ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2020, VOLUME 10, ISSUE 1, p.73-81

A GENERALIZATION OF THE TOTAL GRAPH OF SEMIMODULES OVER COMMUTATIVE SEMIRINGS

FATEMEH ESMAEILI KHALIL SARAEI

Abstract. Let M be a semimodule over a commutative semiring R and U a nonempty proper subset of M. In this paper, a generalization of the total graph $T(\Gamma(M))$, denoted by $T_U(\Gamma(M))$ is presented, where U is a *multiplicative-prime* subset of M. It is the graph with all elements of M as vertices, and for distinct $m, n \in M$, the vertices m and n are adjacent if and only if $m + n \in U$. Among other things, the diameter and the girth of $T(\Gamma(M))$ are also studied.

Article history:
Received 28 February 2018
Accepted 5 April 2019

1. Introduction

Throughout this paper all semirings are commutative with nonzero identity. Recently, there has been considerable in the literature to associating graphs with algebraic structures (see [2], [3], [8] and [11]). The concept of a total graph goes back to Anderson and Badawi in [1]. The total torsion element graph of semimodule over commutative semirings (denoted by $T(\Gamma(M))$ was introduced in [9]. The set of vertices of $T(\Gamma(M))$ is all elements of M, and two distinct vertices m and n are adjacent whenever $m+n \in T(M)$ (that is, $T(M) = \{m \in M : rm = 0 \ for\ some \ 0 \neq r \in R\}$).

Let R be a commutative semiring, M be an R-semimodule and let U be a nonempty subset of M. The subset $\{r \in R : rM \subseteq U\}$ will be denoted by $(U :_R M)$ or $(U : M)$. It is clear that if U is a subsemimodule of M, then $(U : M)$ is an ideal of R. We define a nonempty subset U of M to be a *multiplicative-prime* subset of M if the following two conditions hold: (i) $rm \in U$ for every $r \in R$ and $m \in U$; (ii) if $sx \in U$ for some $s \in R$ and $x \in M$, then $x \in U$ or $s \in (U : M)$. It is clear that $0 \in U$. Note that if U is a subsemimodule of M, then U is necessarily a prime subsemimodule of M. In the present paper, we introduce and investigate the generalized total graph of M, denoted by $T_U(\Gamma(M))$, as a (undirected) graph with all elements of M as vertices, and for distinct $m, n \in M$, the vertices m and n are adjacent if and only if $m + n \in U$.

Let $T_U(\Gamma(U))$ be the (induced) subgraph of $T_U(\Gamma(M))$ with vertex set U, and let $T_U(\Gamma(M \setminus U))$ be the (induced) subgraph $T_U(\Gamma(M))$ with vertices consisting of $M \setminus U$.

The study of $T_U(\Gamma(M))$ breaks naturally into two cases depending on whether or not U is a subsemimodule of M. In Section 3, we investigate the homomorphic character of semimodule which epicts the correcponding graphical character. In the forth section, we handle the case when U is a subsemimodule of M; in Section 5, we do the case when U is not a subsemimodule of M. For every case, we characterize the girths and diameters of $T_U(\Gamma(M))$, $T_U(\Gamma(U))$ and $T_U(\Gamma(M \setminus U)$.

2010 *Mathematics Subject Classification.* 16Y60, 05C75.

Key words and phrases. Multiplicative-prime, k-subsemimodule, Q_M-subsemimodule.

2. Preliminaries

For the sake of completeness, we state some definitions and notations used throughout. For a graph Γ, by $E(\Gamma)$ and $V(\Gamma)$, we denote the set of all edges and vertices, respectively. We recall that a graph is connected if there exists a path connecting any two distinct vertices. The distance between two distinct vertices a and b, denoted by $\mathrm{d}(a,b)$, is the length of a shortest path connecting them (if such a path does not exist, then $d(a,a)=0$ and $d(a,b)=\infty$). The diameter of a graph Γ, denoted by $\mathrm{diam}(\Gamma)$, is equal to $\sup\{d(a,b) : a,b \in V(\Gamma)\}$. The degree of a vertex of a graph G is the number of edges incident to the vertex. The degree of a vertex ν is denoted by $deg(\nu)$. The minimum degree of a graph G denoted by $\delta(G)$ is the minimum degree of its vertices. A graph is complete if it is connected with diameter less than or equal to one. The girth of a graph Γ, denoted $\mathrm{gr}(\Gamma)$, is the length of a shortest cycle in Γ, provided Γ contains a cycle; otherwise; $\mathrm{gr}(\Gamma)=\infty$. We denote the complete graph on n vertices by K^n and the complete bipartite graph on m and n vertices by $K^{m,n}$ (we allow m and n to be infinite cardinals). We say that two (induced) subgraphs Γ_1 and Γ_2 of Γ are disjoint if Γ_1 and Γ_2 have no common vertices and no vertex of Γ_1 (respectively, Γ_2) is adjacent (in Γ) to any vertex not in Γ_1 (respectively, Γ_2).

Throughout this paper R is a commutative semiring with identity. In order to make this paper easier to follow, we recall in this section various notions from semimodule theory which will be used in the sequel. For the definitions of monoid, semirings, semimodules and subsemimodules we refer [4],[5] and [6]. Let M be a semimodule over a commutative semiring R.

(1) A subtractive subsemimodule (= k-subsemimodule) N is a subsemimodule of M such that if $x, x+y \in N$, then $y \in N$ (so $\{0_M\}$ is a k-subsemimodule of M).

(2) An element x of M is called a zero-sum in M if $x+y=0$ for some $y \in M$. We use $S(M)$ to denote the set of all zero-sum elements of M.

(3) A subsemimodule N of a semimodule M over a semiring R is called a partitioning subsemimodule (= Q_M-subsemimodule) if there exists a subset Q_M of M such that $M = \cup\{q+N : q \in Q_M\}$ and if $q_1, q_2 \in Q_M$ then $(q_1+N)\cap(q_2+N) \neq \emptyset$ if and only if $q_1 = q_2$. Let N be a Q_M-subsemimodule of M and let $M/N = \{q+N : q \in Q_M\}$. Then M/N forms an R-semimodule under the operations $\oplus$ and $\odot$ defined as follows: $(q_1+N)\oplus(q_2+N) = q_3+N$, where $q_3 \in Q_M$ is the unique element such that $q_1+q_2+N \subseteq q_3+N$ and $r\odot(q_1+N) = q_4+I$, where $r \in R$ and $q_4 \in Q_M$ is the unique element such that $rq_1+N \subseteq q_4+N$. This R-semimodule M/N is called the quotient semimodule of M by N [7]. By [7, Lemma 2.3], there exists a unique element $q_0 \in Q_M$ such that $q_0+N = N$. Thus q_0+N is the zero element of M/N.

3. Total graph under semimodule homomorphism

In this section we investigate the homomorphic character of semimodule which epicts the correcponding graphical character. We begin with the following lemma.

Lemma 3.1. *Let M be a semimodule over a commutative semiring R and U be a multiplicative-prime subset of M. If $f : M \to M'$ is a semimodule epimorphism with $Kerf \subseteq U$, then $f(U)$ is a multiplicative-prime subset of M'.*

Proof. Suppose that $r \in R$ and $y \in f(U)$. So $y = f(x)$ for some $x \in U$. Then $rx \in U$ since U is a *multiplicative-prime* subset of M. Thus $ry = rf(x) = f(rx) \in f(U)$. Now assume that $rm' \in f(U)$ for some $r \in R$ and $m' \in M'$. So $rm' = f(u)$ for some $u \in U$. Then $m' = f(x)$ for some $m \in M$ and $f(rm) = rm' = f(u)$.Then $rm - u \in Kerf \subseteq U$, hence $rm \in U$. Since U is a *multiplicative-prime* subset of M, then either $r \in (U : M)$ or $m \in U$. On the other hand, it is easy to see that $(U : M) = (f(U) : M')$ since f is an epimorphism. Then either $r \in (f(U) : M')$ or $m' \in f(U)$. So $f(U)$ is a *multiplicative-prime* subset of M'. □

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2020, VOLUME 10, ISSUE 1, p.73-81

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2020, VOLUME 10, ISSUE 1, p.73-81

Lemma 3.2. *Let M be a semimodule over a commutative semiring R and U be a multiplicative-prime subset of M. If $f : M \to M'$ is a semimodule epimorphism with $Ker f \subseteq U$. If x is adjacent to y in $T_U(\Gamma(M))$, then $f(x)$ is adjacent to $f(y)$ in $T_{f(U)}(\Gamma(M'))$.*

Proof. Suppose that x is adjacent to y, so $x + y \in U$. Thus $f(x) + f(y) = f(x + y) \in f(U)$. □

Theorem 3.3. *Let M be a semimodule over a commutative semiring R and U be a multiplicative-prime subset of M. If $f : M \to M'$ is a semimodule epimorphism with $Ker f \subseteq U$ and $T_U(\Gamma(M))$ is a complete graph, then so is $T_{f(U)}(\Gamma(M'))$.*

Proof. Let $m', n' \in M'$. Then there exist $m, n \in M$, such that $f(m) = m'$ and $f(n) = n'$. Since $T_U(\Gamma(M))$ is a complete graph, so $m + n \in U$. Hence $m' + n' = f(m + n) \in f(U)$ and $T_{f(U)}(\Gamma(M'))$ is a complete graph. □

We end this section with a theorem that shows the relationship of the diameters and the girths between $T_U(\Gamma(M))$ and $T_{f(U)}(\Gamma(M'))$ for an R-semimodule's epimorphism $f : M \to M'$.

Theorem 3.4. *Let M be a semimodule over a commutative semiring R and U be a multiplicative-prime subset of M. If $f : M \to M'$ is a semimodule epimorphism with $Ker f \subseteq U$. Then the following hold:*
(1) If $T_U(\Gamma(M))$ is connected, then $T_{f(U)}(\Gamma(M'))$ is connected.
(2) If $diam(T_U(\Gamma(M))) = n$, then $diam(T_{f(U)}(\Gamma(M'))) \leq n$.
(3) If $gr(T_U(\Gamma(M))) = n$, then $gr(T_{f(U)}(\Gamma(M'))) \leq n$.

Proof. (1) Let $m', n' \in M'$. Then $f(m) = m'$ and $f(n) = n'$ for some $m, n \in M$. Since $T_U(\Gamma(M))$ is connected, so there exists a path $m - m_1 - m_2 - ... - m_k - n$ from m to n. So $m + m_1, m_i + m_{i+1}, m_k + n \in U$ for each $i = 1, 2, ..., k - 1$. Then $f(m) + f(m_1), f(m_i) + f(m_{i+1}), f(m_k) + f(n) \in f(U)$ for each $i = 1, 2, ..., k - 1$. So $m' = f(m) - f(m_1) - f(m_2) - ... - f(m_k) - f(n) = n'$ is a path from m' to n'. Hence $T_{f(U)}(\Gamma(M'))$ is connected.
(2) It is clear by part (1).
(3) Let $gr(T_U(\Gamma(M))) = n$. So there exists a cycle $x_1 - x_2 - ... - x_n - x_1$ in $T_U(\Gamma(M))$. Then $x_n + x_1, x_i + x_{i+1} \in U$ for each $i = 1, 2, .., n - 1$. So $f(x_1) - f(x_2) - ... - f(x_n) - f(x_1)$ is a cycle of lenght n in $T_{f(U)}(\Gamma(M'))$. So by definition, we have $gr(T_{f(U)}(\Gamma(M'))) \leq n$. □

4. The case when U is a subsemimodule of M

Let M be a semimodule over a commutative semiring R. In this section, we study the case when U is a subsemimodule of M. Note that since U is a subsemimodule, then U is necessarily a prime subsemimodule.

Theorem 4.1. *Let M be a semimodule over a semiring R such that U is a subsemimodule of M. Then:*
(i) $T_U(\Gamma(M))$ is complete if and only if $U = M$.
(ii) $T_U(\Gamma(M))$ is totally disconnected if and only if $U = S(M) = \{0\}$.

Proof. (i) If $U = M$, then for any two vertices $x, y \in M$, one has $x + y \in U$; hence they are adjacent in $T_U(\Gamma(M))$. Conversely, assume that $T_U(\Gamma(M))$ is complete and let $m \in M$. Then m is adjacent to 0. Thus $m = m + 0 \in U$, and hence we have equality.

(ii) Let $T_U(\Gamma(M))$ be totally disconnected. Then 0 is not adjacent to any vertex; hence $x = x + 0 \notin U$ for every non-zero element x of M. Thus $U = \{0\}$. If there is a non-zero element m of $S(M)$, then there exists $0 \neq m' \in M$ such that $m + m' = 0 \in U$, which is a contradiction. Thus $S(M) = \{0\}$. Conversely, assume that there exist distinct $a, b \in M$ such that $a + b \in U = \{0\}$. Then $a, b \in S(M)$, a contradiction. Hence $T_U(\Gamma(M))$ is totally disconnected. □

Proposition 4.2. *Let M be a semimodule over a commutative semiring R such that U is a subsemimodule of M. Then $T_U(\Gamma(U))$ is a complete (induced) subgraph of $T_U(\Gamma(M))$.*

Proof. The proof is straightforward. □

By [10, Theorem 3.1], if U is a prime submodule of M, then $T_U(\Gamma(U))$ is disjoint from $T_U(\Gamma(M \setminus U))$. But the following theorem shows that for R-semimodules, it is necessary to have U is a k-subsemimodule of M.

Theorem 4.3. *Let M be a semimodule over a commutative semiring R such that U is a subsemimodule of M. Then:*

(i) If U is a k-subsemimodule of M, then $T_U(\Gamma(U))$ is disjoint from $T_U(\Gamma(M \setminus U))$.

(ii) If U is not a k-subsemimodule of M, then $T_U(\Gamma(U))$ is not disjoint from $T_U(\Gamma(M \setminus U))$.

Proof. (i) Let U be a k-subsemimodule of M. If $T_U(\Gamma(U))$ is not disjoint from $T_U(\Gamma(M \setminus U))$, then there exist $a \in U$ and $b \in M \setminus U$ such that $a + b \in U$. Thus $b \in U$ since U is a k-subsemimodule of M which is a contradiction. Thus $T_U(\Gamma(U))$ is disjoint from $T_U(\Gamma(M \setminus U))$.

(ii) Assume that U is not a k-subsemimodule of M. So there exist $a \in U$ and $b \in M \setminus U$ such that $a + b \in U$. So $T_U(\Gamma(U))$ is not disjoint from $T_U(\Gamma(M \setminus U))$. □

Example 4.4. *Let $R = \mathbb{Z}^* = \mathbb{Z}^+ \cup \{0\}$ and $M = \mathbb{Z}^* \cup \{\infty\}$. Then $(R, +, .)$ is a commutative semiring and (M, max) is an R-semimodule and $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, \infty\}$ is a subsemimodule of M by [4, Example 2.2].*
It is easy to see that U is a prime subsemimodule of M with $(U : M) = 0$, but U is not a k-subsemimodule, since ∞, $\infty + 9 \in U$ but $9 \notin U$. So $T_U(\Gamma(U))$ is not disjoint from $T_U(\Gamma(M \setminus U))$ by Theorem 4.3.

Now, we show that if U is not a k-subsemimodule of M, the connectivity of $T_U(\Gamma(M \setminus U))$ leads to the connectivity of $T_U(\Gamma(M))$.

Theorem 4.5. *Let M be a semimodule over a commutative semiring R and let U be a subsemimodule of M such that U is not a k-subsemimodule. If $T_U(\Gamma(M \setminus U))$ is connected, then $T_U(\Gamma(M))$ is connected.*

Proof. Suppose that $T_U(\Gamma(M \setminus U))$ is connected, it suffices to show that there is a path between m and n in $T_U(\Gamma(M))$ for every $m \in U$ and $n \in M \setminus U$ by Proposition 4.2. By Theorem 4.3, there exist $a \in U$ and $b \in M \setminus U$ such that $a + b \in U$. Since $T_U(\Gamma(U))$ is complete, there is an edge between a and m. Also since $T_U(\Gamma(M \setminus U))$ is connected, then there is a path from b to n. So there is a path from m to n in $T_U(\Gamma(M))$ and so $T_U(\Gamma(M))$ is connected. □

Now, we give the main result of this section. The next theorem gives a complete description of $T_U(\Gamma(M))$. We allow α and β to be infinite cardinals.

Theorem 4.6. *Let M be a semimodule over a commutative semiring R such that U is a subsemimodule of M and let $|U| = \alpha$ and $|Q_M \setminus \{q_0\}| = \beta$. Then:*

(i) If U is a k-subsemimodule of M and $2 \in (U : M)$, then $T_U(\Gamma(M \setminus U))$ is the union of disjoint complete subgraphs.

(ii) If U is a k-subsemimodule of M and $2 \notin (U : M)$, then $T_U(\Gamma(M \setminus U))$ is the union of totally disconnected subgraphs and some connected subgraphs.

(iii) If U is a Q_M-subsemimodule of M and $2 \in (U : M)$, then $T_U(\Gamma(M \setminus U))$ is the union of β disjoint K^λ's such that $\lambda \leq \alpha$.

(iv) If U is a Q_M-subsemimodule of M and $2 \notin (U : M)$, then $T_U(\Gamma(M \setminus U))$ is the union of totally disconnected subgraphs and complete bipartite subgraphs.

Proof. (i) Let $2 \in (U : M)$. We set up an equivalence relation $\sim$ on $M \setminus U$ as follows: for $y, y' \in M \setminus U$, we write $y \sim y'$ if and only if $y + y' \in U$. It is straightforward to check that $\sim$ is an equivalence relation on $M \setminus U$, since U is a prime k-subsemimodule. For $y \in M \setminus U$, we denote the equivalence class which contains y by $[y]$. Now let $y \in M \setminus U$. If $[y] = \{y\}$, then $(y + a) + (y + b) = 2y + (a + b) \in U$ for every $a, b \in U$ since $2 \in (U : M)$ and U is a k-subsemimodule. So $y + U$ is a complete subgraph with at most α vertices. If $|[y]| = \gamma > 1$, then for every $y' \in [y]$ we have $(y + a) + (y' + b) = (y + y') + a + b \in U$, where $a, b \in U$. Thus $y + U$ is a part of a complete graph K^ν with $\nu \leq \alpha\gamma$ vertices. Therefore, $T_U(\Gamma(M \setminus U))$ is the union of disjoint complete subgraphs.

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2020, VOLUME 10, ISSUE 1, p.73-81

(ii) Let $2 \notin (U : M)$ and $y \in M \setminus U$. Set

$$A_y = \{y' \in M \setminus U : y + y' \in U\}.$$

If $A_y = \emptyset$, then $y+y' \notin U$ for every $y' \in M \setminus U$. In this case, we show that $y+U$ is a totally disconnected subgraph of $T_U(\Gamma(M \setminus U))$. If $(y+a)+(y+b) \in U$ for some $a, b \in U$, then $2y+a+b = (y+a)+(y+b) \in U$; so $2y \in U$, which is a contradiction since U is a prime k-subsemimodule. Therefore, $y + U$ is a totally disconnected subgraph of $T_U(\Gamma(M \setminus U)$. We may assume that $A_y \neq \emptyset$. Then $y + y' \in U$ for some $y' \in M \setminus U$. Thus $(y+a)+(y'+b) = (y+y')+(a+b) \in U$ for every $a, b \in U$; hence each element of $y+U$ is adjacent to each element of $y' + U$. If $|A_y| = \nu$, then we have a connected subgraph of $T_U(\Gamma(M \setminus U))$ with at most $\alpha\nu$ vertices. Hence, if $2 \notin (U : M)$, then $T_U(\Gamma(M \setminus U))$ is the union of totally disconnected subgraphs and some connected subgraphs.

(iii) First we show that $Q_M \setminus \{q_0\} \subseteq M \setminus U$. Let $q \in Q_M \setminus \{q_0\}$. If $q \in U$, then $q \in U \cap Q_M$. So $q + q_0 \in (q + U) \cap (q_0 + U)$, so $(q + U) \cap (q_0 + U) \neq \emptyset$ and so $q = q_0$ which is a contradiction. Then $q + U \subseteq M \setminus U$ for every $q \in Q_M \setminus \{q_0\}$ since U is a k-subsemimodule.
Let $2 \in (U : M)$ and $q \in Q_M \setminus \{q_0\}$. Then each coset $q + U$ is a complete subgraph of $T_U(\Gamma(M \setminus U))$ with λ vertices such that $\lambda \leq \alpha$ (note that $(q_1 + U) \cap (q_2 + U) \neq \emptyset$ if and only if $q_1 = q_2$) since $(q + a) + (q + b) = 2q + (a + b) \in U$ for all $a, b \in U$ since $2 \in (U : M)$ and U is a k-subsemimodule. Now we show that distinct cosets form disjoint subgraphs of $T_U(\Gamma(M \setminus U)$. If $q_1 + a$ and $q_2 + b$ are adjacent for some $q_1, q_2 \in Q_M \setminus \{q_0\}$ and $a, b \in U$, then $(q_1 + a) + (q_2 + b) \in U$ gives $q_1 + q_2 \in U$ since U is a k-subsemimodule of M. So $q_2 + 2q_1 = q_1 + (q_1 + q_2) \in q_1 + U$. Likewise, $q_2 + 2q_1 \in q_2 + U$ since $2 \in (U : M)$. So $q_2 + 2q_1 \in (q_1 + U) \cap (q_2 + U)$; hence $q_1 = q_2$. Thus $T_U(\Gamma(M \setminus U))$ is the union of β disjoint induced subgraphs $q + U$, each of which is a K^λ such that $\lambda \leq \alpha$.

(iv) Now assume that $2 \notin (U : M)$ and let $q \in Q_M \setminus \{q_0\}$. If $q + q' \notin U$ for every $q' \in Q_M \setminus \{q_0\}$, then $A_q = \emptyset$. Then by part (ii), $q + U$ is a totally disconnected subgraph of $T_U(\Gamma(M \setminus U))$, since $Q_M \setminus \{q_0\} \subseteq M \setminus U$ by part (iii) . So we may assume that $q + q' \in U$ for some $q' \in Q_M \setminus \{q_0\}$. Then by (ii) each element of $q + U$ is adjacent to each element of $q' + U$; we show that q' is the unique element. Let $q + q'' \in U$ for some $q'' \in Q_M \setminus \{q_0\}$. Therefore, $q + q' + q'' = q' + (q + q'') \in q' + U$. Likewise, $q+q'+q'' = q''+(q+q') \in q''+U$. Thus $(q'+U)\cap(q''+U) \neq \emptyset$ gives $q' = q''$. Therefore $(q+U)\cup(q'+U)$ is a complete bipartite subgraph of $T_U(\Gamma(M \setminus U))$. So $T_U(\Gamma(M \setminus U))$ is the union of totally disconnected subgraphs and complete bipartite subgraphs. □

Example 4.7. *Let $R = \mathbb{Z}^* = \mathbb{Z}^+ \cup \{0\}$ and $M = \mathbb{Z}^* \times \mathbb{Z}^*$. Then $(R, +, .)$ is a commutative semiring and $(M, +)$ is an R-semimodule. Assume that $U = 2\mathbb{Z}^* \times 2\mathbb{Z}^*$. One can show that $(U : M) = 2R$ and U is a prime k-subsemimodule of M. Then by Theorem 4.6, $T_U(\Gamma(M \setminus U))$ is the union of disjoint complete subgraphs.*

Example 4.8. *Let $R = \mathbb{Z}^* = \mathbb{Z}^+ \cup \{0\}$ and $M = R$. Then for each $m \in R \setminus \{0\}$, Rm is a Q_M-subsemimodule of M where $Q_M = \{0, 1, 2, ..., m - 1\}$.*
(1) If $U = 2R = \{0, 2, 4, 6, ...\}$, then U is a Q_M-subsemimodule of M and $2 \in (U : M)$. So $T_U(\Gamma(M \setminus U))$ is the union of disjoint K^λ by Theorem 4.6.
(2) If $U = 3R = \{0, 3, 6, 9, ...\}$, then U is a Q_M-subsemimodule of M and $2 \notin (U : M)$. $T_U(\Gamma(M \setminus U))$ is the union of totally disconnected subgraphs and complete bipartite subgraphs by Theorem 4.6.

We obtain some result concerning when $T_U(\Gamma(M \setminus U))$ is complete, connected or totally disconnected graph by the following theorem.

Theorem 4.9. *Let M be a semimodule over a commutative semiring R such that U is a Q_M-subsemimodule of M and let $|U| = \alpha$ and $|Q_M \setminus \{q_0\}| = \beta$. Then the following hold:*
(i) $T_U(\Gamma(M \setminus U))$ is complete if and only if either $2 \in (U : M)$ and $|M/U| = 2$ or $2 \notin (U : M)$, $|M/U| = 3$, $|q + U| = |q' + U| = 1$ and $q + q' \in U$ for $q, q' \in Q_M \setminus \{q_0\}$.
(ii) $T_U(\Gamma(M \setminus U))$ is connected if and only if either $2 \in (U : M)$ and $|M/U| = 2$ or $2 \notin (U : M)$, $|M/U| = 3$ and $q + q' \in U$ for $q, q' \in Q_M \setminus \{q_0\}$.

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2020, VOLUME 10, ISSUE 1, p.73-81

(iii) $T_U(\Gamma(M \setminus U))$ is totally disconnected if and only if either $2 \in (U : M)$ and $|q + U| = 1$ for any $q \in Q_M \setminus \{q_0\}$ or $2 \notin (U : M)$ and $q + q' \in M \setminus U$ for every $q, q' \in Q_M \setminus \{q_0\}$.

Proof. (i) Let U be a Q_M-subsemimodule of M and $T_U(\Gamma(M \setminus U))$ be a complete graph. By Theorem 4.6 $T_U(\Gamma(M \setminus U))$ is complete if and only if $T_U(\Gamma(M \setminus U))$ is either a single complete graph K^λ such that $\lambda \leq \alpha$ or $K^{1,1}$. If $2 \in (U : M)$, then $T_U(\Gamma(M \setminus U))$ is a single graph K^λ and so $\beta = 1$. This implies that $M = U \cup (q + U)$ for $q \in Q_M \setminus \{q_0\}$ and thus $|M/U| = 2$. If $2 \notin (U : M)$, then $T_U(\Gamma(M \setminus U))$ is a complete bipartite graph $K^{1,1}$. Thus $M = U \cup (q+U) \cup (q'+U)$ and $|q+U| = |q'+U| = 1$ and $q+q' \in U$ for $q, q' \in Q_M \setminus \{q_0\}$.
Conversely, suppose that $2 \in (U : M)$ and $|M/U| = 2$, so $M = U \cup (q + U)$ for $q \in Q_M \setminus \{q_0\}$. Assume that $m, n \in M \setminus U$. Then $m, n \in q + U$. Thus there are $u_1, u_2 \in U$ such that $m = q + u_1$ and $n = q + u_2$. Therefore $m + n = (q + u_1) + (q + u_2) = 2q + (u_1 + u_2) \in U$ since U is a subsemimodule and $2 \in (U : M)$. Hence $T_U(\Gamma(M \setminus U))$ is a complete graph. Now, assume that $2 \notin (U : M)$ and $|M/U| = 3$, then $M = U \cup (q + U) \cup (q' + U)$. Let m and m' be distinct elements of $M \setminus U$. So $m \in q + U$ and $m' \in q' + U$ since $|q + U| = |q' + U|$ and $q + q' \in U$. Thus $m = q + u$ and $m' = q' + u'$ for some $u, u' \in U$. Thus $m + m' = (q + q') + (u + u') \in U$ by assumption and since U is a subsemimodule. Hence $T_U(\Gamma(M \setminus U))$ is a complete graph.
(ii) Let U be a Q_M-subsemimodule of M and $T_U(\Gamma(M \setminus U))$ be a connected graph. By Theorem 4.6 $T_U(\Gamma(M \setminus U))$ is either a single complete graph K^λ or a complete bipartite graph. If $2 \in (U : M)$, then $\beta = 1$. So $M = U \cup (q + U)$ for $q \in Q_M \setminus \{q_0\}$ and thus $|M/U| = 2$. If $2 \notin (U : M)$, then $M = U \cup (q + U) \cup (q' + U)$ and $q + q' \in U$ for $q, q' \in Q_M \setminus \{q_0\}$ since $T_U(\Gamma(M \setminus U))$ is a complete bipartite graph. Hence $|M/U| = 3$.
Conversely, suppose that $2 \in (U : M)$ and $|M/U| = 2$, so $M = U \cup (q + U)$ for $q \in Q_M \setminus \{q_0\}$. Since $T_U(\Gamma(M \setminus U))$ is a complete graph by part (i), then it is a connected graph. If $2 \notin (U : M)$, $|M/U| = 3$ and $q + q' \in U$ for $q, q' \in Q_M \setminus \{q_0\}$, then $M = U \cup (q+U) \cup (q'+U)$. By Theorem 4.6 , $(q+U) \cup (q'+U)$ is a complete bipartite graph, so $T_U(\Gamma(M \setminus U))$ is a complete graph.
(iii) Let U be a Q_M-subsemimodule of M and $T_U(\Gamma(M \setminus U))$ be a totally disconnected graph. If $2 \in (U : M)$, since $T_U(\Gamma(M \setminus U))$ is the union of β disjoint K^λ's such that $\lambda \leq \alpha$ by Theorem 4.6, then $|q + U| = 1$ for any $q \in Q_M \setminus \{q_0\}$. If $2 \notin (U : M)$, then $q + q' \in M \setminus U$ for every $q, q' \in Q_M \setminus \{q_0\}$ since $T_U(\Gamma(M \setminus U))$ is totally disconnected by Theorem 4.6.
Conversely, suppose that $2 \in (U : M)$ and $|q + U| = 1$ for any $q \in Q_M \setminus \{q_0\}$, thus $T_U(\Gamma(M \setminus U))$ is the union of β disjoint K^1's by Theorem 4.6 and so $T_U(\Gamma(M \setminus U))$ is totally disconnected. If $2 \notin (U : M)$ and $q + q' \in M \setminus U$ for every $q, q' \in Q_M \setminus \{q_0\}$, then $T_U(\Gamma(M \setminus U))$ is the union of totally disconnected subgraphs by Theorem 4.6 and so $T_U(\Gamma(M \setminus U))$ is totally disconnected. □

Example 4.10. *Let $R = \mathbb{Z}^* = \mathbb{Z}^+ \cup \{0\}$ and $M = \mathbb{Z}_6$. Then $(R, +, .)$ is a commutative semiring and $(M, +_6)$ is an R-semimodule. Set $U = \{0, 2, 4\}$ and $Q_M = \{0, 1\}$. Then U is a prime Q_M-subsemimodule of M. It is easy to see that $2 \in (U : M)$. Since $|M/U| = 2$, so $T_U(\Gamma(M \setminus U))$ is a complete graph by Theorem 4.9.*

Proposition 4.11. *Let M be a semimodule over a commutative semiring R such that U is a prime k-subsemimodule of M. Then the following hold:*
(1) If $2 \in (U : M)$, then $diam(T_U(\Gamma(M \setminus U))) = 0, 1$ or ∞.
(2) If $2 \notin (U : M)$, then $diam(T_U(\Gamma(M \setminus U))) = 0, 1, 2$ or ∞.

Proof. The proof is clear by Theorem 4.6 and Theorem 4.9. □

The next theorem gives a more explicit description of $diam(T_U(\Gamma(M \setminus U)))$.

Theorem 4.12. *Let M be a semimodule over a commutative semiring R such that U is a prime k-subsemimodule of M. Then the following hold:*
(i) $diam(T_U(\Gamma(M \setminus U))) = 0$ if and only if $|M/U| = 1$.
(ii) $diam(T_U(\Gamma(M \setminus U))) = 1$ if and only if $2 \in (U : M)$ and $|M/U| = 2$ or $2 \notin (U : M)$, $|M/U| = 3$,

$|q+U| = |q'+U| = 1$ *and* $q+q' \in U$ *for* $q, q' \in Q_M \setminus \{q_0\}$.
(iii) $diam(T_U(\Gamma(M\setminus U))) = 2$ *if and only if* $2 \notin (U:M)$, $|M/U| = 3$ *and* $q+q' \in U$ *for* $q, q' \in Q_M \setminus \{q_0\}$ *and there is a* $q \in Q_M \setminus \{q_0\}$ *such that* $|q+U| \geq 2$.
(iv) Otherwise $diam(T_U(\Gamma(M \setminus U))) = \infty$

Proof. (i) If $diam(T_U(\Gamma(M\setminus U))) = 0$, then $T_U(\Gamma(M\setminus U))$ is a complete graph K^1 and so $|M/U| = 1$.
(ii) It is clear that $T_U(\Gamma(M\setminus U))$ is a complete graph if and only if $diam(T_U(\Gamma(M\setminus U))) = 1$. So the proof is clear by Theorem 4.9 .
(iii) Let $diam(T_U(\Gamma(M\setminus U))) = 2$. Then $T_U(\Gamma(M\setminus U))$ is a complete bipartite graph $K^{m,n}$ such that $m \geq 2$ or $n \geq 2$. Thus $2 \notin (U:M)$ and $|Q_M \setminus \{q_0\}| = 2$ by Theorem 4.6. Therefore $|M/U| = 3$ and $q+q' \in U$ for $q, q' \in Q_M \setminus \{q_0\}$. Since $m \geq 2$ or $n \geq 2$, we have $|q+U| \geq 2$ or $|q'+U| \geq 2$.
Conversely, let $2 \notin (U:M)$, $|M/U| = 3$. Then $M = U \cup (q+U) \cup (q'+U)$ and $T_U(\Gamma(M\setminus U))$ is a complete bipartite graph since $q+q' \in U$ for $q, q' \in Q_M \setminus \{q_0\}$. Hence $diam(T_U(\Gamma(M\setminus U))) = 2$, since $|q+U| \geq 2$ or $|q'+U| \geq 2$. □

Proposition 4.13. *Let* M *be a semimodule over a commutative semiring* R *such that* U *is a* k*-subsemimodule of* M*. Then* $gr(T_U(\Gamma(M\setminus U))) = 3, 4$ *or* ∞*. In particular, if* $T_U(\Gamma(M\setminus U))$ *contains a cycle,* $gr(T_U(\Gamma(M\setminus U))) \leq 4$.

Proof. Let $T_U(\Gamma(M\setminus U))$ contains a cycle. Then $T_U(\Gamma(M\setminus U))$ is not a totally disconnected graph, so by the proof of Theorem 4.6, $T_U(\Gamma(M\setminus U))$ has either a complete or a complete bipartite subgraph. Therefore, it must contain either a 3-cycle or a 4-cycle. Thus $gr(T_U(\Gamma(M\setminus U))) \leq 4$. □

Now, we explicitly determine $gr(T_U(\Gamma(M\setminus U)))$. The proof breaks naturally into two cases depending on whether or not $2 \in (U:M)$.

Theorem 4.14. *Let* M *be a semimodule over a commutative semiring* R *such that* U *be a* k*-subsemimodule of* M*. Then:*
(i) $gr(T_U(\Gamma(M\setminus U))) = 3$ *if and only if* $2 \in (U:M)$ *and* $|y+U| \geq 3$ *for some* $y \in M\setminus U$.
(ii) $gr(T_U(\Gamma(M\setminus U))) = 4$ *if and only if* $2 \notin (U:M)$ *and* $y+y' \in U$ *for some* $y, y' \in M\setminus U$.
(iii) Otherwise $gr(T_U(\Gamma(M\setminus U))) = \infty$.

Proof. (i) Assume that $gr(T_U(\Gamma(M\setminus U))) = 3$. Then by Theorem 4.6, $T_U(\Gamma(M\setminus U))$ is a complete graph K^λ with $3 \leq \lambda$. Therefore, $2 \in (U:M)$ and $|y+U| \geq 3$ for some $y \in M\setminus U$.
(ii) If $gr(T_U(\Gamma(M\setminus U))) = 4$, then by Theorem 4.6, $T_U(\Gamma(M\setminus U))$ has a complete bipartite subgraph; hence $2 \notin (U:M)$ and $y+y' \in U$ for some $y, y' \in M\setminus U$ by Theorem 4.6. The other implications of (i) and (ii) follows directly from Theorem 4.6. □

We end this section with the following theorem.

Theorem 4.15. *Let* M *be a semimodule over a commutative semiring* R *such that* U *be a* k*-subsemimodule of* M*. Then:*
(i) $gr(T_U(\Gamma(M))) = 3$ *if and only if* $|U| \geq 3$.
(ii) $gr(T_U(\Gamma(M))) = 4$ *if and only if* $2 \notin (U:M)$, $|U| < 3$ *and* $y+y' \in U$ *for some* $y, y' \in M\setminus U$.
(iii) Otherwise, $\mathrm{gr}(T_U(\Gamma(M))) = \infty$.

Proof. (i) This follows from Proposition 4.2.
(ii) Since $gr(T_U(\Gamma(U)) = 3$ or ∞, then $gr(T_U(\Gamma(M\setminus U))) = 4$. Therefore, $2 \notin (U:M)$ and $y+y' \in U$ for some $y, y' \in M\setminus U$ by Theorem 4.14. On the other hand, $gr(T(\Gamma(M)) \neq 3$; so $|U| < 3$. The other implication follows from Theorem 4.6. □

5. The case when U is not a subsemimodule of M

We continue to use the notation already established, so M is a semimodule over a commutative semiring R. In this section , we study the graph $T_U(\Gamma(M))$ when U is not a subsemimodule of M.

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2020, VOLUME 10, ISSUE 1, p.73-81

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2020, VOLUME 10, ISSUE 1, p.73-81

First we have the following examples of *multiplicative-prime* subsets of modules that are not a subsemimodule.

Example 5.1. *Let $R = \mathbb{Z}^* = \mathbb{Z}^+ \cup \{0\}$ and $M = \mathbb{Z}^* \times \mathbb{Z}^*$. Assume that $U_1 = \mathbb{Z}^* \times 2\mathbb{Z}^*$, $U_2 = \mathbb{Z}^* \times 3\mathbb{Z}^*$ and $U = U_1 \cup U_2$. One can show that $(U : M) = 2R \cup 3R$. It is easy to see that U is a multiplicative-prime subset of M but U is not a subsemimodule since $(1, 2), (1, 3) \in U$ and $(1, 2) + (1, 3) = (2, 5) \notin U$.*

Example 5.2. *Let $R = \mathbb{Z}^* = \mathbb{Z}^+ \cup \{0\}$ and $M = R$. Then for each $m \in R \setminus \{0\}$, Rm is a Q_M-subsemimodule of M where $Q_M = \{0, 1, 2, ..., m - 1\}$. Now let $U = 5R \cup 7R$. One can show that $(U : M) = 5R \cup 7R$. It is easy to see that U is a multiplicative-prime subset of M but U is not a subsemimodule since $5, 7 \in U$ and $5 + 7 = 12 \notin U$.*

Lemma 5.3. *Let M be a semimodule over a commutative semiring R such that U is not a subsemimodule of M. Then there are distinct $m, m' \in U^*$ such that $m + m' \in M \setminus U$.*

Proof. It is clear, since U is always closed under scalar multiplication of its elements by elements of R. □

Theorem 5.4. *Let M be a semimodule over a commutative semiring R such that U is not a subsemimodule of M. Then the following hold:*
(i) $T_U(\Gamma(U))$ is connected with $diam(T_U(\Gamma(U))) = 2$.
(ii) Either $gr(T_U(\Gamma(U))) = 3$ or $gr(T_U(\Gamma(U))) = \infty$.
(iii) If $2 \in (0 : M)$ and $gr(T_U(\Gamma(M))) = 3$, then $gr(T_U(\Gamma(U))) = 3$.

Proof. (i) Let $x \in U^*$. Then x is adjacent to 0. Thus $x - 0 - y$ is a path in $T_U(\Gamma(U))$ of length two between any two distinct $x, y \in U^*$. Moreover, there exist nonadjacent $x, y \in U^*$ by Lemma 5.3; thus $diam(T_U(\Gamma(U))) = 2$.
(ii) If $x+y \in U$ for some distinct $x, y \in U^*$, then $0-x-y-0$ is a 3-cycle in $T_U(\Gamma(U))$; so $\mathrm{gr}(T_U(\Gamma(U))) = 3$. Otherwise, $x + y \in M \setminus U$ for all distinct $x, y \in U$. Therefore, in this case, each $x \in U^*$ is adjacent to 0, and no two distinct $x, y \in U^*$ are adjacent. Thus $T_U(\Gamma(U))$ is a star graph with center 0; hence $\mathrm{gr}(T_U(\Gamma(U))) = \infty$.
(iii) Let $m_1 - m_2 - m_3 - m_1$ be a 3-cycle in $T_U(\Gamma(M))$, so $m_1 + m_2, m_2 + m_3, m_3 + m_1 \in U$. If $m_i + m_j = 0$ for some $i \neq j$ where $i, j \in \{1, 2, 3\}$, then $m_j = 2m_i + m_j = m_i + m_i + m_j = m_i$ which is a contradiction. Similarly if $m_i + m_j = m_i + m_k$ for some $i \neq j, k$ and $j \neq k$ where $i, j, k \in \{1, 2, 3\}$, then $m_j = 2m_i + m_j = m_i + (m_i + m_j) = m_i + (m_i + m_k) = 2m_i + m_k = m_k$ which is a contradiction. So $(m_1 + m_2) - (m_2 + m_3) - (m_3 + m_1) - (m_1 + m_2)$ is a 3-cycle in $T_U(\Gamma(U))$., □

Theorem 5.5. *Let M be a semimodule over a commutative semiring R such that U is not a subsemimodule of M. If $U \cap S(M) \neq \{0\}$ and $\delta(T_U(\Gamma(U))) = 1$, then the following hold:*
(i) Some vertex of $T_U(\Gamma(U))$ is adjacent to a vertex of $T_U(\Gamma(M \setminus U))$. In particular, the subgraphs $T_U(\Gamma(U))$ and $T_U(\Gamma(M \setminus U))$ are not disjoint.
(ii) If $T_U(\Gamma(M \setminus U))$ is connected, then $T_U(\Gamma(M))$ is connected.

Proof. (i) Let $0 \neq x \in U \cap S(M)$. Then there exists $y \in M$ such that $x + y = 0$. If $y \in M \setminus U$, Then $x \in U$ and $y \in M \setminus U$ are adjacent vertices in $T_U(\Gamma(M))$. So assume that $y \in U$. Since $\delta(T_U(\Gamma(U))) = 1$, so $deg(z) = 1$ for some $z \in U$. Then z is not adjacent to x, since z is adjacent to zero and $deg(z) = 1$. Therefore $x + z \notin U$. But $(x + z) + y = z \in U$ so $y \in U$ and $x + z \in M \setminus U$ are adjacent vertices in $T_U(\Gamma(M))$. Hence the subgraphs $T_U(\Gamma(U))$ and $T_U(\Gamma(M \setminus U))$ are not disjoint.
(ii) Suppose that $T_U(\Gamma(M \setminus U))$ is connected, it suffices to show that there is a path between m and n in $T_U(\Gamma(M))$ for every $m \in U$ and $n \in M \setminus U$ by Theorem 5.4. By part (i), there exist $a \in U$ and $b \in M \setminus U$ such that $a + b \in U$. Since $T_U(\Gamma(U))$ is connected, there is an edge between a and m. Also since $T_U(\Gamma(M \setminus U))$ is connected, then there is a path from b to n. So there is a path from m to n in $T_U(\Gamma(M))$ and so $T_U(\Gamma(M))$ is connected. □

Lemma 5.6. *Let M be a semimodule over a commutative semiring R such that U is not a subsemimodule of M. Then $|U| \geq 3$.*

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2020, VOLUME 10, ISSUE 1, p.73-81

Proof. There are distinct elements $m, m' \in U^*$ such that $m+m' \in M \setminus U$ by Lemma 5.3. Thus $|U| \geq 3$. □

Theorem 5.7. *Let M be a semimodule over a commutative semiring R such that U is not a subsemimodule of M. If $T_{(U:M)}(\Gamma(R))$ is connected, then $T_U(\Gamma(M))$ is connected.*

Proof. It is clear that if $r \in (U : M)$ and $x \in M$, then $rx \in U$. Assume that $y \in M$ and let $0 - a_1 - a_2 - \cdots - a_k - 1$ be a path from 0 to 1 in $T_{(U:M)}(\Gamma(R))$. So a_1, $a_k + 1$, $a_i + a_{i+1} \in (U : M)$ for each $i = 1, ..., k-1$. Now let $y \in M$. Then $0 - a_1 y - a_2 y - \cdots - a_k y - y$ is a path from 0 to y in $T_U(\Gamma(M))$. Since all vertices may be connected via 0, $T_U(\Gamma(M))$ is connected. □

References

[1] D. F. Anderson and A. Badawi, *The total graph of a commutative ring*, J. Algebra, **320** (2008), 2706-2719.

[2] D. F. Anderson and P. F. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra, **217** (1999), 437-447.

[3] I. Beck, *Coloring of a commutative ring*, J. Algebra, **116** (1988), 208-226.

[4] J. N. Chaudhari and D. R. Bonde, *On subtractive extension of subsemimodules of semimodules* , Journal of the Chungcheong Mathematical Society, **20 (1)** (2013), 37-44.

[5] R. Ebrahimi Atani and S. Ebrahimi Atani,*Ideal theory in commutative semirings*, Bul. Acad. Stiinte Repub. Mold. Mat., **2** (2008), 14-23.

[6] R. Ebrahimi Atani and S. Ebrahimi Atani, *On subsemimodules of semimodules*, Bul. Acad. Stiinte Repub. Mold. Mat., **2 (63)** (2010), 20-30.

[7] S. Ebrahimi Atani and F. Esmaeili Khalil Saraei, *The total graph of a commutative semiring*, An. St. Univ. Ovidius Constanta, **21(2)** (2013), 21-33.

[8] S. Ebrahimi Atani and F. Esmaeili Khalil Saraei , *The total torsion element graph of semimodules over commutative semirings*, Algebra and Discrete Mathematics, **16(1)** (2013), 1-15.

[9] S. Ebrahimi Atani and S. Habibi, *The total torsion element graph of a module over a commutative ring*, An. St. Univ. Ovidius Constanta, **19(1)** (2011), 23-34.

[10] F. Esmaeili Khalil Saraei, H. Heydarinejad Astaneh and R. Navidinia *The total graph of a module with respect to multiplicative-prime subsets*, Romanian Journal of Mathematics and Computer Science, **4 (2)** (2014), 151-166.

[11] Y. Talebi and A. Darzi, *The generalized total graph of a commutative semiring*, Ricerche di matematica, **66 (2)** (2017), 579-589.

Fouman Faculty of Engineering, College of Engineering, University of Tehran,P.O. Box 43515-1155, Fouman, Iran

Email address: `f.esmaeili.kh@ut.ac.ir`