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ON A REDUCED CROSSED PRODUCT OF A GROUP BY A C^*-ALGEBRA. THE CASES OF CONTINUOUS TRACE AND TYPE I REDUCED CROSSED PRODUCT

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Abstract

This paper analyzes two special cases of C^*-algebras, the cases of universal crossed product and reduced crossed product of a group by a C^*-algebra. In the hypothesis that the universal crossed product is a continuous trace C^*-algebra or a type I C^*-algebra, it is proved that the reduced crossed product is a continuous trace C^*-algebra or, respectively, a type I C^*-algebra. Moreover, these results can be extended in the case when the crossed products are obtained from a groupoid and a bundle of C^*-algebras with a constant fiber.

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1. Introduction

The *C**-algebras were considered, primarily, for their use in quantum mechanics to model algebras of physical observables, with physical observables represented by self-adjoint operators acting on a Hilbert space („state space"). For example, in quantum mechanics, an action of the set of real numbers on an operator algebra gives the time evolution of a system and it is known that the classical theory of *C**-algebras accomodates actions of groups by automorphismes. Some special cases of noncomutative *C**-algebras are the continuous trace *C**-algebras and type I *C**-algebras.

According to Dixmier, [1, Definition II.9] a *C**-algebra *A* is a continuous trace *C**-algebra if the ideal with the positive part *the continuous trace elements* (the positive elements a from A such that the mapping $\pi \to Tr(\pi(a))$ is finite and continuous on the spectrum $\hat{A}$ of *A*) is dense in *A*. In Section 2.1. of this paper, we present some properties of continuous trace *C**-algebras. Considering, (A, G, α) a

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dynamical system, we describe in Section 2.2 the construction of the universal crossed product and the reduced crossed product *C**-algebras. This construction is a classical one, and the author of this paper has used, in general, the notations from [5, Chapter 2]. The main purpose of this paper is to show that if the universal crossed product is a continuous trace *C**-algebra, then the reduced crossed product has the same propety (Proposition 3.1). Moreover, if we consider the universal crossed product obtained in the case of *G* being a groupoid, and the universal crossed product is with continuous trace, then the reduced crossed product has the same property. In Section 2.2, we present a general description of type I *C**-algebras. We recall that a *C**-algebra *A* is a type I algebra if for every representation π of *A*, the von Neumann algebra generated by $\pi(A)$ is of type I (Dixmier). Using a general characterization of type I *C**-algebras given by Proposition 2.2.5, we obtain in Proposition 3.3 a similar result to the one concerning continuous trace *C**-algebras, i.e., if the universal crossed product is a type I *C**-algebra, then the reduced crossed product is a type I *C**-algebra, too.

2. Preliminaries

2.1 Continuous trace *C**-algebras

Definition 2.1.1 *Let A be a C*-algebra. A mapping* $tr : A^+ \to [0,\infty]$, *where* A^+ *is the set of positive elements of A, is called trace if the following conditions hold:*

a) *if* $x, y \in A^+ \Rightarrow tr(x+y) = tr(x) + tr(y)$

b) *if* $x \in A^+$, λ *a positive, real number* $\Rightarrow tr(\lambda x) = \lambda tr(x)$

c) *if* $x \in A \Rightarrow tr(xx^*) = tr(x^* x)$

Remark 2.1.2 On positive operators on a Hilbert space *H* a trace can be defined, by choosing an ortonormal basis $\{e_n\}$ and by taking $tr(T) = \sum_{n\geq 1} \langle Te_n, e_n \rangle$. This trace is independent by the choice of ortonormal basis and it extends to a mapping on the ideal of continuous trace operators: $I(H) := \{T \in B(H) / tr(|T|) < \infty\}$.

Remark 2.1.3 If *A* is a *C**-algebra, by $\hat{A}$ is denoted the spectrum of *A*, i.e., the set of unitary equivalence classes of irreducible representations, endowed with the Jacobson topology. As in Remark 2.1.2, if π is an element from $\hat{A}$ we can define a trace for every positive element *x* from *A*.

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Definition 2.1.4 *If A is a C*-algebra, an element x from* A^+ *is called a continuous trace element if the map* $\pi \to Tr(\pi(x))$ *defined on* $\hat{A}$ *is finite and continuous on* $\hat{A}$.

Remark 2.1.5 Let *P* denote the set of the positive, continuous trace elements of *A*. Dixmier showed in [1] that *P* is the positive part of a two sided, self -adjoint ideal of *A* denoted *M* and that *M* is the set of linear combinations of elements from *P*. The closure of the ideal *M* is denoted by $J(A)$.

Definition 2.1.6 [Dixmier] *A continuous trace algebra A is a C*-algebra such that* $J(A) = A$.

The following theorem gives a characterization of continuous trace *C**-algebras and has been proved by Dixmier in [1].

Theorem 2.1.7 *If A is a C*-algebra, then the spectrum* $\hat{A}$ *is Hausdorff in Jacobson topology and for every* $\pi_0 \in \hat{A}$, *there exists a neighbourhood V of* π_0 *and* $a \in A^+$ *such that, for every* $\pi \in V$, $\pi(a)$ *is a rank one projector.*

2.2. Type I *C**-algebras

Definition 2.2.1 *A von Neumann algebra A is a type I von Neumann algebra if A is isomorphic to a von Neumann algebra B and the comutant B' is abelian.*

Definition 2.2.2 *A C*-algebra A is a type I C*-algebra if for every representation* π *of A, the von Neumann algebra generated by* $\pi(A)$ *is of type I.*

Remark 2.2.3 If *A* is a involutive subalgebra of $B(H)$, $B = A''$ is the von Neumann algebra generated by *A*.

Proposition 2.2.4 [2, Proposition 5.4.1] *If A is a type I C*-algebra and* π *a representation of A, then the von Neumann algebra* $\pi(A)'$ *is of type I.*

The next proposition is important for our following considerations concerning the main result about type I crossed product, and it is part of Theorem 9.1 from [2]. Using 9.5.9 from [2], we eliminate the condition of separability of *A* from Theorem 9.1 of [2].

Proposition 2.2.5 *Let A be a C*-algebra. Then the following conditions are equivalent:*

- *i)* *A is of type I;*
- *ii)* *If π is a irreducible representation of A, $\pi(A)$ contains the set of compact operators on H_{π}.*
- *iii)* *every non-zero quotient C^*-algebra of A possesses a non-zero, closed, two sided ideal I such that for every irreducible representation π of I and each $x \in A$, $\pi(x)$ is a compact operator.*

2.3. Crossed product of a group by a C^*-algebra

2.3.1. Dynamical systems

Definition 2.3.1.1 *A group G with unit e acts on a set X if there is a mapping $(s,x) \to s \cdot x$ from $G \times X$ to X such that $e \cdot x = x; s \cdot (r \cdot x) = (sr) \cdot x$ for every $s, r \in G, x \in X$.*

Remark 2.3.1.2 If *A* is a *C**-algebra, $Aut(A)$ denotes the group of automorphisms of *A*. The point norm topology on $Aut(A)$ is the topology in which a sequence $(\alpha_n)_n \in Aut(A)$ converges to $\alpha \in Aut(A)$ if and only if, for every $a \in A$, $\alpha_n(a) \to \alpha(a)$.

Definition 2.3.1.3 *A triplet (A, G, α) is a C*-dynamical system if A is a C*-algebra, G is a locally compact group and $\alpha : G \to Aut(A)$ is a continuous homomorphism.*

2.3.2. Covariant representations

Definition 2.3.2.1 *Let (A, G, α) a C*-dynamical system. A pair (π, U) is called covariant representation of (A, G, α) if $\pi : A \to B(H)$ is a representation of A on a Hilbert space H and $U : G \to U(H)$ is a unitary representation of group G on the same Hilbert space H such that $\pi(\alpha_s(a)) = U_s \pi(a) U_s^*$. We say that (π, U) is a nondegenerate covariant representation if π is a nondegenerate representation.*

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The following example of covariant representation is important for the construction of reduced crossed product.

Example 2.3.2.2 Let (A, G, α) a *C**-dynamical system and $\rho: A \rightarrow B(H_\rho)$ a representation of *A* on the Hilbert space H_ρ. Then define $Ind_e^G \rho$ to be the pair $(\overline{\rho}, U)$, where $\overline{\rho}$ is a representation of *A*, and *U* is a representation of *G* on the Hilbert space $L^2(G, H_\rho)$ and the following relations are hold:

$$\overline{\rho}(a)h(r) = \rho(\alpha_r{}^{-1}(a))(h(r)), U_s h(r) = h(s^{-1}r).$$

The representation $Ind_e^G \rho = (\overline{\rho}, U)$ is a covariant representation of (A, G, α) called the regular representation of (A, G, α).

2.3.3 The crossed product of a group by a *C**-algebra

In the following propositions, we consider (A, G, α) a *C**-dynamical system and we denote by $C_c(G, A)$ the linear space of continuous, compactly supported functions from *G* to *A*. As in [5], we define the following operations of *convolution* and *involution* on $C_c(G, A)$:

$$(f * g)(s) = \int_G f(r)\alpha_r(g(r^{-1}s))d\mu(r) \text{ (convolution)}$$

$$f^*(s) = \Delta(s^{-1})\alpha_s(f(s^{-1})^*) \text{ (involution), where }$$ μ is a Haar measure asociated to the group *G* and Δ is its modular function. Moreover, the mapping $\|\cdot\|_1 : C_c(G, A) \rightarrow R$, $\| f \|_1 = \int_G \| f(s) \| d\mu(s)$ is a norm on $C_c(G, A)$ and the properties of Haar measure guarantee that $\| f^* \|_1 = \| f \|_1, \| f * g \|_1 \leq \| f \|_1 \| g \|_1$.

Together with the norm $\|\cdot\|_1$, the convolution and involution defined above give $C_c(G, A)$ a structure of involutive algebra.

Definition 2.3.3.1 *A* $*$*-representation from* $C_c(G, A)$ *to Hilbert space H is a* $*$*-homomorphism* $\pi: C_c(G, A) \rightarrow B(H)$. *We call* π *a nondegenerate representation if the set* $\{\pi(f)h / f \in C_c(G, A), h \in H\}$ *is dense in H.*

Proposition 2.3.3.2 *Let* (A, G, α) *a C*-dynamical system and* (π, U) *a covariant representation of* (A, G, α) *in the Hilbert space H. Then the mapping* $(\pi \times U)(f) := \int_G \pi(f(s))U_s d\mu(s)$ *from* $C_c(G, A)$ *to* $B(H)$ *defines a* $*$*-representation called the integrated form of the covariant representation* (π, U). *Moreover,* $\| \pi \times U(f) \| \leq \| f \|_1$, *and if* π *is nondegenerate,* $\pi \times U$ *is nondegenerate too.*

Proposition 2.3.3.3 *Let* (A, G, α) *a C*-dynamical system, and for every* $f \in C_c(G, A)$ *we define:* $\| f \| = \sup\{ \| \pi \times U(f) \|, (\pi, U)$ *covariant representation of* $(A, G, \alpha)\}$.

Then the mapping $\|\cdot\|$ *is a norm on* $C_c(G, A)$ *called the universal norm. The universal norm is dominated by* $\|\cdot\|_1$, *and the completion of* $C_c(G, A)$ *in* $\|\cdot\|$ *is a C*-algebra, called the crossed product of G by A and denoted* $A \times_\alpha G$.

Remark 2.3.3.4 $C_c(G, A)$ is a $*$-subalgebra of $A \times_\alpha G$.

Proposition 2.3.3.5 *If* (A, G, α) *is a C*-dynamical system, the mapping sending a covariant pair* (π, U) *to its integrated form* $\pi \times U$ *is a one to one correspondence between nondegenerate covariant representations of* (A, G, α) *and nondegenerate representations of the crossed product* $A \times_\alpha G$. *This correspondence preserves irreducibility and equivalence.*

Definition 2.3.3.6 *If* (A, G, α) *is a C*-dynamical system, the mapping on* $C_c(G, A)$ *defined by* $\| f \|_r = \| Ind_e^G \rho \|$ *where* ρ *is a faithful, nondegenerate representation of A is a norm called the reduced norm. The completion of* $C_c(G, A)$ *in* $\|\cdot\|_r$, *denoted* $A \times_{\alpha, r} G$, *is a C*-algebra called the reduced crossed product.*

Remark 2.3.3.7 If *G* is a topological, locally compact, second countable groupoid with the Haar measure system $\{\lambda^u\}_{u \in G^{(0)}}$, where $G^{(0)}$ denotes the unit space of *G*, $\mathcal{A}$ a bundle of separable C^*-algebras over $G^{(0)}$, and σ a continuous homomorphism

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from G to the space $Iso(\mathcal{A}) = \{(u, V, v) / V : A(v) \to A(u)$ an isomorphism of C^*-algebras, $v, u \in G^{(0)}\}$, following [4,Theorem 4.1], the above construction of the crossed product of a group by a C^*-algebra is extended to the crossed product of the groupoid G by the bundle $\mathcal{A}$, denoted $C^*(G, \mathcal{A})$. Moreover, if $\mathcal{A}$ is a bundle with constant fiber the C^*-algebra A, the triplet (A,G, σ) with G groupoid is a C^*-*groupoid dynamical system*. The construction of crossed product $C^*(G, A)$ is similar to the construction of the crossed product $A \times_\alpha G$, in the case of G being group. In [3], Tetsuya Masuda defines the reduced crossed product of a C^*-groupoid dynamical system, by taking the closure of the space $C_c(G, A)$ in the following norm: $\| f \| = \sup_{x \in G^{(0)}} \| \pi_x(f) \|$, where:

$$[\pi_x(f)\xi](g) = \int_{G^x} p_{g_1}(f(g_1^{-1}g))\xi(g_1)d\nu^x(g_1),\ g \in G^x, \xi \in L^2(G^x, \nu^x) \otimes H .$$

This reduced crossed product is denoted $C_r^*(G, A)$ and it will be called the Masuda crossed product.

3. The main results

Proposition 3.1. *If* (A, G, α) *is a dynamical system and the crossed product given by universal norm,* $A \times_\alpha G$, *is a continuous trace C*-algebra, then the reduced crossed product* $A \times_{\alpha,r} G$ *is a continuous trace C*-algebra.*

Proof To prove that $A \times_{\alpha,r} G$ is a continuous trace C^*-algebra we are using Dixmier's definition, and we prove that, if the crossed product $A \times_\alpha G$ is a continuous trace C^*-algebra, then the reduced crossed product $A \times_{\alpha,r} G$ has enough continuous trace elements so that these elements are dense in it. From the construction of $A \times_\alpha G$ and $A \times_{\alpha,r} G$, the following inclusion of spectra results: $(A \times_{\alpha,r} G)\hat{} \subseteq (A \times_\alpha G)\hat{}$. Moreover, all the continuous trace elements from $A \times_\alpha G$, the elements a from $A \times_\alpha G$ such that the map $\pi \to Tr(\pi(a))$ is continuous and finite on $(A \times_\alpha G)\hat{}$, are continuous trace elements for the reduced crossed product, because the restriction of the map $\pi \to Tr(\pi(a))$ to $(A \times_{\alpha,r} G)\hat{}$

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will be continuous and finite (if the continuous trace elements a are in the reduced crossed product, i.e. $a \in A \times_{\alpha,r} G$). Every x from the reduced crossed product $A \times_{\alpha,r} G$ belongs to $A \times_{\alpha} G$, too. Since $A \times_{\alpha} G$ is a continuous trace C^*-algebra, the continuous trace elements of it will be dense, and it follows that x will be a limit, in universal norm, of a sequence of continuous trace elements, $(x_n)_{n>0}$. If all terms of the sequence $(x_n)_{n>0}$ are in the reduced crossed product, since the reduced norm is majorated by the universal norm, it follows that $\| x_n - x \|_r \leq \| x_n - x \| \to 0$ when $n \to \infty$, and all continuous trace elements of the sequence $(x_n)_{n>0}$ are dense in $A \times_{\alpha,r} G$, hence the reduced crossed product will be a continuous trace C^*-algebra. If there is an infinite number of $(x_n)_{n>0}$ sequence's terms such that these terms don't belong to $A \times_{\alpha,r} G$, for every $n > 0$ such that the term x_n doesn't belong to the reduced crossed product and it is a continuous trace element of $A \times_{\alpha} G$, from the construction of the crossed product $A \times_{\alpha} G$ follows that the term x_n is the limit in universal norm of a functions sequence $(f_{n,m})_{m>0}$ with elements from $C_c(G,A)$, i.e. $\| f_{n,m} - x_n \| \to 0$ when $m \to \infty$. Since the universal norm is greater than the reduced norm, the sequence $(f_{n,m})_{m>0}$ will have the limit x_n in the reduced norm, and because this sequence is chosen from $C_c(G,A)$, it belongs to the reduced crossed product. It remains to show that, starting from some point on, the terms of $f_{n,m}$ are continuous trace elements, which means the map $\pi \to Tr(\pi(f_{n,m}))$ will be continuous on the spectrum of $A \times_{\alpha} G$. The map $\pi \to Tr(\pi(f_{n,m}))$ is also finite, because its limit is a continuous trace element. For this we show that $| Tr(\pi_i(f_{n,m})) - Tr(\pi(f_{n,m})) | \to 0$, when $\pi_i \to \pi$ on the crossed product's spectrum.

Indeed, $| Tr(\pi_i(f_{n,m})) - Tr(\pi(f_{n,m})) | \leq | Tr(\pi_i(f_{n,m})) - Tr(\pi_i(x_n)) | +$
$+ | Tr(\pi_i(x_n)) - Tr(\pi_i(x)) | + | Tr(\pi_i(x)) - Tr(\pi(x)) | + | Tr(\pi(x)) - Tr(\pi(x_n)) | +$
$+ | Tr(\pi(x_n)) - Tr(\pi(f_{n,m})) | \to 0$, when $n, m, i \to \infty$.

Corollary 3.2 *If (A, G, α) is a C*-groupoid dynamical system, and $C^*(G,A)$ is a continuous trace C*-algebra, then Masuda crossed product is a continuous trace C*-algebra.*

Proof From the construction of $C^*(G,A)$ and Masuda crossed product described in Remark 2.3.3.7, follows that a similar proof with the one from Proposition 3.1 can be used in this case.

Proposition 3.3 *If* (A,G,α) *is a dynamical system and the crossed product given by universal norm,* $A\times_\alpha G$, *is a type I C*-algebra, then the reduced crossed product* $A\times_{\alpha,r} G$ *is a type I C*-algebra.*

Proof For the beginning, we will make some considerations about reduced crossed product. According to Lemma 5.15 from [5], if we denote by $I(A)$ the set of closed, two-sided ideals of the *C**-algebra A, it exists a continuous map $IND_e^G : I(A) \to I(A\times_\alpha G)$ such that $IND_e^G \ker\pi = \ker Ind_e^G \pi$ for all nondegenerate representation π of *A* (the map $Ind_e^G \pi$ has been defined in example 2.3.2.2). If π and π' are two faithful, nondegenerate representations of *A*, $\ker Ind_e^G \pi = \ker Ind_e^G \pi' = IND_e^G\{0\}$, because π and π' are faithful representations. Moreover, $\| Ind_e^G \pi(a)\| = \| Ind_e^G \pi'(a)\|$ for all $a\in A\times_\alpha G$.

In the quotient $(A\times_\alpha G)/\ker(Ind_e^G\pi)$, for $a\in A\times_\alpha G$, because $(A\times_\alpha G)/\ker(Ind_e^G\pi)$ is isomorphic to the range of $Ind_e^G\pi$,

$\| Ind_e^G \pi(a)\| = \| a + \ker Ind_e^G \pi\| = \| a + IND_e^G \ker\pi\| = \| a + IND_e^G\{0\}\|$.

It follows that the reduced crossed product is isomorphic to the quotient $(A\times_\alpha G)/\ker(Ind_e^G\pi)$ for every faithful nondegenerate representation π of *A*.

According to Dixmier [2, Proposition 2.11.2, Proposition 3.2.1], the spectrum of the quotient $(A\times_\alpha G)/\ker(Ind_e^G\pi)$ is homeomorphic to the set:

$$(A\times_\alpha G)\hat{}_{\ker(Ind_e^G\pi)} = \{\rho\in (A\times_\alpha G)\hat{}\,/\, \rho(\ker(Ind_e^G\pi)) = 0\} .$$

According to Proposition 2.2.5 and because $A\times_\alpha G$ is a type I *C**-algebra, for every representation $\rho\in (A\times_\alpha G)\hat{}$, $\rho(A\times_\alpha G)$ contains the set of compact operators on H_ρ. Because $A\times_{\alpha,r} G$ is isomorphic to $(A\times_\alpha G)/\ker(Ind_e^G\pi)$, a irreducible representation from $(A\times_{\alpha,r} G)\hat{}$ is in corespondence with a irreducible representation of $A \times_{\alpha} G$, which is zero on $\ker(Ind_e^G \pi)$ for every faithful nondegenerate representation π of *A*. But the irreducible representations of $A \times_{\alpha} G$, contains the set of compact operators, and it results that every irreducible representation from $(A \times_{\alpha,r} G)\hat{}$ contains the set of compact operators. That means $A \times_{\alpha,r} G$ is a type I *C**-algebra.

Using the same argument as in Corollary 3.2, we obtain:

Corollary 3.4 *If* (A, G, α) *is a C*-groupoid dynamical system, and* $C^*(G, A)$ *is a type I* C^* *-algebra, then Masuda crossed product is a type I C*-algebra.*

4. Conclusions

In this paper, we have studied the connection between the universal crossed product and the reduced crossed product, for the case when the universal crossed product is a continuous trace *C**-algebra or a type I *C**-algebra. If the universal crossed product is a continuous trace *C**-algebra, we have proved in Proposition 3.1 that the reduced crossed product is a continuous trace *C**-algebra, too. We have obtained in Proposition 3.3 a similar result to the one in Proposition 3.1 for the case of the universal crossed product being a type I *C**-algebra. Moreover, we have extended in Corollaries 3.2 and 3.4 the results from Propositions 3.1 and 3.3, respectively, to the case of *C**-groupoid dynamical systems.

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