, y_{5,2}y_{6,2}, y_{6,2}y_{7,2}, y_{7,2}x_1, x_1z_{2,1}, z_{2,1}z_{3,1}, z_{3,1}z_{4,1}, z_{4,1}z_{5,1}, z_{5,1}x_1)$.

Now, we prove the main results of this section.

Theorem 4.2. *Let G be a graph consisting of the union of cycle graphs $C_{3r_1}, \dots, C_{3r_{k_1}}, C_{3s_1+1}, \dots, C_{3s_{k_2}+1}$ and $C_{3t_1+2}, \dots, C_{3t_{k_3}+2}$ with a common vertex x_1, where $k_i \geq 0$ for $i = 1, 2, 3$. Let J be its edge ideal. Then*

$$sdepth(\frac{S}{J}) \geq depth(\frac{S}{J}) = \begin{cases} \sum_{i=1}^{k_2} s_i, & \text{if } k_1 = k_3 = 0, \\ \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i - k_1 + 1, & \text{otherwise.} \end{cases}$$

In particular, S/J satisfies the Stanley conjecture.

Proof. Notice that $(J : x_1) = (x_{2,1}, \dots, x_{2,k_1}, x_{3r_1,1}, \dots, x_{3r_{k_1},k_1}, y_{2,1}, \dots, y_{2,k_2}, y_{3s_1+1,1}, \dots, y_{3s_{k_2}+1,k_2}, z_{2,1}, \dots, z_{2,k_3}, z_{3t_1+2,1}, \dots, z_{3t_{k_3}+2,k_3}, x_{3,1}x_{4,1}, \dots, x_{3r_1-2,1}x_{3r_1-1,1}, \dots, x_{3,k_1}x_{4,k_1}, \dots, x_{3r_{k_1}-2,k_1}x_{3r_{k_1}-1,k_1}, y_{3,1}y_{4,1}, \dots, y_{3s_1-1,1}y_{3s_1,1}, \dots, y_{3,k_2}y_{4,k_2}, \dots, y_{3s_{k_2}-1,k_2}y_{3s_{k_2},k_2}, z_{3,1}z_{4,1}, \dots, z_{3t_1,1}z_{3t_1+1,1}, \dots, z_{3,k_3}z_{4,k_3}, \dots, z_{3t_{k_3},k_3}z_{3t_{k_3}+1,k_3})$ and $(J, x_1) = (x_{2,1}x_{3,1}, \dots, x_{3r_1-1,1}x_{3r_1,1}, \dots, x_{2,k_1}x_{3,k_1}, \dots, x_{3r_{k_1}-1,k_1}x_{3r_{k_1},k_1}, y_{2,1}y_{3,1}, \dots, y_{3s_1,1}y_{3s_1+1,1}, \dots, y_{2,k_2}y_{3,k_2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2}, z_{2,1}z_{3,1}, \dots, z_{3t_1+1,1}z_{3t_1+2,1}, \dots, z_{2,k_3}z_{3,k_3}, \dots, z_{3t_{k_3}+1,k_3}z_{3t_{k_3}+2,k_3}, x_1)$, thus we get that

$$\frac{S}{(J:x_1)} \cong \frac{K[x_{3,1}, \dots, x_{3r_1-1,1}]}{(x_{3,1}x_{4,1}, \dots, x_{3r_1-2,1}x_{3r_1-1,1})} \otimes_K \dots \otimes_K \frac{K[x_{3,k_1}, \dots, x_{3r_{k_1}-1,k_1}]}{(x_{3,k_1}x_{4,k_1}, \dots, x_{3r_{k_1}-2,k_1}x_{3r_{k_1}-1,k_1})}$$
$$\otimes_K \frac{K[y_{3,1}, \dots, y_{3s_1,1}]}{(y_{3,1}y_{4,1}, \dots, y_{3s_1-1,1}y_{3s_1,1})} \otimes_K \dots \otimes_K \frac{K[y_{3,k_2}, \dots, y_{3s_{k_2},k_2}]}{(y_{3,k_2}y_{4,k_2}, \dots, y_{3s_{k_2}-1,k_2}y_{3s_{k_2},k_2})}$$
$$\otimes_K \frac{K[z_{3,1}, \dots, z_{3t_1+1,1}]}{(z_{3,1}z_{4,1}, \dots, z_{3t_1,1}z_{3t_1+1,1})} \otimes_K \dots \otimes_K \frac{K[z_{3,k_3}, \dots, z_{3t_{k_3}+1,k_3}]}{(z_{3,k_3}z_{4,k_3}, \dots, z_{3t_{k_3},k_3}z_{3t_{k_3}+1,k_3})} \otimes_K K[x_1],$$

and

$$\frac{S}{(J,x_1)} \cong \frac{K[x_{2,1}, \dots, x_{3r_1,1}]}{(x_{2,1}x_{3,1}, \dots, x_{3r_1-1,1}x_{3r_1,1})} \otimes_K \dots \otimes_K \frac{K[x_{2,k_1}, \dots, x_{3r_{k_1},k_1}]}{(x_{2,k_1}x_{3,k_1}, \dots, x_{3r_{k_1}-1,k_1}x_{3r_{k_1},k_1})}$$
$$\otimes_K \frac{K[y_{2,1}, \dots, y_{3s_1+1,1}]}{(y_{2,1}y_{3,1}, \dots, y_{3s_1,1}y_{3s_1+1,1})} \otimes_K \dots \otimes_K \frac{K[y_{2,k_2}, \dots, y_{3s_{k_2}+1,k_2}]}{(y_{2,k_2}y_{3,k_2}, \dots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})}$$

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2015, VOLUME 5, ISSUE 2, p.118-129

$$\otimes_K \frac{K[z_{2,1},\dots,z_{3t_1+2,1}]}{(z_{2,1}z_{3,1},\dots,z_{3t_1+1,1}z_{3t_1+2,1})}\otimes_K\cdots\otimes_K \frac{K[z_{2,k_3},\dots,z_{3t_{k_3}+2,k_3}]}{(z_{2,k_3}z_{3,k_3},\dots,z_{3t_{k_3}+1,k_3}z_{3t_{k_3}+2,k_3})}.$$

Therefore, by Lemmas 2.6, 2.7, 3.2 and [13, Proposition 2.2.20, Theorem 2.2.21], we obtain that $\operatorname{sdepth}(\frac{S}{J:x_1}) \geq \operatorname{depth}(\frac{S}{J:x_1}) = \sum_{i=1}^{k_1}\lceil\frac{3r_i-3}{3}\rceil + \sum_{i=1}^{k_2}\lceil\frac{3s_i-2}{3}\rceil + \sum_{i=1}^{k_3}\lceil\frac{3t_i-1}{3}\rceil + 1 = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i - k_1 + 1$, and $\operatorname{sdepth}(\frac{S}{(J,x_1)}) \geq \operatorname{depth}(\frac{S}{(J,x_1)}) = \sum_{i=1}^{k_1}\lceil\frac{3r_i-1}{3}\rceil + \sum_{i=1}^{k_2}\lceil\frac{3s_i}{3}\rceil + \sum_{i=1}^{k_3}\lceil\frac{3t_i+1}{3}\rceil = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i + k_3$.

Using Lemma 2.5 on the short exact sequence

$$(3) \qquad 0 \longrightarrow S/(J:x_1) \xrightarrow{\cdot x_1} S/J \longrightarrow S/(J,x_1) \longrightarrow 0,$$

we conclude that $\operatorname{sdepth}(\frac{S}{J}) \geq \begin{cases} \sum_{i=1}^{k_2} s_i, & \text{if } k_1 = k_3 = 0,\\ \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i - k_1 + 1, & \text{otherwise.}\end{cases}$

If $k_1 \neq 0$ or $k_3 \neq 0$, then $\operatorname{depth}(\frac{S}{(J,x_1)}) \geq \operatorname{depth}(\frac{S}{(J:x_1)})$. Using Lemma 2.4 on the short exact sequence (3), it follows that $\operatorname{depth}(\frac{S}{J}) = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i - k_1 + 1$.

Assume that $k_1 = k_3 = 0$. We claim that there exists the S-module isomorphism

$$\begin{aligned}
\frac{(J:x_1)}{J} \cong\ & y_{2,1}\Big(\frac{K[y_{4,1},\dots,y_{3s_1+1,1},y_{2,2},\dots,y_{3s_2+1,2},\dots,y_{2,k_2},\dots,y_{3s_{k_2}+1,k_2}]}{(y_{4,1}y_{5,1},\dots,y_{3s_1,1}y_{3s_1+1,1},y_{2,2}y_{3,2},\dots,y_{2,k_2}y_{3,k_2},\dots,y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})}\Big)[y_{2,1}]\\
&\bigoplus y_{3s_1+1,1}\Big(\frac{K[y_{3,1},\dots,y_{3s_1-1,1},y_{2,2},\dots,y_{3s_2+1,2},\dots,y_{2,k_2},\dots,y_{3s_{k_2}+1,k_2}]}{(y_{3,1}y_{4,1},\dots,y_{3s_1-2,1}y_{3s_1-1,1},\dots,y_{2,k_2}y_{3,k_2},\dots,y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})}\Big)[y_{3s_1+1,1}]\\
&\bigoplus y_{2,2}\Big(\frac{K[y_{3,1},\dots,y_{3s_1,1},y_{4,2},\dots,y_{3s_2+1,2},\dots,y_{2,k_2},\dots,y_{3s_{k_2}+1,k_2}]}{(y_{3,1}y_{4,1},\dots,y_{3s_1-1,1}y_{3s_1,1},y_{4,2}y_{5,2},\dots,y_{3s_2,2}y_{3s_2+1,2},\dots,y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})}\Big)[y_{2,2}]\\
&\bigoplus y_{3s_2+1,2}\Big(\frac{K[y_{3,1},\dots,y_{3s_1,1},y_{3,2},\dots,y_{3s_2-1,2},\dots,y_{2,k_2},\dots,y_{3s_{k_2}+1,k_2}]}{(y_{3,1}y_{4,1},\dots,y_{3,2}y_{4,2},\dots,y_{3s_2-2,2}y_{3s_2-1,2},\dots,y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})}\Big)[y_{3s_2+1,2}]\\
&\bigoplus\cdots\\
&\bigoplus y_{2,k_2}\Big(\frac{K[y_{3,1},\dots,y_{3s_1,1},\dots,y_{3,k_2-1},\dots,y_{3s_{k_2-1},k_2-1},y_{4,k_2},\dots,y_{3s_{k_2}+1,k_2}]}{(y_{3,1}y_{4,1},\dots,y_{3s_1-1,1}y_{3s_1,1},\dots,y_{4,k_2}y_{5,k_2},\dots,y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})}\Big)[y_{2,k_2}]\\
&\bigoplus y_{3s_{k_2}+1,k_2}\Big(\frac{K[y_{3,1},\dots,y_{3s_1,1},\dots,y_{3,k_2-1},\dots,y_{3s_{k_2-1},k_2-1},y_{3,k_2},\dots,y_{3s_{k_2}-1,k_2}]}{(y_{3,1}y_{4,1},\dots,y_{3s_1-1,1}y_{3s_1,1},\dots,y_{3,k_2}y_{4,k_2},\dots,y_{3s_{k_2}-2,k_2}y_{3s_{k_2}-1,k_2})}\Big)[y_{3s_{k_2}+1,k_2}]
\end{aligned}$$

Indeed, if $u \in (J:x_1)$ is a monomial such that $u \notin J$, then there exists some $i \in \{1,\dots,k_2\}$ such that $y_{j,i} \mid u$, where $j = 2$ or $3s_i+1$.

If $y_{2,1}|u$, then we can write u as $u = y_{2,1}^{\alpha}v$ with $\alpha \geq 1$ and $y_{2,1} \nmid v$. Since $u \notin J$, we have that $v \in K[y_{4,1},\dots,y_{3s_1+1,1},y_{2,2},\dots,y_{3s_2+1,2},\dots,y_{2,k_2},\dots,y_{3s_{k_2}+1,k_2}]$ and $v \notin (y_{4,1}y_{5,1},\dots,y_{3s_1,1}y_{3s_1+1,1},y_{2,2}y_{3,2},\dots,y_{3s_2,2}y_{3s_2+1,2},\dots,y_{2,k_2}y_{3,k_2},\dots,y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})$. Similarly, if $y_{3s_1+1,1}|u$ and $y_{2,1} \nmid u$, then $u = y_{3s_1+1,1}^{\alpha}v$ with $\alpha \geq 1$ and $v \in K[y_{3,1},\dots,y_{3s_1-1,1},y_{2,2},\dots,y_{3s_2+1,2},\dots,y_{2,k_2},\dots,y_{3s_{k_2}+1,k_2}]$ and $v \notin (y_{3,1}y_{4,1},\dots,y_{3s_1-2,1}y_{3s_1-1,1},y_{2,2}y_{3,2},\dots,y_{3s_2,2}y_{3s_2+1,2},\dots,y_{2,k_2}y_{3,k_2},\dots,y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})$. If u is a monomial such that $y_{2,2}|u$, $y_{2,1} \nmid u$ and $y_{3s_1+1,1} \nmid u$, then we have that $u = y_{2,2}^{\alpha}v$ with $\alpha \geq 1$ and $v \in K[y_{3,1},\dots,y_{3s_1,1},y_{4,2},\dots,y_{3s_2+1,2},\dots,y_{2,k_2},\dots,y_{3s_{k_2}+1,k_2}]$ and $v \notin (y_{3,1}y_{4,1},\dots,y_{3s_1-1,1}y_{3s_1,1},y_{4,2}y_{5,2},\dots,y_{3s_2,2}y_{3s_2+1,2},\dots,y_{2,k_2}y_{3,k_2},\dots,y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})$. Other cases can be shown in a similar way as the above.

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2015, VOLUME 5, ISSUE 2, p.118-129

Therefore, by [13, Proposition 2.2.20, Theorem 2.2.21] and Lemma 2.6, it follows that depth $(\frac{J:x_1}{J}) = \sum_{i=1}^{k_2-1}\lceil\frac{3s_i-2}{3}\rceil + \lceil\frac{3s_{k_2}-3}{3}\rceil + 1 = \sum_{i=1}^{k_2} s_i$.

Now, using Lemma 2.4 on the short exact sequence

$$0 \longrightarrow (J : x_1)/J \longrightarrow S/J \longrightarrow S/(J : x_1) \longrightarrow 0, \tag{4}$$

this completes the proof. □

Theorem 4.3. *Let G be a graph as in Theorem 4.2 and J be its edge ideal. Then*

$$sdepth(\frac{S}{J}) \leq \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i - k_1 + 1.$$

Proof. Let $d = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i - k_1 + 1$ and $\sigma = (a(1), a(2,1), \ldots, a(3r_1, 1), \ldots, a(2, k_1), \ldots, a(3r_{k_1}, k_1),$ $b(2,1), \ldots, b(3s_1+1, 1), \ldots, b(2, k_2), \ldots, b(3s_{k_2}+1, k_2), c(2,1), \ldots, c(3t_1+2, 1), \ldots, c(2, k_3), \ldots, c(3t_{k_3}+2, k_3))$ $\in \mathbb{N}^n$, where $a(1) = 1$, for any $1 \leq i \leq k_1$, $1 \leq j \leq r_i - 1$, $a(l,i) = \begin{cases} 1 : l = 3j+1 \\ 0 : otherwise \end{cases}$, for any $1 \leq i \leq k_2$, $1 \leq j \leq s_i - 1$, $b(l,i) = \begin{cases} 1 : l = 3j+1 \text{ or } l = 3s_i \\ 0 : otherwise \end{cases}$, and for any $1 \leq i \leq k_3$, $1 \leq j \leq t_i$, $c(l,i) = \begin{cases} 1 : l = 3j+1 \\ 0 : otherwise \end{cases}$. From the proof of Theorem 3.4, it is enough to prove that $\mathcal{P}_{d+1,\sigma} = \emptyset$. Indeed, if monomial

$$u = x_1 \prod_{i=1}^{k_1}(\prod_{j=1}^{r_i-1} x_{3j+1,i}) \prod_{i=1}^{k_2}(y_{3s_i,i} \prod_{j=1}^{s_i-1} y_{3j+1,i}) \prod_{i=1}^{k_3}(\prod_{j=1}^{t_i} z_{3j+1,i}),$$

one can easily see that if $a = (a(1), a(2,1), \ldots, a(3r_1, 1), \ldots, a(2, k_1), \ldots, a(3r_{k_1}, k_1), b(2,1), \ldots,$ $b(3s_1+1, 1), \ldots, b(2, k_2), \ldots, b(3s_{k_2}+1, k_2), c(2,1), \ldots, c(3t_1+2, 1), \ldots, c(2, k_3), \ldots, c(3t_{k_3}+2, k_3)) \in \mathbb{N}^n$ such that $a(l,i) \neq 0$ for some $l \notin \{3j+1 \mid 1 \leq j \leq r_i - 1, 1 \leq i \leq k_1\}$ or $b(l,i) \neq 0$ for some $l \notin \{3j+1, 3s_i \mid 1 \leq j \leq s_i, 1 \leq i \leq k_2\}$ or $c(l,i) \neq 0$ for some $l \notin \{3j+1 \mid 1 \leq j \leq t_i, 1 \leq i \leq k_3\}$, then $u \cdot x^a \in I$. Therefore $\mathcal{P}_{d+1,\sigma} = \emptyset$, thus we obtain the required result. □

Theorem 4.4. *Let G be a graph as in Theorem 4.2. Then*

$$sdepth(\frac{J}{I}) \geq depth(\frac{J}{I}) = \begin{cases} \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_3} t_i - k_1 + 2, & \textit{if } k_2 = 0, \\ \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i - k_1 + 1, & \textit{otherwise.} \end{cases}$$

In particular, J/I satisfies the Stanley conjecture.

Proof. We have the S-module isomorphism:

$$\frac{J}{I} \cong \bigoplus_{i=1}^{k_1} x_1 x_{3r_i,i}\Big(\frac{K[x_{3,1},\ldots,x_{3r_1-1,1},\ldots,x_{3,i-1},\ldots,x_{3r_{i-1}-1,i-1},x_{3,i},\ldots,\widehat{x}_{3r_{i-1},i},x_{3,i+1},\ldots,x_{3r_{i+1},i+1},\ldots,x_{3,k_1},\ldots,x_{3r_{k_1},k_1}]}{(x_{3,1}x_{4,1},\ldots,x_{3r_1-2,1}x_{3r_1-1,1},\ldots,x_{3,i}x_{4,i},\ldots,x_{3r_i-3,i}x_{3r_i-2,i},x_{3,i+1}x_{4,i+1},\ldots,x_{3r_{k_1}-1,k_1}x_{3r_{k_1},k_1})}$$

$$\otimes_K \frac{K[y_{3,1},\ldots,y_{3s_1+1,1},\ldots,y_{3,k_2},\ldots,y_{3s_{k_2}+1,k_2}]}{(y_{3,1}y_{4,1},\ldots,y_{3s_1,1}y_{3s_1+1,1},\ldots,y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})} \otimes_K \frac{K[z_{3,1},\ldots,z_{3t_1+2,1},\ldots,z_{3,k_3},\ldots,z_{3t_{k_3}+2,k_3}]}{(z_{3,1}z_{4,1},\ldots,z_{3t_1+1,1}z_{3t_1+2,1},\ldots,z_{3s_{k_3}+1,k_3}z_{3t_{k_3}+2,k_3})}\Big)[x_1, x_{3r_i,i}]$$

$$\oplus \Big(\bigoplus_{i=1}^{k_2} x_1 y_{3s_i+1,i}\Big(\frac{K[y_{3,1},\ldots,y_{3s_1,1},\ldots,y_{3,i-1},\ldots,y_{3s_{i-1},i-1},y_{3,i},\ldots,\widehat{y}_{3s_i,i},y_{3,i+1},\ldots,y_{3s_{i+1}+1,i+1},\ldots,y_{3,k_2},\ldots,y_{3s_{k_2}+1,k_2}]}{(y_{3,1}y_{4,1},\ldots,y_{3s_1-1,1}y_{3s_1,1},\ldots,y_{3,i}y_{4,i},\ldots,y_{3s_i-2,i}y_{3s_i-1,i},y_{3,i+1}y_{4,i+1},\ldots,y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2})} \otimes_K$$

$$\frac{K[x_{3,1},\ldots,x_{3r_1-1,1},\ldots,x_{3,k_1},\ldots,x_{3r_{k_1}-1,k_1}]}{(x_{3,1}x_{4,1},\ldots,x_{3r_1-2,1}x_{3r_1-1,1},\ldots,x_{3r_{k_1}-2,k_1}x_{3r_{k_1}-1,k_1})} \otimes_K \frac{K[z_{3,1},\ldots,z_{3t_1+2,1},\ldots,z_{3,k_3},\ldots,z_{3t_{k_3}+2,k_3}]}{(z_{3,1}z_{4,1},\ldots,z_{3t_1+1,1}z_{3t_1+2,1},\ldots,z_{3s_{k_3}+1,k_3}z_{3t_{k_3}+2,k_3})})[x_1,y_{3s_i+1,i}])$$

$$\oplus\,(\bigoplus_{i=1}^{k_3} x_1z_{3t_i+2,i}(\frac{K[z_{3,1},\ldots,z_{3t_1+1,1},\ldots,z_{3,i-1},\ldots,z_{3t_{i-1}+1,i-1},z_{3,i},\ldots,\widehat{z}_{3t_i+1,i},z_{3,i+1},\ldots,z_{3t_{i+1}+2,i+1},\ldots,z_{3,k_3},\ldots,z_{3t_{k_3}+2,k_3}]}{(z_{3,1}z_{4,1},\ldots,z_{3t_1,1}z_{3t_1+1,1},\ldots,z_{3,i}z_{4,i},\ldots,z_{3t_i-1,i}z_{3t_i,i},z_{3,i+1}z_{4,i+1},\ldots,z_{3t_{k_3}+1,k_3}z_{3t_{k_3}+2,k_3})} \otimes_K$$

$$\frac{K[x_{3,1},\ldots,x_{3r_1-1,1},\ldots,x_{3,k_1},\ldots,x_{3r_{k_1}-1,k_1}]}{(x_{3,1}x_{4,1},\ldots,x_{3r_1-2,1}x_{3r_1-1,1},\ldots,x_{3r_{k_1}-2,k_1}x_{3r_{k_1}-1,k_1})} \otimes_K \frac{K[y_{3,1},\ldots,y_{3s_1,1},\ldots,y_{3,k_2},\ldots,y_{3s_{k_2},k_2}]}{(y_{3,1}y_{4,1},\ldots,y_{3s_1-1,1}y_{3s_1,1},\ldots,y_{3s_{k_2}-1,k_2}y_{3s_{k_2},k_2})})[x_1,z_{3t_i+2,i}]),$$

where $x_{i,0} = y_{j,0} = z_{l,0} = 0$ for $3 \le i \le k_1$, $3 \le j \le k_2$ and $3 \le l \le k_3$.

Indeed, let $u \in J$ be a monomial such that $u \notin I$, then $x_1x_{3r_i,i}|u$ or $x_1y_{3s_j+1,j}|u$ or $x_1z_{3t_l+2,l}|u$ for some $1 \le i \le k_1$ or $1 \le j \le k_2$ or $1 \le l \le k_3$.

If $x_1x_{3r_1,1}|u$, then we can write u as $u = x_1^{\alpha}x_{3r_1,1}^{\beta}v$ with $\alpha, \beta \ge 1$, $x_1 \nmid v$ and $x_{3r_1,1} \nmid v$. Since $u \notin I$, we have that $v \in K[x_{3,1}, \ldots, x_{3r_1-2,1}, x_{3,2}, \ldots, x_{3r_2,2}, \ldots, x_{3,k_1}, \ldots, x_{3r_{k_1},k_1}, y_{3,1}, \ldots, y_{3s_1+1,1}, \ldots, y_{3,k_2}, \ldots, y_{3s_{k_2}+1,k_2}, z_{3,1}, \ldots, z_{3t_1+2,1}, \ldots, z_{3,k_3}, \ldots, z_{3t_{k_3}+2,k_3}]$ and $v \notin (x_{3,1}x_{4,1}, \ldots, x_{3r_1-3,1}x_{3r_1-2,1}, x_{3,2}x_{4,2}, \ldots, x_{3r_2-1,2}x_{3r_2,2}, \ldots, x_{3,k_1}x_{4,k_1}, \ldots, x_{3r_{k_1}-1,k_1}x_{3r_{k_1},k_1}, y_{3,1}y_{4,1}, \ldots, y_{3s_1,1}y_{3s_1+1,1}, \ldots, y_{3,k_2}y_{4,k_2}, \ldots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2}, z_{3,1}z_{4,1}, \ldots, z_{3t_1+1,1}z_{3t_1+2,1}, \ldots, z_{3,k_3}z_{4,k_3}, \ldots, z_{3t_{k_3}+1,k_3}z_{3t_{k_3}+2,k_3})$. Similarly, if $x_1x_{3r_2,2}|u$ and $x_1x_{3r_1,1} \nmid u$, then $u = x_1^{\alpha}x_{3r_2,2}^{\beta}v$ with $\alpha, \beta \ge 1$ and $v \in K[x_{3,1}, \ldots, x_{3r_1-1,1}, x_{3,2}, \ldots, x_{3r_2-2,2}, x_{3,3}, \ldots, x_{3r_3,3}, \ldots, x_{3,k_1}, \ldots, x_{3r_{k_1},k_1}, y_{3,1}, \ldots, y_{3s_1+1,1}, \ldots, y_{3,k_2}, \ldots, y_{3s_{k_2}+1,k_2}, z_{3,1}, \ldots, z_{3t_1+2,1}, \ldots, z_{3,k_3}, \ldots, z_{3t_{k_3}+2,k_3}]$ and $v \notin (x_{3,1}x_{4,1}, \ldots, x_{3r_1-2,1}x_{3r_1-1,1}, x_{3,2}x_{4,2}, \ldots, x_{3r_2-3,2}x_{3r_2-2,2}, x_{3,3}x_{4,3}, \ldots, x_{3r_3-1,3}x_{3r_3,3}, \ldots, x_{3,k_1}x_{4,k_1}, \ldots, x_{3r_{k_1}-1,k_1}x_{3r_{k_1},k_1}, y_{3,1}y_{4,1}, \ldots, y_{3s_1,1}y_{3s_1+1,1}, \ldots, y_{3,k_2}y_{4,k_2}, \ldots, y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2}, z_{3,1}z_{4,1}, \ldots, z_{3t_1+1,1}z_{3t_1+2,1}, \ldots, z_{3,k_3}z_{4,k_3}, \ldots, z_{3t_{k_3}+1,k_3}z_{3t_{k_3}+2,k_3})$. Other cases can be shown in a similar way as the above.

Therefore, by Lemmas 2.4–2.7, 3.2 and [13, Proposition 2.2.20, Theorem 2.2.21], if $k_2 \neq 0$, then $\operatorname{sdepth}(\frac{J}{I}) \ge \operatorname{depth}(\frac{J}{I}) = \min\{\sum_{i=1}^{k_1-1}\lceil\frac{3r_i-3}{3}\rceil + \lceil\frac{3r_{k_1}-4}{3}\rceil + \sum_{i=1}^{k_2}\lceil\frac{3s_i-1}{3}\rceil + \sum_{i=1}^{k_3}\lceil\frac{3t_i}{3}\rceil, \sum_{i=1}^{k_1}\lceil\frac{3r_i-3}{3}\rceil + \sum_{i=1}^{k_2-1}\lceil\frac{3s_i-2}{3}\rceil + \lceil\frac{3s_{k_s}-3}{3}\rceil + \sum_{i=1}^{k_3}\lceil\frac{3t_i}{3}\rceil, \sum_{i=1}^{k_1}\lceil\frac{3r_i-3}{3}\rceil + \sum_{i=1}^{k_2}\lceil\frac{3s_i-2}{3}\rceil + \sum_{i=1}^{k_3-1}\lceil\frac{3t_i-1}{3}\rceil + \lceil\frac{3t_{k_s}-2}{3}\rceil\} + 2 = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_3} t_i - k_1 + 1.$

If $k_2 = 0$, then $\operatorname{sdepth}(\frac{J}{I}) \ge \operatorname{depth}(\frac{J}{I}) = \min\{\sum_{i=1}^{k_1-1}\lceil\frac{3r_i-3}{3}\rceil + \lceil\frac{3r_{k_1}-4}{3}\rceil + \sum_{i=1}^{k_3}\lceil\frac{3t_i}{3}\rceil, \sum_{i=1}^{k_1}\lceil\frac{3r_i-3}{3}\rceil + \sum_{i=1}^{k_3-1}\lceil\frac{3t_i-1}{3}\rceil + \lceil\frac{3t_{k_s}-2}{3}\rceil\} + 2 = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_3} t_i - k_1 + 2$. This completes the proof. □

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ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2015, VOLUME 5, ISSUE 2, p.118-129