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BI-CONJUGATIVE RELATIONS

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Abstract. In this paper the concept of bi-conjugative relations on sets is introduced. Characterizations of this relations are obtained. In addition, particulary we show that the anti-order relation $\nleqslant$ in poset $(L, \leqslant)$ is not a bi-conjugative relation.

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1. Introduction and preliminaries

The regularity of binary relations was first characterized by Zareckiǐ ([11]). Further criteria for regularity were given by Markowsky ([8]), Schein ([10]) and Xu Xiao-quan and Liu Yingming ([12]) (see also [1] and [2]). The concepts of conjugative relations, dually conjugative relations and dually normal relations were introduced by Guanghao Jiang and Luoshan Xu ([3], [4]), and a characterization of normal relations was introduced and analyzed by Jiang Guanghao, Xu Luoshan, Cai Jin and Han Guiwen in [5].

In this paper, we introduce and analyze bi-conjugative relations on sets.

The following are some basic concepts needed in the sequel, for other nonexplicitly stated elementary notions please refer to papers [1] – [6] and [11], and to book [7].

For a set X, we call ρ a binary relation on X, if $\rho \subseteq X \times X$. Let $\mathcal{B}(X)$ denote the set of all binary relations on X. For $\alpha, \beta \in \mathcal{B}(X)$, define

$$\beta \circ \alpha = \{(x, z) \in X \times X : (\exists y \in X)((x, y) \in \alpha \wedge (y, z) \in \beta)\}.$$

The relation $\beta \circ \alpha$ is called the composition of α and β. It is well known that $(\mathcal{B}(X), \circ)$ a semigroup. The relation $\triangle_X = \{(x, x) : x \in X\}$ is the identity. For a binary relation α on a set X, define $\alpha^{-1} = \{(x, y) \in X \times X : (y, x) \in \alpha\}$ and $\alpha^C = X \times X \setminus \alpha$.

The following classes of elements in the semigroup $\mathcal{B}(X)$, given in the following definition, have been investigated:

Definition 1.1. For relation $\alpha \in \mathcal{B}(X)$ we say that it is:

– *regular* if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha \circ \beta \circ \alpha.$$

– *normal* ([5]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha \circ \beta \circ (\alpha^C)^{-1}.$$

– *dually normal* ([4]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = (\alpha^C)^{-1} \circ \beta \circ \alpha.$$

– *conjugative* ([3]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha^{-1} \circ \beta \circ \alpha.$$

– *dually conjugative* ([3]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha \circ \beta \circ \alpha^{-1}.$$

– *quasi-regular* ([9]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha^C \circ \beta \circ \alpha.$$

– *dually quasi-regular* ([9]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha \circ \beta \circ \alpha^C.$$

Besides that, for $\alpha, \beta \in \mathcal{B}(X)$ and $x, y \in X$, we define the *box product* of relation α and relation β by

$$\begin{aligned}(\alpha \square \beta)(x, y) &= \alpha x \times \beta y \\ &= \{(u, v) \in X \times X : u \in \alpha x \wedge v \in \beta y\}.\end{aligned}$$

Let $\alpha, \beta, \gamma \in \mathcal{B}(X)$ be arbitrary relations, then

$$\gamma \circ \beta \circ \alpha = (\alpha \square \gamma^{-1})(\beta) \tag{1.1}$$

holds. Indeed, we have

$$\begin{aligned}(u, v) \in \gamma \circ \beta \circ \alpha &\iff (\exists a, b \in X)((u, a) \in \alpha \wedge (a, b) \in \beta \wedge (b, v) \in \gamma) \\ &\iff (\exists (a, b) \in \beta)(u \in \alpha a \wedge v \in \gamma^{-1} b) \\ &\iff (\exists (a, b) \in \beta)((u, v) \in \alpha a \times \gamma^{-1} b) \\ &\iff (\exists (a, b) \in \beta)((u, v) \in (\alpha \square \gamma^{-1})(a, b)) \\ &\iff (u, v) \in (\alpha \square \gamma^{-1})(\beta).\end{aligned}$$

Now, we can equations, introduced in Definition 1.1, represent in the new way. For example, a conjugative relation α satisfies the following equation $\alpha = (\alpha \square \alpha)(\beta)$, and if α is a dually conjugative relation, then the following equation $\alpha = (\alpha^{-1} \square \alpha^{-1})(\beta)$ holds. Analogously, a normal relation α is described by $\alpha = ((\alpha^C)^{-1} \square \alpha^{-1})(\beta)$, and a dually normal relation α satisfies the following equation $\alpha = (\alpha \square \alpha^C)(\beta)$. Descriptions of quasi-regular relations and dually quasi-regular relations now appear in the following way: $\alpha = (\alpha \square (\alpha^C)^{-1})(\beta)$ and $\alpha = (\alpha^C \square \alpha^{-1})(\beta)$.

2. Bi-conjugative relations

Put $\alpha^1 = \alpha$. It is easy to see that $(\alpha^{-1})^C = (\alpha^C)^{-1}$ holds. Definition 1.1 describes equalities

$$\alpha = (\alpha^a)^i \circ \beta \circ (\alpha^b)^j$$

for some $\beta \in \mathcal{B}(X)$ where $i, j \in \{-1, 1\}$ and $a, b \in \{1, C\}$. We should investigate all other possibilities since some of possibilities given in the previous equation have been investigated. According to this attitude, in the following definition we introduce a new class of elements in $\mathcal{B}(X)$.

Definition 2.1. For relation $\alpha \in \mathcal{B}(X)$ we say that it is a *bi-conjugative* relation on X if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha^{-1} \circ \beta \circ \alpha^{-1}. \tag{2.1}$$

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It is easy to see that the dual of a bi-conjugative relation α is again a bi-conjugative relation. Besides, for bi-conjugative relation α on a set X the following $Dom(\alpha) = R(\alpha)$ holds.

The family $\mathcal{BC}(X)$ of all bi-conjugative relations on set X is not empty. For example, $\triangle_X \in \mathcal{BC}(X)$ and $\nabla_X = \triangle_X^C \in \mathcal{BC}(X)$. Besides, since for any bijective relation ψ on X

$$\psi = \triangle_X \circ \psi \circ \triangle_X = (\psi^{-1} \circ \psi) \circ \psi \circ (\psi \circ \psi^{-1}) = \psi^{-1} \circ (\psi \circ \psi \circ \psi) \circ \psi^{-1}$$

holds, we have $\psi \in \mathcal{BC}(X)$. For symmetric and idempotent relation α on set X we have

$$\alpha = \alpha^2 = \alpha \circ \triangle_X \circ \alpha = \alpha^{-1} \circ \triangle_X \circ \alpha^{-1}.$$

Therefore, this relation is a bi-conjugative relation on X. Further on, the following implication $\alpha \in \mathcal{BC}(X) \Longrightarrow \alpha^{-1} \in \mathcal{BC}(X)$ holds also.

According to the equation (1.1), the condition (2.1) is equivalent to the following condition

$$\alpha = (\alpha^{-1} \,\square\, \alpha)(\beta). \tag{2.2}$$

Our first proposition is an adaptation of Schein's result exposed in [10], Theorem 1. (See, also, [2], Lemma 1.)

Theorem 2.2. *For a binary relation $\alpha \in \mathcal{B}(X)$, relation*

$$\alpha^* = (\alpha \circ \alpha^C \circ \alpha)^C$$

is the maximal element in the family of all relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha^{-1} \circ \beta \circ \alpha^{-1} \subseteq \alpha.$$

Proof. First, remember ourself that

$$\max\{\beta \in \mathcal{B}(X) : \alpha^{-1} \circ \beta \circ \alpha^{-1} \subseteq \alpha\} = \cup\{\beta \in \mathcal{B}(X) : \alpha^{-1} \circ \beta \circ \alpha^{-1} \subseteq \alpha\}.$$

Let $\beta \in B(X)$ be an arbitrary relation such that $\alpha^{-1} \circ \beta \circ \alpha^{-1} \subseteq \alpha$. We will prove that $\beta \subseteq \alpha^*$. If not, there is $(x, y) \in \beta$ such that $\neg((x, y) \in \alpha^*)$. The last gives:

$(x, y) \in \alpha \circ \alpha^C \circ \alpha \Longleftrightarrow$

$(\exists u, v \in X)((x, u) \in \alpha \wedge (u, v) \in \alpha^C \wedge (v, y) \in \alpha) \Longleftrightarrow$

$(\exists u, v \in X)((u, x) \in \alpha^{-1} \wedge (u, v) \in \alpha^C \wedge (y, v) \in \alpha^{-1} \Longrightarrow$

$(\exists u, v \in X)((u, x) \in \alpha^{-1} \wedge (x, y) \in \beta \wedge (y, v) \in \alpha^{-1} \wedge (u, v) \in \alpha^C) \Longrightarrow$

$(\exists u, v) \in X)((u, v) \in \alpha^{-1} \circ \beta \circ \alpha^{-1} \subseteq \alpha \,\wedge\, (u, v) \in \alpha^C)$

We got a contradiction. So, there must be $\beta \subseteq \alpha^*$.

On the other hand, we should prove that

$$\alpha^{-1} \circ \alpha^* \circ \alpha^{-1} \subseteq \alpha.$$

Let $(x, y) \in \alpha^{-1} \circ \alpha^* \circ \alpha^{-1}$ be an arbitrary element. Then, there are elements $u, v \in X$ such that $(x, u) \in \alpha^{-1}$, $(u, v) \in \alpha^*$ and $(v, y) \in \alpha^{-1}$. So, from

$$(u, x) \in \alpha,\ \neg((u, v) \in \alpha \circ \alpha^C \circ \alpha),\ (y, v) \in \alpha,$$

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we have $\neg((x,y) \in \alpha^C)$. Suppose that $(x,y) \in \alpha^C$. Then, we have $(u,v) \in \alpha \circ \alpha^C \circ \alpha$, which is impossible. Hence, we have to $(x,y) \in \alpha$ and therefore, $\alpha^{-1} \circ \alpha^* \circ \alpha^{-1} \subseteq \alpha$.

Finally, we conclude that α^* is the maximal element of the family of all relations $\beta \in \mathcal{B}(X)$ such that $\alpha^{-1} \circ \beta \circ \alpha^{-1} \subseteq \alpha$. □

It is easy to see that holds
$\alpha^\star = \{(x,y) \in X \times X : \alpha^{-1} \circ \{(x,y)\} \circ \alpha^{-1} \subseteq \alpha\}$
$\quad = \{(x,y) \in X \times X : \alpha^{-1}x \times \alpha^{-1}y \subseteq \alpha\}$.
Also, we have $\alpha^\star = ((\alpha \,\Box\, \alpha^{-1})(\alpha^C))^C$ by the concept exposed in the equation (1.1).

In the following proposition we give a characterization of bi-conjugative relations. It is our adaptation of concept exposed in [6], Theorem 7.2.

Theorem 2.3. *For a binary relation α on a set X, the following conditions are equivalent:*
(1) *α is a bi-conjugative relation.*
(2) *For all $x, y \in X$, if $(x,y) \in \alpha$, there exist $u, v \in X$ such that:*
(a) $(u,x) \in \alpha \wedge (y,v) \in \alpha$,
(b) $(\forall s,t \in X)((u,s) \in \alpha \wedge (t,v) \in \alpha \Longrightarrow (s,t) \in \alpha)$.
(3) $\alpha \subseteq \alpha^{-1} \circ \alpha^* \circ \alpha^{-1}$.

Proof. (1) $\Longrightarrow$ (2). Let α be a bi-conjugative relation, i.e. let there exists a relation β such that $\alpha = \alpha^{-1} \circ \beta \circ \alpha^{-1}$. Let $(x,y) \in \alpha$. Then there exist elements $u, v \in X$ such that

$$(x,u) \in \alpha^{-1},\ (u,v) \in \beta,\ (v,y) \in \alpha^{-1}.$$

From this follows that there exist elements $u, v \in X$ such that

$$(u,x) \in \alpha \ \wedge\ (y,v) \in \alpha.$$

This proves condition (a).

Now, we check the condition (b). Let $s, t \in X$ be arbitrary elements such that $(u,s) \in \alpha$ and $(t,v) \in \alpha$. Now, from $(s,u) \in \alpha^{-1}$, $(u,v) \in \beta$ and $(v,t) \in \alpha^{-1}$ follows $(s,t) \in \alpha^{-1} \circ \beta \circ \alpha^{-1} = \alpha$.
(2) $\Longrightarrow$ (1). Define a binary relation

$$\alpha' = \{(u,v) \in X \times X : (\forall s,t \in X)((u,s) \in \alpha \wedge (t,v) \in \alpha \Longrightarrow (s,t) \in \alpha)\}$$

and show that $\alpha^{-1} \circ \alpha' \circ \alpha^{-1} = \alpha$ is valid. Let $(x,y) \in \alpha$. Then there exist elements $u, v \in X$ such that the conditions (a) and (b) are hold. We have $(u,v) \in \alpha'$ by definition of relation α'.

Further, from $(x,u) \in \alpha^{-1}$, $(u,v) \in \alpha'$ and $(v,y) \in \alpha^{-1}$ follows $(x,y) \in \alpha^{-1} \circ \alpha' \circ \alpha^{-1}$. Hence, we have $\alpha \subseteq \alpha^{-1} \circ \alpha' \circ \alpha^{-1}$. Contrary, let $(x,y) \in \alpha^{-1} \circ \alpha' \circ \alpha^{-1}$ be an arbitrary pair. There exist elements $u, v \in X$ such that $(x,u) \in \alpha^{-1}$, $(u,v) \in \alpha'$ and $(v,y) \in \alpha^{-1}$, i.e. such that $(u,x) \in \alpha$ and $(y,v) \in \alpha$, Hence, by definition of relation α', follows $(x,y) \in \alpha$ since $(u,v) \in \alpha'$. Therefore, $\alpha^{-1} \circ \alpha' \circ \alpha^{-1} \subseteq \alpha$. So, the relation α is a bi-conjugative relation on X since there exists a relation α' such that $\alpha^{-1} \circ \alpha' \circ \alpha^{-1} = \alpha$.
(1) $\Longleftrightarrow$ (3). Let α be a bi-conjugative relation. Then there a relation β such that $\alpha = \alpha^{-1} \circ \beta \circ \alpha^{-1}$. Since $\alpha^* = \max\{\beta \in \mathcal{B}(X) : \alpha^{-1} \circ \beta \circ \alpha^{-1} \subseteq \alpha\}$, we have $\beta \subseteq \alpha^*$ and

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$\alpha = \alpha^{-1} \circ \beta \circ \alpha^{-1} \subseteq \alpha^{-1} \circ \alpha^* \circ \alpha^{-1}$. Contrary, let holds $\alpha \subseteq \alpha^{-1} \circ \alpha^* \circ \alpha^{-1}$, for a relation α. Then, we have $\alpha \subseteq \alpha^{-1} \circ \alpha^* \circ \alpha^{-1} \subseteq \alpha$. So, the relation α is bi-conjugative relation on set X. $\square$

Corollary 2.4. *Let $(L, \leqslant)$ be a poset. Relation $\nleqslant$ is not a bi-conjugative relation on L.*

Proof. Let $\nleqslant$ be a bi-conjugative relation on set X, and let $x, y \in X$ be elements such that $x \nleqslant y$. Then, by previous theorem, there exist elements $u, v \in X$ such that:
(a) $u \nleqslant x \wedge y \nleqslant v$;
(b) $(\forall s, t \in L)((u \nleqslant s \wedge t \nleqslant v) \Longrightarrow s \nleqslant t)$.
Let z be an arbitrary element and if we put $z = s = t$ in formula (b), we have

$$(u \nleqslant z \wedge z \nleqslant v) \Longrightarrow z \nleqslant z.$$

It is a contradiction. Hence, $\neg(u \nleqslant z \wedge z \nleqslant v)$. Follows $u \leqslant z \vee z \leqslant v$. Further on, let $s, t \in L$ be arbitrary elements such that $u \nleqslant s$ and $t \nleqslant v$. For $z = s$, from the last disjunction we have $u \leqslant s \vee s \leqslant v$ and also for $z = t$ we have $u \leqslant t \vee t \leqslant v$. So, there are fourth possibilities:
(1) $u \leqslant s \wedge u \leqslant t \wedge, u \nleqslant s \wedge t \nleqslant v$.
(2) $u \leqslant s \wedge t \leqslant v \wedge u \nleqslant s \wedge t \nleqslant v$.
(3) $s \leqslant v \wedge t \leqslant v \wedge u \nleqslant s \wedge t \nleqslant v$.
(4) $s \leqslant v \wedge u \leqslant t \wedge u \nleqslant s \wedge t \nleqslant v$.
Since, options (1), (2) and (3) are contradictions, it left the possibility (4). In this case, since $u \nleqslant s \Longrightarrow (u \nleqslant t \vee t \nleqslant s)$ holds as a contraposition of the transitivity $(u \leqslant t \wedge t \leqslant s) \Longrightarrow u \leqslant s$, we have $s \leqslant v \wedge u \leqslant t \wedge (u \nleqslant t \vee t \nleqslant s) \wedge t \nleqslant v$. Finally, since the option $u \nleqslant t$ is in contradiction with $u \leqslant t$, we have to $t \nleqslant s$ which is in contradiction with the consequence $s \nleqslant t$ of implication (b). Therefore, the relation $\nleqslant$ cannot satisfies the condition (b) of Theorem 2.3.

$\square$

Example 2.5. Let α be a bi-conjugative relation on set X. Then there exists a relation β on X such that $\alpha = \alpha^{-1} \circ \beta \circ \alpha^{-1}$. If θ is an equivalence relation on X and $\gamma \in \mathcal{B}(X)$, we define relation

$$\gamma/\theta = \{(a\theta, b\theta) \in X/\theta \times X/\theta : (\exists a' \in X)(\exists b' \in X)((a, a') \in \theta \wedge (a', b') \in \gamma \wedge (b, b') \in \theta)\}.$$

It is easy to that

$$\alpha/\theta = (\alpha/\theta)^{-1} \circ \beta/\theta \circ (\alpha/\theta)^{-1}$$

holds. So, the relation α/θ is a bi-conjugative relation on X/θ. Therefore, for any equivalence relation θ on X there is a correspondence $\Phi_\theta : \mathcal{BC}(X) \longrightarrow \mathcal{BC}(X/\theta)$.

Example 2.6. Let α' be a bi-conjugative element in $\mathcal{B}(X')$. Then there exists a relation $\beta' \in \mathcal{B}(X')$ such that $\alpha' = (\alpha')^{-1} \circ \beta' \circ (\alpha')^{-1}$. For a mapping $f : X \longrightarrow X'$ and a relation $\gamma' \in \mathcal{B}(X')$ we define $f^{-1}(\gamma')$ by

$$(x, y) \in f^{-1}(\gamma') \Longleftrightarrow (f(x), f(y)) \in \gamma'.$$

If f is a surjective mapping, we have:

$$(x,y) \in f^{-1}(\alpha') \Longleftrightarrow (x,y) \in (f^{-1}(\alpha'))^{-1} \circ f^{-1}(\beta') \circ (f^{-1}(\alpha'))^{-1}.$$

So, the relation $f^{-1}(\alpha')$ is a bi-conjugative relation in $\mathcal{B}(X)$. Since for any equivalence relation θ on X, the mapping $\pi : X \longrightarrow X/\theta$ is a surjective, there is a correspondence $\Psi_\theta : \mathcal{BC}(X/\theta) \longrightarrow \mathcal{BC}(X)$ also.

Further on, if $\mathcal{E}(X)$ is the family of all equivalence relations on set X, then for any bi-conjugative relation α in X there is the family $\mathcal{BC}(\alpha) = \{\pi^{-1}(\alpha/\theta) : \theta \in \mathcal{E}(X)\}$ of bi-conjugative relations on X. Such that subfamily is this one $\mathcal{BC}(\nabla_{X/\theta}) = \{\pi^{-1}(\nabla_{X/\theta}) : \theta \in \mathcal{E}(X)\}$.

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