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FINITELY BI-CONJUGATIVE RELATIONS

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Abstract. In this paper, the concept of finitely bi-conjugative relations is introduced. A characterization of this relations is obtained. Particulary we show when the anti-order relation $\nleqslant$ is a finitely bi-conjugative relation.

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1. Introduction

The concept of finitely conjugative relations was introduced by Guanghao Jiang and Luoshan Xu in [1], the concept of finitely dual normal relations was introduced and analyzed by Jiang Guanghao and Xu Luoshan in [2] and the concept of finitely quasi-conjugative relations was introduced and analyzed by these authors in [3]. In this article, we introduce and analyze the notion of finitely bi-conjugative relations as a continuation of our article [4].

For a set X, we call ρ a relation on X, if $\rho \subseteq X \times X$. Let $\mathcal{B}(X)$ be denote the set of all binary relations on X. For $\alpha, \beta \in \mathcal{B}(X)$, define

$$\beta \circ \alpha = \{(x, z) \in X \times X : (\exists y \in X)((x, y) \in \alpha \wedge (y, z) \in \beta)\}.$$

The relation $\beta \circ \alpha$ is called the composition of α and β. It is well known that $(\mathcal{B}(X), \circ)$ is a semigroup. The latter family, with the composition, is not only a semigroup, but also a monoid. Namely, $\triangle_X = \{(x, x) : x \in X\}$ is its identity element. For a relation α on a set X, define $\alpha^{-1} = \{(x, y) \in X \times X : (y, x) \in \alpha\}$ and $\alpha^C = (X \times X) \setminus \alpha$.

Let A and B be subsets of X. For $\alpha \in \mathcal{B}(X)$, set

$$A\alpha = \{y \in X : (\exists a \in A)((a, y) \in \alpha)\}, \ \ \alpha B = \{x \in X : (\exists b \in B)((x, b) \in \alpha)\}.$$

It is easy to see that $A\alpha = \alpha^{-1}A$ holds. Specially, we put $a\alpha$ if $\{a\}\alpha$ and αb if $\alpha\{b\}$.

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2. Bi-conjugative relations

The following classes of elements in the semigroup $\mathcal{B}(X)$ have been investigated:
– *regular* if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha \circ \beta \circ \alpha.$$

– *dually normal* ([2]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = (\alpha^C)^{-1} \circ \beta \circ \alpha.$$

– *conjugative* ([1]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha^{-1} \circ \beta \circ \alpha.$$

– *dually conjugative* ([1]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha \circ \beta \circ \alpha^{-1}.$$

The notion of *bi-conjugative relation* was introduced in the paper [4] by the following way:

Definition 2.1. For a relation $\alpha \in \mathcal{B}(X)$ we say that it is a *bi-conjugative* relation on X if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha^{-1} \circ \beta \circ \alpha^{-1}.$$

The family $\mathcal{BC}(X)$ of all bi-conjugative relations on set X is not empty. For example, $\triangle_X \in \mathcal{BC}(X)$ and $\nabla_X = \triangle_X^C \in \mathcal{BC}(X)$.

3. Finitely bi-conjugative relations

In this section we introduce the concept of finitely bi-conjugative relations and give a characterization of this relations. For that we need the concept of *finite extension* of a relation. That notion and belonging notation we borrow from articles [1] and [2]. For any set X, let $X^{(<\omega)} = \{F \subseteq X : F \text{ is finite and nonempty }\}$.

Definition 3.1. ([1], Definition 3.3; [2], Definition 3.4) Let α be a binary relation on a set X. Define a binary relation $\alpha^{(<\omega)}$ on $X^{(<\omega)}$, called the *finite extension* of α, such that

$$(\forall F, G \in X^{(<\omega)})((F,G) \in \alpha^{(<\omega)} \iff G \subseteq F\alpha).$$

From Definition 3.1, we immediately obtain that

$$(\forall F, G \in X^{(<\omega)})((F,G) \in (\alpha^{-1})^{(<\omega)} \iff G \subseteq F\alpha^{-1} = \alpha F).$$

Now, we can introduce concept of *finitely bi-conjugative relation.*

Definition 3.2. A relation α on a set X is called *finitely bi-conjugative* if there exists a relation $\beta^{(<\omega)}$ on $X^{(<\omega)}$ such that

$$\alpha^{(<\omega)} = (\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^{-1})^{(<\omega)}.$$

Although it seems, in accordance with Definition 2.1, it would be better to call a relation α on X to be finitely bi-conjugative if its finite extension to $X^{(<\omega)}$ is a bi-conjugative relation, we will not use that option. That concept is different from our concept given by Definition 3.2.

Now we give an essential characterization of finitely bi-conjugative relations.

Theorem 3.1. *A relation α on a set X if a finitely bi-conjugative relation if and only if for all $F, G \in X^{(<\omega)}$, if $G \subseteq F\alpha$, then there are $U, V \in X^{(<\omega)}$, such that*
(i) *$U \subseteq \alpha F$, $G \subseteq \alpha V$, and*
(ii) *for all $S, T \in X^{(<\omega)}$, if $U \subseteq \alpha S$ and $T \subseteq \alpha V$ then $T \subseteq S\alpha$.*

Proof. (Necessity) Let α be a finitely bi-conjugative relation on set X. Then there is a relation $\beta^{(<\omega)} \subseteq (X^{(<\omega)})^2$ such that $(\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^{-1})^{(<\omega)} = \alpha^{(<\omega)}$. For all $(F, G) \in (X^{(<\omega)})^2$, if $G \subseteq F\alpha$, i.e., $(F, G) \in \alpha^{(<\omega)}$, thus $(F, G) \in (\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^{-1})^{(<\omega)}$. Therefore, there are U and V in $(X^{(<\omega)})$ such that $(F, U) \in (\alpha^{-1})^{(<\omega)}$, $(U, V) \in \beta^{(<\omega)}$ and $(V, G) \in (\alpha^{-1})^{(<\omega)}$, i.e., $U \subseteq F\alpha^{-1} = \alpha F$, $G \subseteq V\alpha^{-1} = \alpha V$. Hence we have got the condition (i).

Now we check the condition (ii). For all $(S, T) \in (X^{(<\omega)})^2$, if $U \subseteq \alpha S$ and $T \subseteq \alpha V$, i.e., $(S, U) \in (\alpha^{-1})^{(<\omega)}$ and $(V, T) \in (\alpha^{-1})^{(<\omega)}$, then by $(U, V) \in \beta^{(<\omega)}$, we have $(S, T) \in (\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^{-1})^{(<\omega)}$, i.e., $(S, T) \in \alpha^{(<\omega)}$. Hence $T \subseteq S\alpha$.

(Sufficiency) Let α be a relation on a set X such that for $F, G \in X^{(<\omega)}$ with $G \subseteq F\alpha$ there are $U, V \in X^{(<\omega)}$ such that conditions (i) and (ii) hold. Define a binary relation $\beta^{(<\omega)} \subseteq X^{(<\omega)} \times X^{(<\omega)}$ by

$$(F, G) \in \beta \Longleftrightarrow (\forall S, T \in X^{(<\omega)})((F \subseteq \alpha S \wedge T \cap \alpha G \neq \emptyset) \Longrightarrow T \cap S\alpha \neq \emptyset).$$

First, check that

$$(\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^{-1})^{(<\omega)} \subseteq \alpha^{(<\omega)} \tag{3.1}$$

holds. For all $H, W \in X^{(<\omega)}$, if $(H, W) \in (\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^{-1})^{(<\omega)}$, then there are $F, G \in X^{(<\omega)}$ with $(H, F) \in (\alpha^{-1})^{(<\omega)}$, $(F, G) \in \beta^{(<\omega)}$ and $(G, W) \in (\alpha^{-1})^{(<\omega)}$. Then $F \subseteq H\alpha^{-1} = \alpha H$ and $W \subseteq G\alpha^{-1} = \alpha G$. For all $w \in W$, let $S = H$, $T = \{w\}$. Then $F \subseteq \alpha S$ and $\alpha G \cap T \neq \emptyset$ because $w \in T$ and $w \in \alpha G$. Since $(F, G) \in \beta^{(<\omega)}$, we have that $F \subseteq \alpha S \wedge \alpha G \cap T \neq \emptyset$ implies $T \cap S\alpha \neq \emptyset$. Hence, $w \in S\alpha$, i.e. $W \subseteq S\alpha$. So, we have $(H, W) = (S, W) \in \alpha^{(<\omega)}$. Therefore, we have $(\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^{-1})^{(<\omega)} \subseteq \alpha^{(<\omega)}$.

The second, check that

$$\alpha^{(<\omega)} \subseteq (\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^{-1})^{(<\omega)} \tag{3.2}$$

holds. For all $H, W \in X^{(<\omega)}$, if $(H, W) \in \alpha^{(<\omega)}$ (i.e., $W \subseteq H\alpha$), there are $A, B \in X^{(<\omega)}$ such that:
(i') $A \subseteq \alpha H$, $W \subseteq \alpha B$, and
(ii') for all $S, T \in X^{(<\omega)}$, if $A \subseteq \alpha S$ and $T \subseteq \alpha B$, then $T \subseteq S\alpha$.

Now, we have to show that $(A, B) \in \beta^{(<\omega)}$. Let be for all $(C, D) \in (X^{(<\omega)})^2$ the following $A \subseteq \alpha D$ and $D \cap \alpha B \neq \emptyset$ hold. From $D \cap \alpha B \neq \emptyset$ follows that there exists an element $d \in D \cap \alpha B (\neq \emptyset)$. So, $d \in D$ and $d \in \alpha B$. Put $S = C$ and $T = \{d\}$. Then, by (ii'), we have

$$(A \subseteq \alpha S \wedge T = \{d\} \subseteq \alpha B) \Longrightarrow \{d\} = T \subseteq S\alpha,$$

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i.e. $\emptyset \neq D \cap S\alpha = T \cap S\alpha$. Therefore, $(A, B) \in \beta^{(<\omega)}$ by definition of $\beta^{(<\omega)}$. Finally, for $(H, A) \in (\alpha^{-1})^{(<\omega)}$, $(A, B) \in \beta^{(<\omega)}$ and $(B, W) \in (\alpha^{-1})^{(<\omega)}$ follows that $(H, W) \in (\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^{-1})^{(<\omega)}$.

By assertion (3.1) and (3.2), we have $\alpha^{(<\omega)} = (\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^{-1})^{(<\omega)}$. So, α is a finitely bi-conjugative relation on X. $\square$

Particulary, if we put $F = \{x\}$ and $G = \{y\}$ in the Theorem 3.1, we give the following consequent.

Corollary 3.1. *Let α be a relation on a set X. Then α is a finitely bi-conjugative relation on X if and only if for all elements $x, y \in X$ such that $(x, y) \in \alpha$ there are finite subsets $U = \{u_1, u_2, ..., u_m\}$ and $V = \{v_1, v_2, ..., v_n\}$ in $X^{(<\omega)}$ such that*
(1^0) $(\forall i \in \{1, 2, ..., m\})((u_i, x) \in \alpha) \wedge (\exists j \in \{1, 2, ..., n\})((y, v_j) \in \alpha)$, and
(2^0) for all $S = \{s_1, s_2, ..., s_p\}$ in $X^{(<\omega)}$ and $t \in X$ holds

$$(U \subseteq \alpha S \wedge (\exists v_j \in V)((t, v_j) \in \alpha)) \Longrightarrow (\exists s_k \in S)((s_k, t) \in \alpha) .$$

Proof. Let α be a finitely bi-conjugative relation on X and let x, y be elements of X such that $(x, y) \in \alpha$. If we put $F = \{x\}$ and $G = \{y\}$ in Theorem 3.1, then there exist finite subsets $U = \{u_1, u_2, ..., u_m\}$ and $V = \{v_1, v_2, ..., v_n\}$ in $X^{(<\omega)}$ such that conditions (1^0) and (2^0) hold.

Opposite, let for all elements x, y in X such that $(x, y) \in \alpha$ there are $U = \{u_1, u_2, ..., u_m\}$ and $V = \{v_1, v_2, ..., v_n\}$ in $X^{(<\omega)}$ such that conditions (1^0) and (2^0) hold. Define binary relation $\beta^{(<\omega)} \subseteq X^{<\omega} \times X^{<\omega}$ by

$$(A, B) \in \beta^{(<\omega)} \Longleftrightarrow (\forall S \in X^{<\omega})(\forall t \in X)((A \subseteq \alpha S \wedge t \in \alpha B) \Longrightarrow t \in S\alpha).$$

The proof that the equality $(\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^{-1})^{(<\omega)} = \alpha^{(<\omega)}$ holds is some as in the Theorem 3.1. So, α is a finitely bi-conjugative relation. $\square$

At end of this note, we show when the anti-order relation $\nleqslant$ on poset $(L, \leqslant)$ is a finitely bi-conjugative idempotent relation.

Theorem 3.2. *Let $(L, \leqslant)$ be a poset. Then the relation $\nleqslant$ on L is a finitely be-conjugative relation if and only if for all $x, y \in L$ such that $x \nleqslant y$, there exist finite subsets $U = \{u_1, u_2, ..., u_m\}$ and $V = \{v_1, v_2, ..., v_n\}$ of L such that*
(a) $(\forall i \in \{1, 2, ..., m\})(u_i \nleqslant x)$ and $(\exists j \in \{1, 2, ..., n\})(y \nleqslant v_j)$ and
(b) $(\forall z \in L)((\exists i \in \{1, 2, ..., m\})(u_i \leqslant z) \vee (\forall j \in \{1, 2, ..., n\})(z \leqslant v_j))$.

Proof. Let $x, y \in L$ such that $x \nleqslant y$. Then, since $\nleqslant$ is a finitely bi-conjugative relation on L, there exist finite subsets $U = \{u_1, u_2, ..., u_m\}$ and $V = \{v_1, v_2, ..., v_n\}$ of L such that
(1) $(\forall i \in \{1, 2, ..., m\})(u_i \nleqslant x) \wedge (\exists j \in \{1, 2, ..., n\})(y \nleqslant v_j)$, and
(2) for all $S = \{s_1, s_2, ..., s_p\}$ in $L^{(<\omega)}$ and $t \in L$ the following holds

$$((\forall u_i \in U)(\exists s_k \in S)(u_i \nleqslant s_k) \wedge (\exists v_j \in V)(t \nleqslant v_j)) \Longrightarrow (\exists s_{k'} \in S)(s_{k'} \nleqslant t).$$

For $z \in L$, let $S = \{z\} = \{t\}$. Then by (2), from

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$$(\forall u_i \in U)(u_i \nleqslant z) \wedge (\exists v_j \in V)(z \nleqslant v_j)$$

implies $z \nleqslant z$. It is a contradiction. Hence, we have

$$\neg((\forall u_i \in U)(u_i \nleqslant z) \wedge (\exists v_j \in V)(z \nleqslant v_j)).$$

So, finally, we have

$$(\exists i \in \{1, 2, ..., m\})(u_i \leqslant z) \vee (\forall j \in \{1, 2, ..., n\})(z \leqslant v_j).$$

Let for $(x, y) \in L^2$ be $x \nleqslant y$ holds and let there exist finite subsets $U = \{u_1, u_2, ..., u_m\}$ and $V = \{v_1, v_2, ..., v_n\}$ of L satisfying conditions (a) and (b). So, the condition (a) is the condition (1^0) in Corollary 3.1.

Let $S \in L^{(<\omega)}$ and $t \in L$ with

$$(\forall u_i \in U)(\exists s_k \in S)(u_i \nleqslant s_k) \text{ and } (\exists v_j \in V)(t \nleqslant v_j)$$

holds. Suppose that $(\forall s_k \in S)(s_k \leqslant t)$ holds. Then, by (b), for $S = \{s\}$ and $z = s$, we have

$$(\exists u_i \in U)(u_i \leqslant s) \vee (\forall v_j \in V)(s \leqslant v_j).$$

The first option is impossible because $(\forall u_i)(u_i \nleqslant s)$. Let the option $(\forall v_j \in V)(s \leqslant v_j)$ be valid. Then from $(\exists v_j \in V)(t \nleqslant v_j)$ and $s \leqslant t$ follows $(\exists v_j \in V)(s \nleqslant v_j)$. It is in contradiction with $(\forall v_j \in V)(s \leqslant v_j)$. So, must to be $\neg(\forall s_k \in S)(s_k \leqslant t)$. Thus $(\exists s_k \in S)(s_k \nleqslant t)$. Hence $\nleqslant$ satisfies also condition (2^0) in Corollary 3.1. Finally, the relation $\nleqslant$ is a finitely bi-conjugative relation on L. □

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