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OSCILLATION THEOREMS FOR SECOND ORDER NONLINEAR NEUTRAL TYPE DIFFERENCE EQUATIONS WITH POSITIVE AND NEGATIVE COEFFICIENTS

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Abstract. In this paper, the authors using two inequalities and Ricatti type transformation obtained some new oscillation results for the second order nonlinear neutral type difference equations of the form

$$\Delta(a_n\Delta(x_n + c_n x_{n-k})) + p_n f(x_{n+1-l}) - q_n g(x_{n+1-m}) = 0,$$

and

$$\Delta(a_n\Delta(x_n - c_n x_{n-k})) + p_n f(x_{n+1-l}) - q_n g(x_{n+1-m}) = 0.$$

The obtained results improve, extend and generalize some of the known results. Further examples are provided to illustrate the importance of the main results.

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1. Introduction

Neutral type difference and differential equations arise in many areas of applied mathematics, such as population dynamics [5], bifurcation analysis [2], circuit theory [3], dynamic behavior of delayed network systems [20], and so on. Hence these equations have attracted a great interest during last few decades. Therefore, in this paper we study the oscillation of solution of the neutral type difference equations of the form

$$\Delta(a_n\Delta(x_n + c_n x_{n-k})) + p_n f(x_{n+1-l}) - q_n g(x_{n+1-m}) = 0 \tag{1.1}$$

and

$$\Delta(a_n\Delta(x_n - c_n x_{n-k})) + p_n f(x_{n+1-l}) - q_n g(x_{n+1-m}) = 0 \tag{1.2}$$

where $n \in \mathbb{N}(n_0) = \{n_0, n_0 + 1, ...\}$, n_0 is a nonnegative integer k, l, m are nonnegative integers, $\{a_n\}, \{c_n\}, \{p_n\}, \{q_n\}$ are real sequences, f and $g : \mathbb{R} \to \mathbb{R}$ are continuous and nondecreasing functions with $uf(u) > 0$, and $ug(u) > 0$ for $u \neq 0$.

Let $\theta = \max\{k, l, m\}$. By a solution of equation (1.1)((1.2)), we mean a real sequence $\{x_n\}$ which is defined for all $n \geq n_0 - \theta$, and satisfies equation (1.1)((1.2)) for all $n \in \mathbb{N}(n_0)$. It is well known that equation (1.1)((1.2)) has a unique solution $\{x_n\}$ if an initial sequence $\{x_0(n)\}$ is given to hold for $x_n = x_0(n), n = n_0 - \theta, n_0 - \theta + 1, ..., n_0$. A nontrivial solution $\{x_n\}$ of equation (1.1)((1.2)) is said to be oscillatory if it is neither eventually positive nor eventually negative, and it is nonoscillatory otherwise.

In [4, 6, 7, 10, 11, 8, 9, 18, 17, 19], the authors obtained some sufficient conditions for the existence of nonoscillatory solutions and oscillation of all solutions of equations (1.1) and (1.2) when $f(u) = g(u) = u$, and $a_n \equiv 1$. In [15], the authors established some sufficient conditions for the oscillation of equations

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(1.1) and (1.2) with $f \equiv g$, and $\frac{f(u)}{u} \geq M_1 > 0$ for $u \neq 0$, and in [13], the authors discussed oscillatory behavior of solutions of equations (1.1) and (1.2) with $a_n \equiv 1$.

Motivated by these results, in this paper we established sufficient conditions for the oscillation of all solutions of equations (1.1) and (1.2) without these types of restrictions. Our results extend and generalize some of the results in [1, 4, 7, 8, 9, 10, 11, 13, 15, 17], and the references cited therein.

In Section 2, we present our main results for equations (1.1) and (1.2), and in Section 3, we present some examples to illustrate our theorems.

2. Oscillation Results

In this section, we obtain some oscillation criteria for equations (1.1) and (1.2), subject to the following conditions:

(H_1) $\{a_n\}$ is a positive real sequence such that $\sum_{n=n_0}^{\infty} \frac{1}{a_n} = \infty$;
(H_2) $\{c_n\}$, $\{p_n\}$ and $\{q_n\}$ are nonnegative real sequences;
(H_3) there exists β, ratio of odd positive integers, and a positive constant M_1 such that $\frac{f(u)}{u^\beta} \geq M_1$ for $u \neq 0$;
(H_4) there are positive constants M and M_2 such that $0 \leq \frac{g(u)}{u} \leq M_2$, and $0 < \frac{g(u)}{f(u)} \leq M$ for $u \neq 0$;
(H_5) there is a constant M_3 such that $p_n - Mq_{n-m+l} \geq M_3 > 0$ for all $n \in \mathbb{N}(n_0)$.

Lemma 2.1. *If b_1 and b_2 are nonnegative, then $(b_1 + b_2)^\beta \leq 2^{\beta-1}(b_1^\beta + b_2^\beta)$ for $\beta \geq 1$, and $(b_1 + b_2)^\beta \leq (b_1^\beta + b_2^\beta)$ for $0 < \beta < 1$.*

Proof. The proof can be found in [16]. □

Theorem 2.2. *Let assumptions $(H_1) - (H_5)$ hold. Further assume that there are constants α_1 and α_2 such that $0 \leq \alpha_1 \leq c_n \leq \alpha_2$ for all $n \in \mathbb{N}(n_0)$. If $l \geq m + 1 \geq k$, and*

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n} \left(\sum_{s=n-l+m}^{n-1} q_s \right) < \infty, \tag{2.1}$$

then every solution of equation (1.1) is oscillatory.

Proof. Assume that $\{x_n\}$ is a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that $x_n > 0$ and $x_{n-\theta} > 0$ for all $n \geq n_1 \in \mathbb{N}(n_0)$. The proof for the case $x_n < 0$ is similar and is omitted. Choose an integer $N > n_1$ so that

$$\sum_{n=N}^{\infty} \frac{1}{a_n} \left(\sum_{s=n-l+m}^{n-1} q_s \right) < \frac{\alpha_1}{M_2}. \tag{2.2}$$

Set

$$z_n = x_n + c_n x_{n-k} - \sum_{s=N}^{n-1} \frac{1}{a_s} \sum_{t=s-l+m}^{s-1} q_t g(x_{t+1-m}) \tag{2.3}$$

for all $n \geq N$. Then from equation (1.1) and conditions (H_3) and (H_4), we have

$$\begin{aligned}
\Delta(a_n \Delta z_n) &= \Delta(a_n \Delta(x_n + c_n x_{n-k})) - p_n f(x_{n+1-m}) + q_{n-l+m} g(x_{n+1-l}) \\
&= -p_n f(x_{n+1-l}) + q_{n-l+m} g(x_{n+1-l}) \\
&\leq -M_1[p_n - Mq_{n-l+m}] x_{n+1-l}^\beta \\
&\leq -M_3 M_1 x_{n+1-l}^\beta \leq 0 \qquad (2.4)
\end{aligned}$$

for all $n \geq N$. Hence $\{a_n \Delta z_n\}$ is eventually nondecreasing. So either $\Delta z_n < 0$ or $\Delta z_n \geq 0$ for all $n \geq N_1$ for some integer $N_1 \geq N$.

If $\Delta z_n < 0$ for all $n \geq N_1$, then (2.4) and (H_1) imply $\lim_{n\to\infty} z_n = -\infty$. We claim that $\{x_n\}$ is bounded from above. If this is not the case, then there is an integer $N_2 \geq N_1 + k$ such that

$$z_{N_2} < 0, \text{ and } \max_{N_1 \leq n \leq N_2} x_n = x_{N_2}. \tag{2.5}$$

Then, we have

$$\begin{aligned} 0 > z_{N_2} &= x_{N_2} + c_{N_2} x_{N_2-k} - \sum_{s=N}^{N_2-1} \frac{1}{a_s} \sum_{t=s-l+m}^{s-1} q_t g(x_{t+1-m}) \\ &\geq \alpha_1 x_{N_2-k} - M_2 x_{N_2-k} \sum_{s=N}^{N_2-1} \frac{1}{a_s} \sum_{t=s-l+m}^{s-1} q_t \\ &\geq [\alpha_1 - M_2 \sum_{n=N}^{\infty} \frac{1}{a_n} \sum_{t=n-l+m}^{n-1} q_s] x_{N_2-k} \geq 0. \end{aligned}$$

This contradiction shows that $\{x_n\}$ must be bounded so there exists a constant $L > 0$ such that $x_n \leq L$ for all $n \geq N_1$. It follows from (2.3) that

$$z_n \geq -LM_2 \sum_{s=N_1}^{n-1} \frac{1}{a_s} \sum_{t=s-l+m}^{s-1} q_t \geq -L\alpha_1 > -\infty,$$

which contradicts the fact that $\lim_{n\to\infty} z_n = -\infty$. Therefore, we have $\Delta z_n \geq 0$ for all $n \geq N_1$. Now, summing (2.3) from N_1 to $n-1$, we obtain

$$\infty > a_{N_1} \Delta z_{N_1} \geq -a_{n+1} \Delta z_{n+1} + a_N \Delta z_N \geq M_3 M_1 \sum_{s=N_1}^{n-1} x_{s+1-l}^{\beta},$$

and therefore $\{x_n^{\beta}\}$ is summable for $n \in \mathbb{N}(N_1)$. Then, by Lemma 2.1, we have

$$y_n^{\beta} = (x_n + c_n x_{n-k})^{\beta} \leq 2^{\beta-1}(x_n^{\beta} + \alpha_2^{\beta} x_{n-k}^{\beta}) \; for \; \beta \geq 1,$$

and

$$y_n^{\beta} = (x_n + c_n x_{n-k})^{\beta} \leq (x_n^{\beta} + \alpha_2^{\beta} x_{n-k}^{\beta}) \; for \; 0 < \beta < 1,$$

so $\{y_n^{\beta}\}$ is also summable. On the other hand, from equation (2.3), we obtain

$$\Delta y_n = \Delta z_n + \frac{1}{a_n} \sum_{s=n-l+m}^{n-1} q_s g(x_{s+1-m}) \geq 0$$

so that Δy_n is nondecreasing for all $n \geq N_1$. But then $y_n^{\beta} \geq y_{N_1}^{\beta}$ for all $n \geq N_1$ implies that $\{y_n^{\beta}\}$ is not summable, a contradiction. This completes the proof of the theorem. □

Remark 2.3. *If $f(u) = g(u)$, then Theorem 2.2 reduced to Theorem 2.1 of* [15].

Next, we establish an oscillation result for equation (1.1) when $l = m$, and for this case the condition $0 < \frac{g(u)}{u} \leq M_2$ is not required.

Theorem 2.4. *Let assumptions $(H_1) - (H_4)$ hold. Further assume that $\beta \geq 1$, and $l = m$,*

$$1 - c_{n+1-l} > 0 \; for \; all \; n \in \mathbb{N}(n_0), \tag{2.6}$$

and

$$Q_n = p_n - Mq_n > 0 \; for \; all \; n \in \mathbb{N}(n_o). \tag{2.7}$$

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If there exists a positive and nondecreasing sequence $\{\rho_n\}$ *such that*

$$\lim_{n\to\infty} \sup \sum_{s=N}^{n-1} \left[M_1 \rho_s Q_s (1 - c_{s+1-l})^\beta - \frac{(\Delta \rho_s)^2 a_{s-l}}{4L\beta\rho_s} \right] = \infty \tag{2.8}$$

for any $L > 0$, *then every solution of equation* (1.1) *is oscillatory.*

Proof. Assume that $\{x_n\}$ is a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that $x_n > 0$ for all $n \geq n_0 + \theta$. The proof for the case $x_n < 0$ is similar and is omitted. Define

$$z_n = x_n + c_n x_{n-k},\ n \geq N \in \mathbb{N}(n_0),$$

then $z_n > 0$, and from equation (1.1), and conditions (H_3) and (H_4), we have

$$\Delta(a_n \Delta z_n) + M_1 Q_n x^\beta_{n+1-l} \leq 0,\ n \geq N. \tag{2.9}$$

From (2.7) and (2.9), we obtain $\Delta(a_n \Delta z_n) \leq 0$ for all $n \geq N_1$. Therefore $\Delta z_n \leq 0$ or $\Delta z_n > 0$ for all $n \geq N$.

If $\Delta z_n \leq 0$ for all $n \geq N_1 \geq N$ then by (H_1), we obtain $z_n \to -\infty$ as $n \to \infty$, which is a contradiction. Hence $\Delta z_n > 0$ for all $n \geq N$. From the definition z_n, we obtain $x_n \geq (1 - c_n) z_n$, and using this in (2.9) we have

$$\Delta(a_n \Delta z_n) + M_1 Q_n (1 - c_{n+1-l})^\beta z^\beta_{n+1-l} \leq 0,\ n \geq N. \tag{2.10}$$

Define

$$w_n = \frac{\rho_n a_n \Delta z_n}{z^\beta_{n-l}},\ n \geq N,$$

then $w_n > 0$, and from (2.10), we obtain

$$\Delta w_n \leq -M_1 \rho_n Q_n (1 - c_{n+1-l})^\beta + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - L\beta \frac{\rho_n}{\rho^2_{n+1} a_{n-l}} w^2_{n+1},\ n \geq N \tag{2.11}$$

where we have used $\{a_n \Delta z_n\}$ is positive and nonincreasing and $L = z^{\beta-1}_{N-l}$. Summing the inequality (2.11) from N to $n-1$ and using completing the square, we have

$$\sum_{s=N}^{n-1} \left[M_1 \rho_s Q_s (1 - c_{s+1-l})^\beta - \frac{(\Delta \rho_s)^2 a_{s-l}}{4L\beta\rho_s} \right] \leq w_N - w_n \leq w_N.$$

Taking lim sup in the last inequality, we obtain a contradiction with (2.8), and the proof of the theorem is complete. □

Remark 2.5. *Let* $q_n \equiv 0$ *in equation* (1.1). *Then Theorem 2.2 reduced to the known oscillation criterion for the equation*

$$\Delta(a_n \Delta(x_n + c_n x_{n-k})) + p_n f(x_{n+1-l}) = 0$$

given in [1], *and the references cited therein.*

In the following we establish oscillation results for equation (1.2).

Theorem 2.6. *Let assumptions* $(H_1) - (H_5)$ *hold. Further assume that there is a constant* α_3 *such that* $0 \leq c_n \leq \alpha_3 < 1$, *for all* $n \in \mathbb{N}(n_0)$. *If* $l \geq m + 1$, *and*

$$\alpha_3 + M_2 \sum_{n=N}^{\infty} \frac{1}{a_n} \sum_{s=n-l+m}^{n-1} q_s \leq 1 \tag{2.12}$$

then any solution $\{x_n\}$ *of equation* (1.2) *is either oscillatory or satisfies* $\lim_{n\to\infty} x_n = 0$.

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Proof. Suppose that $\{x_n\}$ is a nonoscillatory solution of equation (1.2), say $x_n > 0$ for $n \geq N \geq n_0 + \theta$. Define

$$w_n = x_n - c_n x_{n-k} - \sum_{s=N}^{n-1} \frac{1}{a_s} \sum_{t=s-l+m}^{s-1} q_t g(x_{t+1-m}). \tag{2.13}$$

Then as in the proof of Theorem 2.2, we have

$$\Delta(a_n \Delta w_n) \leq -M_3 M_1 x_{n+1-l}^{\beta} \leq 0 \tag{2.14}$$

for all $n \geq N$, and conclude that $\{a_n \Delta w_n\}$ is eventually nonincreasing. Therefore $\Delta w_n < 0$ or $\Delta w_n \geq 0$ for all $n \geq N_1 \geq N$.

Assume that $\Delta w_n < 0$ for all $n \geq N_1$, then by (H_1) we have $\lim_{n\to\infty} w_n = -\infty$. We claim that $\{x_n\}$ is bounded from above. If not, there exists an integer $N_2 \geq N_1 + k$ such that

$$w_{N_2} < 0, \text{ and } \max_{N_1 \leq n \leq N_2 - k} x_n = x_{N_2-k}, \tag{2.15}$$

and we have

$$\begin{aligned} 0 > w_{N_2} &= x_{N_2} - c_{N_2} x_{N_2-k} - \sum_{s=N}^{N_2-1} \frac{1}{a_s} \sum_{t=s-l+m}^{s-1} q_t g(x_{s+1-m}) \\ &\geq \left[1 - \alpha_3 - M_2 \sum_{s=N}^{N_2-1} \frac{1}{a_s} \sum_{t=s-l+m}^{s-1} q_t \right] x_{N_2-k} \geq 0. \end{aligned}$$

This contradiction shows that $\{x_n\}$ must be bounded from above, so there exists a constant $L > 0$ such that $x_n \leq L$ for all $n \geq N_1$. It follows from (2.12) and (2.13) that

$$w_n \geq -L \left[\alpha_3 + M_2 \sum_{s=N}^{N_2-1} \frac{1}{a_s} \sum_{t=s-l+m}^{s-1} q_t \right] \geq -L > -\infty,$$

which contradicts the fact that $\lim_{n\to\infty} w_n = -\infty$. Hence $\Delta w > 0$ for $n \geq N_1$.

In this case, we see that L is a nonnegative constant , where $L = \lim_{n\to\infty} a_n \Delta w_n$. Summing (2.14) from N_1 to ∞, we obtain

$$\infty > a_{N_1} \Delta w_{N_1} - L \geq M_3 M_1 \sum_{n=N_1}^{\infty} x_{n+1-l}^{\beta}$$

which implies that $\{x_n^{\beta}\}$ is summable, and thus $\lim_{n\to\infty} x_n = 0$. This completes the proof. □

Finally we obtain oscillation results for equation (1.2) when $l = m$, and for this case the condition $0 < \frac{g(u)}{u} \leq M_2$, is not needed.

Theorem 2.7. *Let assumptions $(H_1) - (H_4)$ hold. Further assume that $l \geq k + 1$,*

$$0 \leq c_n \leq \alpha_3 < 1 \; for \; n \in \mathbb{N}(N_0), \tag{2.16}$$

$$0 < \beta \leq 1, \tag{2.17}$$

$$\sum_{n=N}^{\infty} Q_n = \infty, \tag{2.18}$$

and

$$\lim_{n\to\infty} \sup \sum_{s=n-l+k}^{n-1} \frac{1}{a_s} \sum_{t=s}^{n-1} Q_t = \infty. \tag{2.19}$$

Then every solution of equation (1.2) is oscillatory.

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Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (1.2). Without loss of generality, we may assume that $x_n > 0$ and $x_{n-\theta} > 0$ for all $n \geq n_1 \in \mathbb{N}(n_0)$. The proof for the case $x_n < 0$ is similar and is omitted. Define

$$z_n = x_n - c_n x_{n-k},\ n \geq n_1.$$

From equation (1.2), and conditions (H_3) and (H_4) we have

$$\Delta(a_n \Delta z_n) \leq -M_1 Q_n x^{\beta}_{n+1-l} \leq 0,\ n \geq N \geq n_1. \tag{2.20}$$

Hence $\{z_n\}$ and $\{a_n \Delta z_n\}$ are eventually of one sign for all $n \geq N$. Then by Lemma 2.1 of [12], and (H_1) the sequence $\{z_n\}$ satisfies one of the following two cases for all $n \geq N$:

(i) $z_n > 0, a_n \Delta z_n > 0, \Delta(a_n \Delta z_n) \leq 0$;
(ii) $z_n < 0, a_n \Delta z_n > 0, \Delta(a_n \Delta z_n) \leq 0$.

Case (i): From the definition of z_n, we have $x_n \geq z_n$, and using this in (2.20), we obtain

$$\Delta(a_n \Delta z_n) + M_1 Q_n z^{\beta}_{n+1-l} \leq 0,\ n \geq N. \tag{2.21}$$

Define

$$w_n = \frac{a_n \Delta z_n}{z^{\beta}_{n-l}},\ n \geq N,$$

then $w_n > 0$ for $n \geq N$, and from (2.21), we obtain

$$\begin{aligned} \Delta w_n &= -M_1 Q_n - \frac{a_n \Delta z_n}{z^{\beta}_{n+1-l} z^{\beta}_{n-l}} \Delta z^{\beta}_{n-l} \\ &\leq -M_1 Q_n - \frac{-\beta a_n \Delta z_n}{z^{\beta}_{n-l} z^{\beta}_{n+1-l}} \Delta z_{n-l} \\ &\leq -M_1 Q_n,\ n \geq N. \end{aligned}$$

Summing the last inequality from N to $n-1$, we have

$$M_1 \sum_{s=N}^{n-1} Q_s \leq w_N - w_n \leq w_N.$$

Letting $n \to \infty$, we obtain a contradiction with (2.18).

Case(ii): From the definition of z_n and (2.16), we have

$$x_{n-k} \geq \left(\frac{-z_n}{\alpha_3} \right). \tag{2.22}$$

Using (2.22) in (2.20), we obtain

$$\Delta(a_n \Delta z_n) - \frac{M_1 Q_n}{\alpha_3^{\beta}} z^{\beta}_{n+1-l+k} \leq 0, n \geq N.$$

Summing the last inequality from s to $n-1$ for $n > s+1$, we have

$$a_n \Delta z_n - a_s \Delta z_s - \frac{M_1}{\alpha_3^{\beta}} \sum_{t=s}^{n-1} Q_t z^{\beta}_{t+1-l+k} \leq 0. \tag{2.23}$$

Summing again the last inequality from $n-l+k$ to $n-1$ for s, we have

$$z_{n-l+k} - z_n \leq \frac{M_1}{\alpha_3^{\beta}} z^{\beta}_{n-l+k} \sum_{s=n-l+k}^{n-1} \frac{1}{a_s} \sum_{t=s}^{n-1} Q_t$$

or

$$\frac{z_{n-l+k}}{z^{\beta}_{n-l+k}} \geq \frac{M_1}{\alpha_3^{\beta}} \sum_{s=n-l+k}^{n-1} \frac{1}{a_s} \sum_{t=s}^{n-1} Q_t.$$

If $\beta = 1$, then from the last inequality, we obtain

$$\frac{\alpha_3}{M_1} \geq \lim_{n\to\infty} \sup \sum_{s=n-l+k}^{n-1} \frac{1}{a_s} \sum_{t=s}^{n-1} Q_t$$

which is a contradiction with (2.19). Next, assume $0 < \beta < 1$. Since z_n is negative and increasing, we have $\lim_{n\to\infty} z_n = \delta \leq 0$. If $\delta = 0$, then from (2.23), we obtain

$$\lim_{n\to\infty} \sup \sum_{s=n-l+k}^{n-1} \frac{1}{a_s} \sum_{t=s}^{n-1} Q_t \leq 0$$

since $1 - \beta > 0$, which is a contradiction with (2.19). Now assume that $\delta < 0$. From (2.23) we have

$$\Delta z_s + \frac{M_1 z_n^\beta}{\alpha_3^\beta a_s} \sum_{t=s}^{n-1} Q_t \geq 0.$$

Summing the last inequality from N to $n-1$, and rearranging we obtain

$$\frac{\alpha_3^\beta z_N}{M_1 z_n^\beta} \geq \sum_{s=N}^{n-1} \frac{1}{a_s} \sum_{t=s}^{n-1} Q_t.$$

In view of $\delta < 0$, $\frac{\alpha_3^\beta z_N}{M_1 z_n^\beta}$ has an upper bound, so

$$\lim_{n\to\infty} \sum_{s=N}^{n-1} \frac{1}{a_s} \sum_{t=s}^{n-1} Q_t < \infty$$

which again contradicts (2.19). This completes the proof of the theorem. □

Remark 2.8. *Let $q_n \equiv 0$ in equation (1.2), then Theorem 2.5 reduced to the known oscillation criteria for the equation*

$$\Delta(a_n \Delta(x_n - c_n x_{n-h})) + p_n f(x_{n+1-l}) = 0$$

given in [1, 14], and the references cited therein.

3. Examples

In this section, we present some examples to illustrate the main results.

Example 3.1. *Consider the second order nonlinear neutral difference equation of the form*

$$\Delta\left(n\Delta\left(x_n + 2x_{n-1}\right)\right) + \left(2n + 1 + \frac{1}{3^n}\right) x_{n-2}^3 \left(1 + x_{n-2}^2\right) - \frac{4}{3^n} \frac{x_n^3}{(1 + x_n^2)} = 0, \ n \geq 2. \tag{3.1}$$

Here $a_n = n$, $c_n = 2$, $p_n = 2n + 1 + \frac{1}{3^n}$, $q_n = \frac{4}{3^n}$, $k = 1$, $l = 3$, $m = 1$, $f(u) = u^3(1 + u^2)$, and $g(u) = \frac{u^3}{1+u^2}$. By taking $\beta = 3$, and $M_1 = M_2 = M = 1$, we see that conditions $(H_1) - (H_4)$ hold. Further,

$$p_n - q_{n-m+l} = 2n + 1 + \frac{1}{3^n} - \frac{4}{3^{n+2}} > 1,$$

and

$$\sum_{n=2}^{\infty} \frac{1}{n} \sum_{s=n-2}^{n-1} \frac{4}{3^s} = \sum_{n=2}^{\infty} \frac{4}{n} \left(\frac{1}{3^{n-2}} + \frac{1}{3^{n-1}}\right) < \infty.$$

Therefore all the conditions of Theorem 2.2 are satisfied, and hence every solution of equation (3.1) is oscillatory. In fact, $\{x_n\} = \{(-1)^n\}$ is one such oscillatory solution of equation (3.1).

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Example 3.2. *Consider the second order nonlinear neutral difference equation of the form*

$$(3.2)\qquad \Delta\left(n\left(\Delta\left(x_n+\frac{1}{2}x_{n-1}\right)\right)\right)+\left(n+\frac{1}{2}+\frac{1}{2^{n+2}}\right)x_{n-1}^{5/3}\left(1+x_{n-1}^4\right)-\frac{1}{2^n}\frac{x_{n-1}^{5/3}}{(1+x_{n-1}^2)}=0,\ n\geq 1.$$

Here $a_n=n,\ c_n=\frac{1}{2},\ p_n=n+\frac{1}{2}+\frac{1}{2^{n+2}},\ q_n=\frac{1}{2^n},\ f(u)=u^{5/3}(1+u^4),\ g(u)=\frac{u^{5/3}}{1+u^4},\ k=1,$ *and* $l=2.$ *With* $\beta=\frac{5}{3},\ M=1,$ *and* $M_1=1$ *we see that the conditions* $(H_1)-(H_4)$ *are satisfied. Further we see that*

$$1-c_{n+1-l}=\frac{1}{2}>0,$$

and

$$Q_n=p_n-Mq_n=n+\frac{1}{2}-\frac{3}{2^{n+2}}\geq\frac{9}{8}>0.$$

By taking $\rho_n\equiv 1$, *we see that the condition* (2.8) *becomes*

$$\lim_{n\to\infty}\sup\sum_{s=1}^{n-1}\left[Q_s\left(1-c_{s+1-l}\right)^{\beta}\right]=\sum_{1}^{\infty}\left(\frac{1}{2}\right)^{5/3}\left(n+\frac{1}{2}-\frac{3}{2^{n+2}}\right)=\infty.$$

Hence all conditions of Theorem 2.4 are satisfied, and therefore every solution of equation (3.2) *is oscillatory. In fact,* $\{x_n\}=\{(-1)^n\}$ *is one such oscillatory solution of equation* (3.2).

Example 3.3. *Consider the second order nonlinear neutral difference equation of the form*

$$(3.3)\qquad \Delta\left(n\Delta\left(x_n-\frac{1}{2}x_{n-2}\right)\right)+\frac{1}{3}\left(4n+2-\frac{1}{4^{n+3}}\right)\frac{x_{n-1}^3\left(2+x_{n-1}^2\right)}{(1+x_{n-1}^2)}-\frac{1}{4^{n+3}}\frac{x_n^3}{(1+x_n^2)}=0,\ n\geq 1.$$

Here $a_n=n,\ c_n=\frac{1}{2},\ p_n=\frac{1}{3}(4n+2-\frac{1}{4^{n+3}}),\ q_n=\frac{1}{4^{n+3}},\ f(u)=\frac{u^3(2+u^2)}{(1+u^2)},\ g(u)=\frac{u^3}{(1+u^2)},\ k=2,\ l=2,\ m=1.$ *With* $\beta=3,\ M_1=1,\ M_2=1,$ *and* $M=1$, *we see that the conditions* $(H_1)-(H_5)$ *hold. Further we see that*

$$\sum_{1}^{\infty}\frac{1}{a_n}=\sum_{1}^{\infty}\frac{1}{n}=\infty$$

and

$$\begin{aligned}\alpha_3+M_2\sum_{n=1}^{\infty}\frac{1}{a_n}\sum_{s=n-l+m}^{n-1}q_s &= \frac{1}{2}+\sum_{1}^{\infty}\frac{1}{n}\left(\frac{1}{4^{n+2}}\right)\\ &< \frac{1}{2}+\frac{1}{48}<1.\end{aligned}$$

Hence by Theorem 2.4, every solution of equation (3.3) *is either oscillatory or tends to zero as* $n\to\infty$. *In fact* $\{x_n\}=\{(-1)^n\}$ *is one such oscillatory solution of equation* (3.3).

Example 3.4. *Consider the second order nonlinear neutral difference equation of the form*

$$(3.4)\qquad \Delta\left(n\Delta\left(x_n-\frac{1}{2}x_{n-1}\right)\right)+(2+2^n)\left(\frac{15}{8}(3n+2)2^{\frac{2n+1}{3}}+\frac{1}{4^{\frac{n}{3}}}\right)\frac{x_{n-1}^{\frac{1}{3}}}{(1+|x_{n-1}|)}-\frac{2}{4^{\frac{n}{3}}}x_{n-1}^{\frac{1}{3}}=0.$$

Here $a_n=n,\ c_n=\frac{1}{2},\ p_n=(2+2^n)[\frac{15}{8}(3n+2)2^{\frac{2n+1}{3}}+\frac{1}{4^{\frac{n}{3}}}],\ q_n=\frac{2}{4^{n/3}},\ k=1,\ l=2,$ *and* $Q_n=(2+2^n)(\frac{15}{8}(3n+2)2^{\frac{2n+1}{3}}+\frac{1}{4^{n/3}})-\frac{2}{4^{\frac{n}{3}}}>0.$ *Further, we see that*

$$\sum_{n=1}^{\infty}Q_n=\sum_{n=1}^{\infty}\left[\frac{15}{8}(2+2^n)(3n+2)2^{\frac{2n+1}{3}}+2^{\frac{n}{3}}\right]=\infty,$$

and

$$\lim_{n\to\infty}\sup\sum_{s=n-l+k}^{n-1}\frac{1}{a_s}\sum_{t=s}^{n-1}Q_t=\lim_{n\to\infty}\sup\left(\frac{1}{n-1}\right)\left[\frac{15}{8}(2+2^{n-1})(3n-1)2^{\frac{2n-1}{3}}+2^{\frac{n-1}{3}}\right]=\infty.$$

Hence all conditions of Theorem 2.7 are satisfied, and therefore every solution of equation (3.4) *is oscillatory. In fact* $\{x_n\} = \{(-1)^n 2^n\}$ *is one such oscillatory solution of equation* (3.4).

4. Conclusion

In this study, we have obtained new sufficient conditions for the oscillation of all solutions of equations (1.1) and (1.2) via Ricatti transformation and inequalities. Further the oscillation criteria obtained here are applicable to deduce oscillation results for the equations (3.1) to (3.4). The same cannot be deducible for equations (3.1) to (3.4) from any previously known oscillation criteria given in [4, 7, 8, 9, 10, 11, 13, 15, 17], since $a_n \not\equiv 1$, and $f \neq g$. Therefore the results obtained here improve, extend and generalize the existing results.

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