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COEFFICIENT ESTIMATES OF NEW CLASSES OF $q-$STARLIKE AND $q-$CONVEX FUNCTIONS OF COMPLEX ORDER WITH RESPECT TO $(j,k)-$SYMMETRIC POINTS

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Abstract. We introduce new classes of $q-$starlike and $q-$convex functions of complex order with respect to $(j,k)-$symmetric points. Furthermore, the application of the results are also illustrated. We find estimates on the coefficients for second and third coefficients of these classes.

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Key words: analytic function; univalent function; Schwarz function; $q-$starlike, $q-$convex, $q-$derivative operator, subordination, Fekete-Szegő inequality

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1. INTRODUCTION

Recently, the area of the $q-$analysis has attracted serious attention of the researchers. The great interest is due to its applications in various branches of mathematics and physics, as for example, in the areas of ordinary fractional calculus, optimal control problems, $q-$difference and $q-$integral equations and in $q-$transform analysis. The generalized $q-$Taylor formula in the fractional $q-$calculus was introduced by Purohit and Raina [32]. The application of $q-$calculus was initiated by Jackson [21, 22]. He was the first to develop the $q-$integral and $q-$derivative in a systematic way. Later, geometrical interpretation of the $q-$analysis has been recognized through studies on quantum groups. Simply, the quantum calculus is ordinary classical calculus without the notion of limits. It defines $q-$calculus and $h-$calculus. Here h ostensibly stands for Planck's constant, while q stands for quantum. Mohammed and Darus [28] studied approximation and geometric properties of these $q-$operators in some subclasses of analytic functions in compact disk. Recently, Purohit and Raina [32, 33] have used the fractional $q-$calculus operators in investigating certain classes of functions which are analytic in the open disk. Also Purohit [31] also studied these $q-$operators, defined by using the convolution of normalized analytic functions and $q-$hypergeometric functions. A comprehensive study on the applications of $q-$calculus in the operator theory may be found in [11]. Ramachandran et al. [34] have used the fractional $q-$calculus operators in investigating certain bound for $q-$starlike and $q-$convex functions with respect to symmetric points.

Let $\mathcal{A}$ denote the class of all analytic function of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

in the open unit disc $\mathcal{U} = \{\ z : z \in \mathbb{C}; |z| < 1\}$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of functions which are univalent in $\mathcal{U}$. Also, let $\mathcal{P}$ denote the class of functions of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \qquad (z \in \mathcal{U})$$

which are analytic and convex in $\mathcal{U}$ and satisfy the condition

$$Re\,(p(z)) > 0, \qquad (z \in \mathcal{U}).$$

We denote by $\mathcal{S}^*$, $\mathcal{C}$, $\mathcal{K}$ and $\mathcal{C}^*$ the familiar subclasses of $\mathcal{A}$ consisting of functions which are respectively starlike, convex, close-to-convex and quasi-convex in $\mathcal{U}$. Our favorite references of the field are [17, 20] which covers most of the topics in a lucid and economical style.

The Bieberbach conjecture about the coefficient of the univalent functions in the unit disk was formulated by Bieberbach [13] in the year 1916. The conjecture states that for every function $f \in \mathcal{S}$, given by (1.1), we have $\mid a_n \mid \leq n$ for every n. Strict inequality holds for all n unless f is the Koebe function or one of its rotation. For many years, this conjecture remained as a challenge to mathematicians. After the proof of $\mid a_3 \mid \leq 3$ by Löwner in 1923, Fekete-Szegö surprised the mathematicians with the complicated inequality

$$\left|a_3 - \mu a_2^2\right| \leq 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right)$$

which holds good for all values $0 \leq \mu \leq 1$. Note that this inequality region was thoroughly investigated by Schaefer and Spencer [39]. For a class functions in $\mathcal{A}$ and a real (or more generally complex) number μ, the Fekete-Szegö problem is all about finding the best possible constant $C(\mu)$ so that $\left|a_3 - \mu a_2^2\right| \leq C(\mu)$ for every function in $\mathcal{A}$.

In univalent function theory, all geometrically defined subclasses does have beautiful analytic characterization defined in terms of differential inequality. So extending the existing subclasses in q-calculus has numerous applications. To provide a unified approach to the study of various properties of the certain subclasses of $\mathcal{A}$, we introduce new classes of $(j,\,k)$ symmetric functions of complex order involving $q-$derivative of f and have obtained the Fekete-Szegö inequality for the classes.

If f and g are analytic in $\mathcal{U}$, we say that the function f is subordinate to g, written as $f(z) \prec g(z)$ in $\mathcal{U}$, if there exist a Schwarz function $\omega(z)$, which is analytic in $\mathcal{U}$ with $w(0) = 0$ and $\mid w(z) \mid < 1$ such that $f(z) = g(w(z))$ for $z \in \mathcal{U}$. Furthermore, if the function $g(z)$ is univalent $\mathcal{U}$, then we have the following equivalence holds(see [14] and [27]):

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \qquad \text{and} \qquad f(\mathcal{U}) \subset g(\mathcal{U}).$$

For function $f \in \mathcal{A}$ given by (1.1) and $0 < q < 1$, the $q-$derivative of a function f is defined by (see [21, 22])

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z} \qquad (z \neq 0), \tag{1.2}$$

$D_q f(0) = f'(0)$ and $D_q^2 f(z) = D_q(D_q f(z))$. From (1.2), we deduce that

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q\, a_n z^{n-1}, \tag{1.3}$$

where

$$[n]_q = \frac{1-q^n}{1-q}. \tag{1.4}$$

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As $q \to 1^-$, $[n]_q \to n$. For a function $h(z) = z^n$, we observe that

$$D_q(h(z)) = D_q(z^n) = \frac{1-q^n}{1-q} z^{n-1} = [n]_q \, z^{n-1},$$

$$\lim_{q\to 1} D_q(h(z)) = \lim_{q\to 1} \left([n]_q \, z^{n-1} \right) = nz^{n-1} = h'(z),$$

where h' is the ordinary derivative.

As a right inverse, Jackson [21] introduced the $q-$integral

$$\int_0^z h(t) d_q t = z(1-q) \sum_{n=0}^{\infty} q^n f\left(zq^n\right),$$

provided that the series converges. For a function $h(z) = z^n$, we observe that

$$\int_0^z h(t) d_q t = \lim_{q\to 1^-} \frac{z^{n+1}}{[n+1]_q} = \frac{z^{n+1}}{n+1} = \int_0^z h(t) dt,$$

where $\int_0^z h(t)dt$ is the ordinary integral.

Making use of $D_q f(z)$, Seoudy and Aouf in [40] introduced the subclsses $\mathcal{S}_j(\alpha)$ and $\mathcal{C}_j(\alpha)$ of the class $\mathcal{A}$ for $0 \le \alpha < 1$ which are defined by

$$\mathcal{S}_j^*(\alpha) = \left\{ f \in \mathcal{A} : Re \frac{zD_q f(z)}{f(z)} > \alpha, z \in \mathcal{U} \right\}, \tag{1.5}$$

$$\mathcal{C}_j(\alpha) = \left\{ f \in \mathcal{A} : Re \frac{D_q(zD_q f(z))}{D_q f(z)} > \alpha, z \in \mathcal{U} \right\}. \tag{1.6}$$

We note that

$$f \in \mathcal{C}_j(\alpha) \Leftrightarrow zD_q f \in \mathcal{S}_j^*, \tag{1.7}$$

and

$$\lim_{q\to 1^-} \mathcal{S}_j^*(\alpha) = \left\{ f \in \mathcal{A} : \lim_{q\to 1^-} Re \frac{zD_q f(z)}{f(z)} > \alpha, z \in \mathcal{U} \right\} = \mathcal{S}^*(\alpha)$$

$$\lim_{q\to 1^-} \mathcal{C}_j(\alpha) = \left\{ f \in \mathcal{A} : \lim_{q\to 1^-} Re \frac{D_q(zD_q f(z))}{D_q f(z)} > \alpha, z \in \mathcal{U} \right\} = \mathcal{C}(\alpha),$$

where $\mathcal{S}(\alpha)$ and $\mathcal{C}(\alpha)$ are respectively, the classes of starlike of order α and convex of order α in $\mathcal{U}$ (see Robertson [36]).

Let k be a positive integer and $\varepsilon = \exp(2\pi i/k)$. A domain $\mathbb{D}$ is said to be $k-$fold symmetric if a rotation of $\mathbb{D}$ about the origin through an angle $2\pi/k$ carries $\mathbb{D}$ onto itself. A function $f \in \mathcal{A}$ is said to be $k-$fold symmetric in $\mathcal{U}$ if for each $z \in \mathcal{U}$

$$f(\varepsilon z) = \varepsilon f(z).$$

The family of all $k-$fold symmetric functions is denoted by $\mathcal{S}^k$ and for $k = 2$, we get class of the odd univalent functions. The notion of $(j,k)-$symmetric functions ($k = 2, 3, \ldots; j = 0, 1, 2, \ldots (k-1)$) is a generalization of even, odd, $k-$symmetrical functions. Let $\varepsilon = \exp(2\pi i/k)$ and $j = 0, 1, 2, \ldots (k-1)$ where $k \ge 2$ is a natural number. A function $f : \mathcal{U} \mapsto \mathbb{C}$ is called $(j,k)-$symmetrical if

$$f(\varepsilon z) = \varepsilon^j f(z), \;\; z \in \mathcal{U}.$$

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We note that the family of all $(j,k)-$symmetric functions is denoted by $\mathcal{S}^{(j,k)}$. Also, $\mathcal{S}^{(0,2)}, \mathcal{S}^{(1,2)}$ and $\mathcal{S}^{(1,k)}$ are called even, odd and $k-$symmetric functions respectively.
We have the following decomposition theorem (see [25]).

For every mapping $f : \mathbb{D} \mapsto \mathbb{C}$, and $\mathbb{D}$ is a $k-$fold symmetric set, there exist exactly the sequence of $(j,k)-$symmetrical functions $f_{j,k}$,

$$f(z) = \frac{1}{k}\sum_{j=0}^{k-1} f_{j,k}(z), \tag{1.8}$$

where

$$f_{j,k}(z) = \frac{1}{k}\sum_{v=0}^{k-1} \varepsilon^{-vj} f(\varepsilon^{v} z), \tag{1.9}$$

$$(f \in \mathcal{A}; k = 1, 2, \ldots ; j = 0, 1, 2, \ldots (k-1)).$$

The decomposition (1.8) is a generalization of the well-known fact that each function defined on a symmetrical subset $\mathcal{U}$ of $\mathbb{C}$ can be uniquely represented as the sum of an even function and an odd function (see Theorem 1 of [25]). From (1.9), we can get

$$f_{j,k}(z) = \frac{1}{k}\sum_{v=0}^{k-1} \varepsilon^{-vj} f(\varepsilon^{v} z) = \frac{1}{k}\sum_{v=0}^{k-1} \varepsilon^{-vj} \left(\sum_{n=1}^{\infty} a_n (\varepsilon^{v} z)^n\right),$$

then

$$f_{j,k}(z) = \sum_{n=1}^{\infty} \psi_n a_n z^n, \qquad a_1 = 1, \qquad \psi_n = \frac{1}{k}\sum_{v=0}^{k-1} \varepsilon^{(n-j)v} = \begin{cases} 1 & n = lk + j; \\ 0 & n \neq lk + j \end{cases}. \tag{1.10}$$

Motivated by Ma and Minda [26], we define a subclass of analytic functions of complex order involving $q-$derivative of f.

Definition 1.1. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}_{j,k}^{q,b}(\phi)$ if it satisfies the following subordination condition:

$$1 + \frac{1}{b}\left(\frac{zD_q f(z)}{f_{j,k}(z)} - 1\right) \prec \phi(z) \qquad (b \in \mathbb{C} - \{0\}; \phi \in \mathcal{P}). \tag{1.11}$$

Definition 1.2. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{C}_{j,k}^{q,b}(\phi)$, if it satisfies the following subordination condition:

$$1 + \frac{1}{b}\left(\frac{D_q(zD_q f(z))}{D_q f_{j,k}(z)} - 1\right) \prec \phi(z) \qquad (b \in \mathbb{C} - \{0\}; \phi \in \mathcal{P}). \tag{1.12}$$

Remark 1.3. *The family $\mathcal{S}_{j,k}^{q,b}(\phi)$ and $\mathcal{C}_{j,k}^{q,b}(\phi)$ is of special interest for it contains many well-known as well as many new classes of analytic univalent functions. If we let $j = k = 1$, the classes $\mathcal{S}_{j,k}^{q,b}(\phi)$ and $\mathcal{C}_{j,k}^{q,b}(\phi)$ reduces to classes recently introduced by Seoudy and Aouf in* [40]. *If we let $q \to 1^-$, the class $\mathcal{S}_{j,k}^{q,b}(\phi)$ and $\mathcal{C}_{j,k}^{q,b}(\phi)$ reduces to the well-known Janowski starlike function and Janowski convex function of complex order respectively. We note that the family $\mathcal{S}^*(\alpha)$ of starlike function of order $\alpha\,(0 \le \alpha < 1)$*[15, 17], *the family $\mathcal{C}(\alpha)$ of convex function of order $\alpha\,(0 \le \alpha < 1)$*[15, 17], *$k - UCV(\alpha)$*[12], *$k - UST(\alpha)$ and many other well known subclasses of $\mathcal{S}$ (see also the work of Kanas and Srivastava* [23], *Goodman* [18, 19] *and Rønning* [37, 38]*) can be obtained as special cases of either $\mathcal{S}_{j,k}^{q,b}(\phi)$ and $\mathcal{C}_{j,k}^{q,b}(\phi)$.*

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Lemma 1.4. [26] *Let $p(z) \in \mathcal{P}$ and also let v be a complex number, then*

$$|c_2 - vc_1^2| \leq 2\ \max\{1, |2v-1|\},$$

the result is sharp for functions given by

$$p(z) = \frac{1+z^2}{1-z^2}, \qquad p(z) = \frac{1+z}{1-z}.$$

Lemma 1.5. [26] *Let $p(z) \in \mathcal{P}$, then*

$$|c_2 - vc_1^2| \leq \begin{cases} -4v+2, & if \quad v \leq 0; \\ 2, & if \quad 0 \leq v \leq 1; \\ 4v-2, & if \quad v \geq 1. \end{cases} \tag{1.13}$$

When $v < 0$ or $v > 1$, the equality holds if and only if $p(z) = (1+z)/(1-z)$ or one of its rotations. If $0 < v < 1$, then the equality if and only if $p(z) = (1+z^2)/(1-z^2)$ or one of its rotations. If $v = 0$, the equality holds if and only if

$$p(z) = \left(\frac{1}{2} + \frac{1}{2}\vartheta\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\vartheta\right)\frac{1-z}{1+z}, (0 \leq \vartheta \leq 1),$$

or one of its rotations. If $v = 1$, the equality holds if and only if

$$\frac{1}{p(z)} = \left(\frac{1}{2} + \frac{1}{2}\vartheta\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\vartheta\right)\frac{1-z}{1+z}, (0 \leq \vartheta \leq 1).$$

Also the above upper bound is sharp and it can be improved as follows when $0 \leq v \leq 1$

$$|c_2 - vc_1^2| + v|c_1|^2 \leq 2, \qquad (0 < v \leq 1/2),$$

$$|c_2 - vc_1^2| + (1-v)|c_1|^2 \leq 2, \qquad (1/2 \leq v < 1).$$

In the present paper, we obtain the Fekete-Szegö inequalities for the class $\mathcal{S}_{j,k}^{q,b}(\phi)$ and $\mathcal{C}_{j,k}^{q,b}(\phi)$. We employ the technique adapted by Ma and Minda [26] to find the coefficient estimates for our class.

2. Main Results

Unless otherwise mentioned, we assume throughout this paper that the function $0 < q < 1, b \in \mathbb{C} - \{0\}, \phi \in \mathcal{P}$, $[n]_q$ is given by (1.4) and $z \in \mathcal{U}$.

Theorem 2.1. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots (B_1 \neq 0)$. If $f(z) \in \mathcal{S}_{j,k}^{q,b}(\phi)$, then*

$$|a_3 - \mu a_2^2| \leq \frac{|B_1 b|}{[3]_q - \psi_3} \max\left\{1; \left|\frac{B_2}{B_1} + \frac{B_1 b}{[2]_q - \psi_2}\left(\psi_2 - \frac{[3]_q - \psi_3}{[2]_q - \psi_2}\mu\right)\right|\right\}. \tag{2.1}$$

The result is sharp.

Proof. If $f \in \mathcal{S}_{j,k}^{q,b}(\phi)$, then there exists a Schwarz function $\omega(z)$, which is analytic in $\mathcal{U}$ with $w(0) = 0$ and $|\ w(z)\ |< 1 \in \mathcal{U}$ such that

$$1 + \frac{1}{b}\left[\frac{zD_q f(z)}{f_{j,k}(z)} - 1\right] = \phi(\omega(z)). \tag{2.2}$$

Define the function $p(z)$ by

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + c_1 z + c_2 z^2 + \cdots,\ z \in \mathcal{U}. \tag{2.3}$$

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Since $\omega(z)$ is Schwarz function, we see that $Re\, p(z) > 0$ and $p(z) = 1$.
Therefore

$$\begin{aligned}\phi\left(\omega(z)\right) &= \phi\left(\frac{p(z)-1}{p(z)+1}\right)\\ &= \phi\left(\frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right)z^3 + \cdots\right]\right)\\ &= 1 + \frac{1}{2}B_1c_1z + \left[\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2\right]z^2 + \cdots . \qquad (2.4)\end{aligned}$$

Now by substituting (2.4) in (2.3), we have

$$1 + \frac{1}{b}\left[\frac{zD_qf(z)}{f_{j,k}(z)} - 1\right] = 1 + \frac{1}{2}B_1c_1z + \left[\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2\right]z^2 + \cdots .$$

From this equation, we obtain

$$\frac{[2]_q - \psi_2}{b}a_2 = \frac{B_1c_1}{2}$$
$$\frac{[3]_q - \psi_3}{b}a_3 - \left(\frac{[2]_q - \psi_2}{b}a_2\right)\psi_2a_2^2 = \frac{B_1c_2}{2} - \frac{B_1c_1^2}{4} + \frac{B_2c_1^2}{4},$$

or equivalently

$$a_2 = \frac{B_1c_1b}{2\left([2]_q - \psi_2\right)}$$
$$a_3 = \frac{B_1b}{2\left([3]_q - \psi_3\right)}\left(c_2 - \frac{c_1^2}{2}\left(1 - \frac{B_2}{B_1} - \frac{B_1b\psi_2}{[2]_q - \psi_2}\right)\right).$$

Therefore,

$$|a_3 - \mu a_2^2| \leq \frac{B_1b}{2\left([3]_q - \psi_3\right)}\left(c_2 - vc_1^2\right), \qquad (2.5)$$

where

$$v = \frac{1}{2}\left[1 - \frac{B_2}{B_1} + \frac{B_1b}{[2]_q - \psi_2}\left(\psi_2 - \frac{[3]_q - \psi_3}{[2]_q - \psi_2}\mu\right)\right]. \qquad (2.6)$$

Our result now follows by an application of Lemma 1.4.
The result is sharp for the functions

$$\frac{zD_qf(z)}{f_{j,k}(z)} = \phi(z^2) \qquad \text{and} \qquad \frac{zD_qf(z)}{f_{j,k}(z)} = \phi(z).$$

This completes the proof of Theorem 2.1. □

Similarly, we can prove the following theorem for the class $\mathcal{C}_{j,k}^{q,b}(\phi)$.

Theorem 2.2. *Let $\phi(z) = 1 + B_1z + B_2z^2 + \cdots$ with $B_1 > 0$. If $f(z)$ given by (1.1) belongs to $\mathcal{C}_{j,k}^{q,b}(\phi)$, then*

$$|a_3 - \mu a_2^2| \leq \frac{|B_1b|}{[3]_q\left([3]_q - \psi_3\right)}\max\left\{1; \left|\frac{B_2}{B_1} + \frac{B_1b}{[2]_q - \psi_2}\left(\psi_2 - \frac{[3]_q\left([3]_q - \psi_3\right)}{\left([2]_q\right)^2\left([2]_q - \psi_2\right)}\mu\right)\right|\right\}. \qquad (2.7)$$

The result is sharp.

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Theorem 2.3. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ with $B_1 > 0$ and $B_2 \geq 0$. Let*

$$\sigma_1 = \frac{([2]_q - \psi_2)\, B_1^2 b \psi_2 + ([2]_q - \psi_2)^2\, (B_2 - B_1)}{([3]_q - \psi_3)\, B_1^2 b}, \tag{2.8}$$

$$\sigma_2 = \frac{([2]_q - \psi_2)\, B_1^2 b \psi_2 + ([2]_q - \psi_2)^2\, (B_2 + B_1)}{([3]_q - \psi_3)\, B_1^2 b}, \tag{2.9}$$

$$\sigma_3 = \frac{([2]_q - \psi_2)\, B_1^2 b \psi_2 + ([2]_q - \psi_2)^2\, B_2}{([3]_q - \psi_3)\, B_1^2 b}. \tag{2.10}$$

If $f(z)$ given by (1.1) belongs to $\mathcal{S}_{j,k}^{q,b}(\phi)$ with $b > 0$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{B_2 b}{[3]_q - \psi_3} + \frac{B_1^2 b^2}{[2]_q - \psi_2} \left(\frac{\psi_2}{[3]_q - \psi_3} - \frac{\mu}{[2]_q - \psi_2} \right) & if \quad \mu \leq \sigma_1, \\ \frac{B_1 b}{[3]_q - \psi_3} & if \quad \sigma_1 \leq \mu \leq \sigma_2, \\ -\frac{B_2 b}{[3]_q - \psi_3} - \frac{B_1^2 b^2}{[2]_q - \psi_2} \left(\frac{\psi_2}{[3]_q - \psi_3} - \frac{\mu}{[2]_q - \psi_2} \right) & if \quad \mu \geq \sigma_2. \end{cases} \tag{2.11}$$

Further, if $\sigma_1 \leq \mu \leq \sigma_3$, then

$$|a_3 - \mu a_2^2| + \frac{([2]_q - \psi_2)^2}{([3]_q - \psi_3) B_1^2 b} \left[B_1 - B_2 - \frac{B_1^2 b}{[2]_q - \psi_2} \left(\psi_2 - \frac{[3]_q - \psi_3}{[2]_q - \psi_2} \mu \right) \right] |a_2|^2 \leq \frac{B_1 b}{[3]_q - \psi_3}. \tag{2.12}$$

If $\sigma_3 \leq \mu \leq \sigma_2$, then

$$|a_3 - \mu a_2^2| + \frac{([2]_q - \psi_2)^2}{([3]_q - \psi_3) B_1^2 b} \left[B_1 + B_2 + \frac{B_1^2 b}{[2]_q - \psi_2} \left(\psi_2 - \frac{[3]_q - \psi_3}{[2]_q - \psi_2} \mu \right) \right] |a_2|^2 \leq \frac{B_1 b}{[3]_q - \psi_3}. \tag{2.13}$$

The result is sharp.

Proof. Applying Lemma 1.5 to (2.5) and (2.6), we can obtain our results. To show that the bounds are sharp, we define the functions $\mathscr{K}_{\phi n} (n = 2, 3, 4 \ldots)$ by

$$1 + \frac{1}{b} \left(\frac{z D_q \mathscr{K}_{\phi n}(z)}{\mathscr{K}_{\phi n}(z)} - 1 \right) = \phi(z^{n-1}), \qquad \mathscr{K}_{\phi n}(0) = 0 = \mathscr{K}_{\phi n}'(0) - 1$$

and the functions $\mathscr{F}_\lambda$ and $\mathscr{G}_\lambda (0 \leq \lambda \leq 1)$ by

$$1 + \frac{1}{b} \left(\frac{z D_q \mathscr{F}_\lambda(z)}{\mathscr{F}_\lambda(z)} - 1 \right) = \phi \left(\frac{z(z + \lambda)}{1 + \lambda z} \right), \qquad \mathscr{F}_\lambda(0) = 0 = \mathscr{F}_\lambda'(0) - 1$$

and

$$1 + \frac{1}{b} \left(\frac{z D_q \mathscr{G}_\lambda(z)}{\mathscr{G}_\lambda(z)} - 1 \right) = \phi \left(-\frac{1 + \lambda z}{z(z + \lambda)} \right), \qquad \mathscr{G}_\lambda(0) = 0 = \mathscr{G}_\lambda'(0) - 1.$$

Clearly, the functions $\mathscr{K}_{\phi n}, \mathscr{F}_\lambda$ and $\mathscr{G}_\lambda \in \mathcal{S}_{j,k}^{q,b}(\phi)$. If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds if and only if f is $\mathscr{K}_{\phi 2}$, or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, the equality holds if and only if f is $\mathscr{K}_{\phi 3}$, or one of its rotations. If $\mu = \sigma_1$, then the equality holds if and only if f is $\mathscr{F}_\lambda$, or one of its rotations. If $\mu = \sigma_2$, then the equality holds if and only if f is $\mathscr{G}_\lambda$, or one of its rotations. □

Similarly, we can obtain the following theorem.

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Theorem 2.4. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ with $B_1 > 0$ and $B_2 \geq 0$. Let*

$$\chi_1 = \frac{([2]_q)^2 ([2]_q - \psi_2) \left[bB_1^2 + ([2]_q - \psi_2)(B_2 - B_1)\right]}{B_1^2 b[3]_q ([3]_q - \psi_3)},$$

$$\chi_2 = \frac{([2]_q)^2 ([2]_q - \psi_2) \left[bB_1^2 + ([2]_q - \psi_2)(B_2 + B_1)\right]}{B_1^2 b[3]_q ([3]_q - \psi_3)},$$

$$\chi_3 = \frac{([2]_q)^2 ([2]_q - \psi_2) \left[bB_1^2 + ([2]_q - \psi_2) B_2\right]}{B_1^2 b[3]_q ([3]_q - \psi_3)}.$$

If $f(z)$ given by (1.1) belongs to $\mathcal{C}_{j,k}^{q,b}(\phi)$ with $b > 0$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{B_2 b}{[3]_q([3]_q-\psi_3)} + \frac{B_1^2 b^2}{[3]_q([3]_q-\psi_3)([2]_q-\psi_2)} \left(\psi_2 - \frac{[3]_q([3]_q-\psi_3)}{([2]_q)^2([2]_q-\psi_2)}\mu\right) & if \ \ \mu \leq \chi_1, \\ \frac{B_1 b}{[3]_q([3]_q-\psi_3)} & if \ \ \chi_1 \leq \mu \leq \chi_2, \\ -\frac{B_2 b}{[3]_q([3]_q-\psi_3)} - \frac{B_1^2 b^2}{[3]_q([3]_q-\psi_3)([2]_q-\psi_2)} \left(\psi_2 - \frac{[3]_q([3]_q-\psi_3)}{([2]_q)^2([2]_q-\psi_2)}\mu\right) & if \ \ \mu \geq \chi_2. \end{cases} \tag{2.14}$$

Further, if $\chi_1 \leq \mu \leq \chi_3$, then

$$|a_3 - \mu a_2^2| + \frac{([2]_q)^2([2]_q - \psi_2)^2}{[3]_q([3]_q - \psi_3)B_1^2 b} \left[B_1 - B_2 - \frac{B_1^2 b}{[2]_q - \psi_2}\left(\psi_2 - \frac{[3]_q([3]_q - \psi_3)}{([2]_q)^2([2]_q - \psi_2)}\mu\right)\right] |a_2|^2 \leq \frac{B_1 b}{[3]_q([3]_q - \psi_3)}.$$

If $\chi_3 \leq \mu \leq \chi_2$, then

$$|a_3 - \mu a_2^2| + \frac{([2]_q)^2([2]_q - \psi_2)^2}{[3]_q([3]_q - \psi_3)B_1^2 b} \left[B_1 + B_2 + \frac{B_1^2 b}{[2]_q - \psi_2}\left(\psi_2 - \frac{[3]_q([3]_q - \psi_3)}{([2]_q)^2([2]_q - \psi_2)}\mu\right)\right] |a_2|^2 \leq \frac{B_1 b}{[3]_q([3]_q - \psi_3)}.$$

The result is sharp.

If we $q \to 1^-$, $j = k = 1$ and for an appropriate choice ϕ in Theorem 2.1, we have the following.

Corollary 2.5. *Let $f(z) \in \mathcal{A}$ satisfy the inequality*

$$\alpha < Re\left\{1 + \frac{1}{b}\left[\frac{zf'(z)}{f(z)} - 1\right]\right\} < \beta, \tag{2.15}$$

then

$$|a_3 - \mu a_2^2| \leq \frac{|b|(\beta - \alpha)}{\sqrt{2}\pi}\sqrt{1 - \cos\left(\frac{2\pi(1-\alpha)}{\beta - \alpha}\right)} \max\left\{1, \left|\frac{B_2}{B_1} + (1-\mu)bB_1\right|\right\},$$

where

$$B_n = \frac{\beta - \alpha}{n\pi}\, i \left[1 - e^{2n\pi i((1-\alpha)/(\beta-\alpha))}\right].$$

Proof. Let

$$\phi(z) = 1 + \frac{\beta - \alpha}{\pi} i \log\left(\frac{1 - e^{2\pi i((1-\alpha)/(\beta-\alpha))}\, z}{1 - z}\right).$$

Clearly, it can be seen that $\phi(z)$ maps $\mathcal{U}$ onto a convex domain conformally and is of the form

$$h(z) = 1 + \sum_{n=1}^{\infty} B_n z^n$$

where $B_n = \frac{\beta-\alpha}{n\pi} i \left(1 - e^{2n\pi i((1-\alpha)/(\beta-\alpha))}\right)$. From the equivalent subordination condition proved by Kuroki and Owa in [24], the inequality (2.15) can be rewritten in the form

$$1 + \frac{1}{b}\left[\frac{zf'(z)}{f(z)} - 1\right] \prec \phi(z).$$

Following the steps as in Theorem 2.1, we get the desired result. □

Taking $q \to 1^-$ in Theorem 2.1, we obtain the following result for the functions belonging to the class $\mathcal{S}^b_{j,k}(\phi)$.

Corollary 2.6. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots (B_1 \neq 0)$. If $f(z)$given by (1.1) belongs to the class $\mathcal{S}^b_{j,k}(\phi)$, then*

$$|a_3 - \mu a_2^2| \leq \frac{|B_1||b|}{3-\psi_3} \max\left\{1; \left|\frac{B_2}{B_1} + \frac{B_1 b}{2-\psi_2}\left(\psi_2 - \frac{3-\psi_3}{2-\psi_2}\mu\right)\right|\right\}.$$

The result is sharp.

Taking $q \to 1^-$ in Theorem 2.2, we obtain the following result for the functions belonging to the class $\mathcal{C}^b_{j,k}(\phi)$.

Corollary 2.7. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots (B_1 \neq 0)$. If $f(z)$ given by (1.1)belongs to the class $\mathcal{C}^b_{j,k}(\phi)$, then*

$$|a_3 - \mu a_2^2| \leq \frac{|B_1||b|}{3^2-3\psi_3} \max\left\{1; \left|\frac{B_2}{B_1} + \frac{B_1 b}{2-\psi_2}\left(\psi_2 - \frac{3^2-3\psi_3}{4(2-\psi_2)}\mu\right)\right|\right\}.$$

The result is sharp.

Taking $q \to 1^-$ in Theorem 2.3, we obtain the following result for the functions belonging to the class $\mathcal{S}^b_{j,k}(\phi)$.

Corollary 2.8. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ with $B_1 > 0$ and $B_2 \geq 0$. Let*

$$\sigma_4 = \frac{B_1^2 b\psi_2(2-\psi_2) + (B_2 - B_1)(2-\psi_2)^2}{B_1^2 b(3-\psi_3)},$$

$$\sigma_5 = \frac{B_1^2 b\psi_2(2-\psi_2) + (B_2 + B_1)(2-\psi_2)^2}{B_1^2 b(3-\psi_3)},$$

$$\sigma_6 = \frac{B_1^2 b\psi_2(2-\psi_2) + B_2(2-\psi_2)^2}{B_1^2 b(3-\psi_3)}.$$

If $f(z)$given by (1.1) belongs to the class $\mathcal{S}^b_{j,k}(\phi)$ with $b > 0$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{B_2 b}{3-\psi_3} + \frac{B_1^2 b^2}{(2-\psi_2)(3-\psi_3)}\left(\psi_2 - \frac{3-\psi_3}{2-\psi_2}\mu\right) & if \ \ \mu \leq \sigma_4, \\ \frac{B_1 b}{3-\psi_3} & if \ \ \sigma_4 \leq \mu \leq \sigma_5, \\ \frac{-B_2 b}{3-\psi_3} - \frac{B_1^2 b^2}{(2-\psi_2)(3-\psi_3)}\left(\psi_2 - \frac{3-\psi_3}{2-\psi_2}\mu\right) & if \ \ \mu \geq \sigma_5. \end{cases}$$

Further, if $\sigma_4 \leq \mu \leq \sigma_6$, then

$$|a_3 - \mu a_2^2| + \frac{(2-\psi)^2}{B_1^2 b(3-\psi_3)}\left[B_1 - B_2 - \frac{B_1^2 b}{2-\psi_2}\left(1 - \frac{3-\psi_3}{2-\psi_2}\mu\right)\right]|a_2|^2 \leq \frac{B_1 b}{3-\psi_3}.$$

If $\sigma_6 \leq \mu \leq \sigma_5$, then

$$|a_3 - \mu a_2^2| + \frac{(2-\psi)^2}{B_1^2 b(3-\psi_3)}\left[B_1 + B_2 + \frac{B_1^2 b}{2-\psi_2}\left(1 - \frac{3-\psi_3}{2-\psi_2}\mu\right)\right]|a_2|^2 \leq \frac{B_1 b}{3-\psi_3}.$$

The result is sharp.

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Taking $q \to 1^-$ in Theorem 2.4, we obtain the following result for the functions belonging to the class $\mathcal{C}^b_{j,k}(\phi)$.

Corollary 2.9. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ with $B_1 > 0$ and $B_2 \geq 0$. Let*

$$\chi_4 = \frac{4(2-\psi_2)[B_1^2 b\psi_2 + (B_2 - B_1)(2-\psi_2)]}{B_1^2 b(3^2 - 3\psi_3)},$$

$$\chi_5 = \frac{4(2-\psi_2)[B_1^2 b\psi_2 + (B_2 + B_1)(2-\psi_2)]}{B_1^2 b(3^2 - 3\psi_3)},$$

$$\chi_6 = \frac{4(2-\psi_2)[B_1^2 b\psi_2 + B_2(2-\psi_2)]}{B_1^2 b(3^2 - 3\psi_3)}.$$

If $f(z)$ given by (1.1) belongs to $\mathcal{C}^{q,b}_{j,k}(\phi)$ with $b > 0$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{B_2 b}{3^2-3\psi_3} + \frac{B_1^2 b^2}{(3^2-3\psi_3)(2-\psi_2)}\left(\psi_2 - \frac{3^2-3\psi_3}{4(2-\psi_2)}\mu\right) & if\ \ \mu \leq \chi_4, \\ \frac{B_1 b}{3^2-3\psi_3} & if\ \ \chi_4 \leq \mu \leq \chi_5, \\ -\frac{B_2 b}{3^2-3\psi_3} - \frac{B_1^2 b^2}{(3^2-3\psi_3)(2-\psi_2)}\left(\psi_2 - \frac{3^2-3\psi_3}{4(2-\psi_2)}\mu\right) & if\ \ \mu \geq \chi_5. \end{cases}$$

Further, if $\chi_4 \leq \mu \leq \chi_6$, then

$$|a_3 - \mu a_2^2| + \frac{4(2-\psi_2)^2}{B_1^2 b(3^2-3\psi_3)}\left[B_1 - B_2 - \frac{B_1^2 b}{2-\psi_2}\left(\psi_2 - \frac{3^2-3\psi_3}{4(2-\psi_2)}\mu\right)\right]|a_2|^2 \leq \frac{B_1 b}{3^2-3\psi_3}.$$

If $\chi_3 \leq \mu \leq \chi_2$, then

$$|a_3 - \mu a_2^2| + \frac{4(2-\psi_2)^2}{B_1^2 b(3^2-3\psi_3)}\left[B_1 + B_2 + \frac{B_1^2 b}{2-\psi_2}\left(\psi_2 - \frac{3^2-3\psi_3}{4(2-\psi_2)}\mu\right)\right]|a_2|^2 \leq \frac{B_1 b}{3^2-3\psi_3}.$$

The result is sharp.

Remark 2.10. *For the special case $j = 1$ and $k = 1$ in Theorem 2.1, 2.2, 2.3 and 2.4, we get the results similar to those obtained by Seoudy and Aouf (see Theorem 1, 2, 3 and 4 of* [40]*).*

Remark 2.11. *For the special case $j = 1$ and $k = 1$ in Corollary 2.6, we get the result similar to those obtained by Ravichandran et al.* [35].

Remark 2.12. *For the special case $j = 1$ and $k = 1$ in Corollary 2.7, 2.8 and 2.9 we get the results similar to those obtained by Seoudy and Aouf (See Corollary 2, 3 and 4 of* [40]*).*

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