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Γ-SEMIRINGS WITH APARTNESS

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ABSTRACT. The settings of this article is the Bishop's constructive algebra including the Intuitionistic logic. With this text we continue our research of algebraic structures with the relation of apartness. In this paper we introduce the concept of Γ-semirings with apartness. We first consider the ideals and co-ideals of a Γ-semiring with apartness. Also, by using the congruences and co-congruences induced by strongly extensional homomorphisms between such Γ-semirings, we establish an isomorphism theorem.

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1. Introduction

The settings of this article is the Bishop's constructive algebra [**Bish**] including the Intuitionistic logic [**IL**] in the sense of books [1, 5, 8] and articles [2, 3, 4, 13, 14, 15, 16]. Let $(S, =, \neq)$ be a relational system, where the relation $' \neq '$ is an apertness relation - a relation on a set S which is consistent, symmetric and co-transitive. This relation is an extensive relation with respect to the equality relation in S ([2, 13, 15, 16]). This relational system is called '*set with apartness*' or, shortly, a *set* in [Bish] orientation. Since in this system the logical principle of the TND is not an axiom in [**IL**] logical system, all formulas that (directly or indirectly) contain the equality have their own non-equivalent doubles. These specifics generate greater complexity in many algebraic structures than is the case in classical algebra.

In this article, our intention is to recognize, understand and describe as precisely as possible these specificity on the example of one complex algebraic structure, Γ-semirings structure.

The concept of Γ-semirings were first introduced and studied by M. K. Rao [9, 10] as a generalization of notion of Γ-rings.

Many authors have studies on these algebraic structures. For example: H. Hedayati and K. P. Shum [6] (2011), R. Jagatap1 and Y. Pawar [7] (2011) and M. K. Rao [11] (2018). There is an interest in the academic community to study and publish the results of these research on these algebraic structures, their internal organization as well as their substructures in general, as well as in many specific cases.

In this article we will deal with this algebraic structure within the specific environment offered by [**Bish**] orientation. So, we will observe the behavior of these algebraic structures, assuming that all carriers of algebraicity are sets with apartness relations, that all relations, operations, and functions that appear in they are strongly extensive with respect to apartneses. Therefore, we will introduce and analyze the concept of Γ-semirings with apartness. We also analyze the doubles of the congruence relations, the order relations, the ideals, and the filters in such introduced Γ-semirings with apartness.

The notions and notations used in this article but not determined in it, we are take over from our previously published articles [3, 4, 18].

2. Γ-SEMIRINGS WITH APARTNESS

2.1. **Concept of Γ-semirings with apartness.** Looking at the definition of Γ-semigring in the classical sense ([9, 10, 6]), we first introduce the concept of Γ-semirings with apartness which will be used throughout this paper. Let $(R, +, \cdot)$ and $(\Gamma, +, \cdot)$ be commutative semigroups with apartness. About the 'apartness' reader can consult the following books [1, 5, 8]. By this we mean that the sets $R \equiv (R, =_R, \neq_R)$ and $\Gamma \equiv (\Gamma, =_\Gamma, \neq_\Gamma)$ are supplied by apartness relations and that the internal operations in them are strongly extensive total functions. In the following, we do not use indices in the equation relations and apartness relations, except in cases where it is necessary to distinguish them so as not to cause confusion. About the relations, functions and operations in the system [**Bish**] a reader can consult some of our previously published articles such as [2, 15, 16], or any of our bibliographic units listed in the literature of this article ([3, 4, 13, 14, 17, 18]).

Definition 2.1. *We call R a* Γ-semiring with apartness *if there exists a map $R \times \Gamma \times R \longrightarrow R$, written image of (x, a, y) by xay, such that it satisfies the following axioms:*
(1) $(\forall x, y, z \in R)(\forall a \in \Gamma)(xa(y + z) = xay + xaz$ *and* $(x + y)az = xaz + yaz)$,
(2) $(\forall x, y \in R)(\forall a, b \in \Gamma)(x(a + b)y = xay + xby)$,
(3) $(\forall x, y, z \in R)(\forall a, b \in \Gamma)((xay)bz = xa(ybz))$.

Remark 2.2. *As can be seen, the definition of Γ-semirings with apartness is completely identical to the definition of Γ-semiring in the classical case. However, they do not determine the same algebraic structure. The reader should always keep in mind that the logical setting are different and that the manipulation with them takes place with the previously acceptance of the various principles-philosophical orientations. In this environment, the following implication is valid*

$$(\forall x, y, u, v \in R)(\forall a, b \in \Gamma)(xay \neq ubv \implies (x \neq u \lor a \neq b\, y \neq v)).$$

A Γ-semiring with apartness R is said to have a *zero element* if there exists an element $0 \in R$ such that the following

$$(\forall x \in R)(\forall a \in \Gamma)(0 + x = x = x + 0 \text{ and } 0ax = 0 = xa0)$$

is valid. Of course, we also have

$$(\forall x, y \in R)(x + y \neq 0 \implies (x \neq 0 \lor y \neq 0))$$

and

$$(\forall x, y \in R)(\forall a \in \Gamma)(xay \neq 0 \implies (x \neq 0 \land y \neq 0)).$$

Also, a Γ-semiring with apartness R is said to be *commutative* if the following holds

$$(\forall x, y \in R)(\forall a \in \Gamma)(xay = yax).$$

About the slogan '*a function f is an embedding*', which we will use in the following definition, the reader can consult with some of our previously published texts [2, 13, 14, 15, 16].

Definition 2.3. *Let R be a Γ-semiring and $T2$ a Λ-semiring. Then $(f, \varphi) : (R, \Gamma) \longrightarrow (T, \Lambda)$ is called a* homomorphism *if $f : R \longrightarrow T$ and $\varphi : \Gamma \longrightarrow \Lambda$ are strongly extensional homomorphisms of semigroups such that*

$$(\forall x, y \in R)(\forall a \in \Gamma)((f, \varphi)(xay) = f(x)\varphi(a)f(y))$$

holds. The mapping (f, φ) is called an epimorphism *if (f, φ) is a homomorphism and f and φ are epimorphisms of semigroups. Similarly, we can define a monomorphism. A homomorphism (f, φ) is an isomorphism if (f, φ) is an epimorphism and a monomorphism and f and φ are embeddings.*

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2.2. **The concept of cosub-Γ-semirings.** By the following definition we introduce the notion cosub-Γ-semiring. To understand the term 'cosub-semigroup', which we will use in this case, the reader can consult our text [12] or [13, 14, 17].

Definition 2.4. *Let R be a Γ-semiring with apartness*

(1) *A non-empty subset A of R is a* sub-Γ-semiring *of R if A is an additive sub-semigroup of R and the following holds*

$$(\forall x, y \in R)(\forall a \in \Gamma)((x \in A \wedge y \in A) \Longrightarrow xay \in A).$$

(2) *A subset B of R is a* cosub-Γ-semiring *of R if B is an additive cosub-semigroup of R and the following holds*

$$(\forall x, y \in R)(\forall a \in \Gamma)(xay \in B \Longrightarrow (x \in A \vee y \in A)).$$

2.3. **The concept of Γ-coideals.**

Definition 2.5. *Let R be a Γ-semiring with apartness.*

(1) *A subset B of R is a* right Γ-coideal *of R if B is a additive cosub-semigroup of R and the following holds*

$$(\forall x, y \in R)(\forall a \in \Gamma)(xay \in B \Longrightarrow y \in B).$$

(2) *A subset B of R is a* left Γ-coideal *of R if B is a additive cosub-semigroup of R and the following holds*

$$(\forall x, y \in R)(\forall a \in \Gamma)(xay \in B \Longrightarrow x \in B).$$

(3) *A subset B of R is a* Γ-coideal *of R if B is a additive cosub-semigroup of R and the following holds*

$$(\forall x, y \in R)(\forall a \in \Gamma)(xay \in B \Longrightarrow (x \in B \wedge y \in B)).$$

If R is a Γ-semiring with zero element 0, then it is mandatory to assume that $0 \lhd B$..

Proposition 2.6. *If B is (left, right) coideal of a Γ-semiring R, then the set $B^{\lhd}$ is a (letf, ringt) ideal of R.*

Theorem 2.7. *The union of any family $\{B_i\}_{i \in I}$ of (right, left) Γ-coideals of a Γ-semigroup (R, Γ) is a (right, left) Γ-coideal of R.*

Proof. If semiring has the zero element 0, then holds $0 \lhd B_i$ for any $i \in I$. Thus, we have $0 \lhd \bigcup_{i \in I} B_i$.

Let $x, y \in R$ be arbitrary elements such that $x + y \in \bigcup_{i \in I} B_i$. Then $(\exists i \in I)(x + y \in B_i)$ holds. Thus $(\exists i \in I)(x \in B_i \vee y \in B_i)$ and $x \in \bigcup_{i \in I} B_i$ or $y \in\in \bigcup_{i \in I} B_i$. So, the set $\bigcup_{i \in I} B_i$ is a cosubsemigrouyp of R.

Let $x, y \in S$ and $a \in \Gamma$ be arbitrary elements such that $xay \in \bigcup_{i \in I} B_i$. Thus $(\exists i \in I)(xay \in B_i)$. Thus $(\exists i \in I)(xay \Longrightarrow (x \in B_i \in \wedge y \in B_i))$ and $x \in \bigcup_{i \in I} B_i$ and $y \in \bigcup_{i \in I} B_i$. So, the set $\bigcup_{i \in I} B_i$ is a Γ-coideal of Γ-semiring R. □

Corollary 2.8. *Let X be a subset of Γ-semigroup (R, Γ). Then there exists the maximal (left, rignt) Γ-coideal of Γ-semiring (R, Γ) included in X.*

Proof. and $\{B_i\}_{i \in I}$ the family of all (left, rignt) Γ-coideals of Γ-semigrin (R, Γ) included in X. Then the union $\bigcup_{i \in I} B_i$ is the maximal (left, right) Γ-coideal includen in X by previous theorem. □

Corollary 2.9. *Let $\mathfrak{L}(R, \Gamma)$ be the family of all (left, right) Γ-cideals of (R, Γ). Then $(\mathfrak{L}(R, \Gamma), \sqcup, \sqcap)$ is a completely lattice, where $B_1 \sqcup B_2 = B_1 \cup B_2$ and $B_1 \sqcap B_2$ is the maximal coideal included in $B_1 \cap B_2$.*

2.4. **The concept of Γ-cocongruence.**

Definition 2.10. *A co-equality relation q on* Γ*-semiring* (R, Γ) *is said to be a* co-congruence *if the following conditions*

$(\forall x, y, z \in R)((x + z, y + z) \in q \implies (x, y) \in q)$ *and*

$(\forall x, y, z \in R)(\forall a \in \Gamma)(((xaz, yaz) \in q \vee (zax, zay) \in q) \implies (x, y) \in q)$

are satisfied.

It is known ([9]: pp. 51, [6]: Theorem 4.5) that if ρ is a congruence relation on a Γ-semiring (R, Γ), then $R/\rho = \{[x]_\rho : x \in R\}$ is also Γ-semiring where it is

$$(\forall x, y \in R)([x]_\rho + [y]_\rho = [x + y]_\rho),$$

$$(\forall x, y \in R)(\forall a \in \Gamma)([x]_\rho a[y]_\rho = [xay]_\rho).$$

Proposition 2.11. *If q is a* Γ*-cocongruence on a* Γ*-semiring* (R, Γ)*, then the relation* $q^\triangleleft$ *is a* Γ*-congruence on* (R, Γ)*.*

Theorem 2.12. *Let q be a* Γ*-cocongruence on a* Γ*-semigrong* (R, Γ)*. Then the family* $R : q = \{[x]_q : x \in R\}$ *is a* Γ*-semiting also with*

$$(\forall x, y \in R)([x]_q =_1 [y]_1 \iff (x, y) \triangleleft q),$$

$$(\forall x, y \in R)([x]_q \neq_1 [y]_q \iff (x, y) \in q),$$

$$(\forall x, y \in R)([x]_q + [y]_q =_1 [x + y]_q),$$

$$(\forall x, y \in R)(\forall a \in \Gamma)([x]_q a[y]_q =_1 [xay]_q).$$

Lemma 2.13. *Let q be a* Γ*-cocongruence on a* Γ*-semigrong* (R, Γ)*. Then the mapping* $(\pi, i) : R \longrightarrow R : q$*, defined by* $\pi(x) = [x]_q$ *and* $i(a) = a$*, is a strongly extensional epimorphism.*

Lemma 2.14. *If the mapping* $(f, \varphi) : (R, \Gamma) \longrightarrow (T, \Lambda)$ *is a strongly extensional homomoephism, then the relation* $q(f)$ *on R, defined by*

$$(\forall x, y \in R)((x, y) \in q(f) \iff f(x) \neq f(y)),$$

is a Γ*-cocongruence on R.*

Without major difficulties, the following theorem can be proved. We can be viewed on this theorem as on the First Theorem on Isomorphisms using co-congruences in Γ-semirigns with apartness.

Theorem 2.15. *Let* $(f, \varphi) : (R, \Gamma) \longrightarrow (T, \Lambda)$ *be a strongly extensional homomorphism, then there exists the strongly extensional injective and embedding homomorphism* $(g, \varphi) : (R : q(f), \Gamma) \longrightarrow (T, \Lambda)$ *such that*

$$(f, \varphi) = (g, \varphi) \circ (\pi, i).$$

3. Final observation

The specificity of Bishop's constructive aspect of looking at algebraic structures provides a more significant complexity of these algebraic structures, but also their significantly richer subculture family than is the case in classical algebra. This complexity as well as the richness of the family of substructure of the observed algebraic structures will increase significantly if we observe the algebraic structures arranged not only by the quasiorder relations, but also by the co-quasiorderes relations. This would allow for the investigation of at least the following two substructure families: (a) ideals and co-ideals, (b) filters and co-filters in Γ-semirings with apartness.

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References

[1] E. Bishop. *Foundations of Constructive Analysis*. New York: McGraw-Hill, 1967.

[2] M. Božić and D. A. Romano. *Relations, functions and operations in constructive mathematics*. Publ. VTS, Ser. Math., **2**(1986), 25–39.

[3] S. Crvenković, M. Mitrović and D. A. Romano. *Semigroups with apartness*. Math. Logic Quarterly, **59**(6)(2013), 407–414.

[4] S. Crvenković, M. Mitrović and D. A. Romano. *Basic notions of (Constructive) semigroups with apartness*. Semigroup Forum, **92**(3)(2016), 659–674.

[5] D. S. Bridges and F. Richman. *Varieties of Constructive Mathematics*. Cambridge: London Mathematical Society Lecture Notes, No. 97, Cambridge University Press, 1987.

[6] H. Hedayati and K. P. Shum. *An introduction to* Γ*-semirings*. Int. J. Algebra, **5**(15)(2011), 709–726.

[7] R. Jagatap and Y. Pawar. *Right ideals of* Γ*-semirings*. Novi Sad J. Math., **43**(2)(2011), 11–19.

[8] R. Mines, F. Richman and W. Ruitenburg. *A course of constructive algebra*. New York: Springer-Verlag, 1988.

[9] M. K. Rao. Γ*-semirings, 1*. Southeast Asian Bull. Math., **19**(1995), 49–54.

[10] M. K. Rao. Γ*-semirings, II*. Southeast Asian Bull. Math., **21**(1997), 281–287.

[11] M. K. Rao. *Ideals in ordered* Γ*-semirings*. Discuss. Math. General Al. Appl., **38**(1)(2018), 47-–68.

[12] D. A. Romano. *Aspect of constructive Abelian groups*. In: Z. Stojaković (Ed.). Proceedings of the 5th conference "Algebra and Logic", Cetinje 1986 (pp. 167-174). Novi Sad: University of Novi Sad, Institute of Mathematics, 1987.

[13] D. A. Romano. *Rings and fields, a constructive view*. Math. Logic Quarterly (Formerly: Z. Math. Logik Grundl. Math.), **34**(1)(1988), 25–40.

[14] D. A. Romano. *Rings and modules, a constructive view*. In: S. Crvenković (Ed.). Proceedings of the 6th conference "Algebra and Logic", Sarajevo 1987 (pp. 113–140). Novi Sad: University of Novi Sad, Institute of Mathematics, 1989.

[15] D. A. Romano. *Coequality relations, a survey*. Bull. Soc. Math. Banja Luka, **3**(1996), 1–36.

[16] D. A. Romano. *A note on two problems in constructive commutative coideal theory*. Sci. Rev. Ser. Math., **19-20**(1996), 91–96.

[17] D. A. Romano. *Semivaluation on Heyting field*. Kragujevac J. Mat., **20**(1998), 24–40.

[18] D. A. Romano. *Semilattice-ordered semiring with apartness*. J. Adv. Math. Stud., **11**(3) (2018), 496–502.

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