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Direct method for variational problems by using hybrid of block-pulse and Bernoulli polynomials

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Abstract

In this paper, a numerical method for solving variational problems is presented. The method is based upon hybrid functions approximation. The properties of hybrid functions consisting of block-pulse functions and Bernoulli polynomials are presented. The operational matrices of integration and product and the integration of the cross product of two hybrid functions of block-pulse and Bernoulli polynomials vectors are then utilized to reduce the variational problems to the solution of algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of the technique.

Mathematics Subject Classification: 33F05; 41A30; 49M30

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1 Introduction

In the large number of problems arising in analysis, mechanics, geometry, etc., it is necessary to determine the maximal and minimal of a certain functional. Such Problems are called variational problems [28]. The direct method of Ritz and Galerkin in solving variational problems has been of considerable concern and is well covered in many textbooks [6, 7, 8].

The available sets of orthogonal functions can be divided into three classes. The first class includes sets of piecewise constant basis functions (PCBF's)(e.g., block-pulse, Haar, Walsh, etc.). The second class consists of sets of orthogonal polynomials (e.g., Chebyshev, Laguerre, Legendre, etc.). The third class is the set of sine-cosine functions in the

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Fourier series. Orthogonal functions have been used when dealing with various problems of the dynamical systems. The main advantage of using orthogonal functions is that they reduce the dynamical system problems to those of solving a system of algebraic equations. The approach is based on converting the underlying differential equation into an integral equation through integration, approximating various signals involved in the equation by truncated orthogonal functions, and using the operational matrix of integration P to eliminate the integral operations. Special attention has been given to applications of Walsh functions [4], Chebyshev polynomials [9], Laguerre polynomials [12], Legendre polynomials [3] and Fourier series [23]. Among orthogonal polynomials, the shifted Legendre polynomials $p_m(t)$, $m = 0, 1, 2, ...,$ where $0 \le t \le 1$, is computationally more effective [20]. The Bernoulli polynomials and Taylor series are not based on orthogonal functions, nevertheless, they possess the operational matrix of integration. However, since the integration of the cross product of two Taylor series vectors is given in terms of a Hilbert matrix [24], which are known to be ill-posed, the applications of Taylor series are limited.

For approximating an arbitrary time function the advantages of Bernoulli polynomials $\beta_m(t)$, $m = 0, 1, 2, ..., M$, where $0 \le t \le 1$, over shifted Legendre polynomials $p_m(t)$, $m = 0, 1, 2, ..., M$, are:

a) the operational matrix P, in Bernoulli polynomials has less errors than P for shifted Legendre polynomials for $1 < M < 10$. This is because for P in $\beta_m(t)$ we ignore the term $\frac{\beta_{M+1}(t)}{M+1}$ while for P in $p_m(t)$ we ignore the term $\frac{p_{M+1}(t)}{2(2M+1)}$;

b) the Bernoulli polynomials have less terms than shifted Legendre polynomials. For example $\beta_6(t)$, has 5 terms while $p_6(t)$, has 7 terms, and this difference will increase by increasing m. Hence for approximating an arbitrary function we use less CPU time by applying Bernoulli polynomials as compared to shifted Legendre polynomials;

c) the coefficient of individual terms in Bernoulli polynomials $\beta_m(t)$, are smaller than the coefficient of individual terms in the shifted Legendre polynomials $p_m(t)$. Since the computational errors in the product are related to the coefficients of individual terms, the computational errors are less by using Bernoulli polynomials.

In recent years the hybrid functions consisting of the combination of block-pulse functions with Chebyshev polynomials [13, 22, 29], Legendre polynomials [11, 16, 21], or Taylor series [15, 17, 18] have been shown to give excellent results for discretization of selected problems. Among these three hybrid functions, the hybrid functions of block-pulse and Legendre polynomials have shown to be computationally more effective.

The outline of this paper is as follows: In sections 2 we introduce properties of hybrid functions. Section 3 is devoted to the problem statement. In section 4 the numerical method is used to approximate the variational problems and in Section 5 we report our

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numerical findings and demonstrate the accuracy of the proposed numerical scheme by considering four numerical examples.

2 Properties of Hybrid Functions

2.1 Hybrid of block-pulse and Bernoulli polynomials

Hybrid functions $b_{nm}(t),\ n = 1, 2, \ldots, N,\ m = 0, 1, \ldots, M$ are defined on the interval $[0, t_f)$ as

$$b_{nm}(t) = \begin{cases} \beta_m(\frac{N}{t_f}t - n + 1), & t \in [\frac{n-1}{N}t_f, \frac{n}{N}t_f), \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where n and m are the order of block-pulse functions and Bernoulli polynomials, respectively. In Eq. (1), $\beta_m(t),\ m = 0, 1, 2, \ldots$ are the Bernoulli polynomials of order m, which can be defined by [5]

$$\beta_m(t) = \sum_{k=0}^{m} \binom{m}{k} \alpha_k t^{m-k},$$

where $\alpha_k,\ k = 0, 1, \ldots, m$ are Bernoulli numbers. These numbers are a sequence of signed rational numbers which arise in the series expansion of trigonometric functions [1] and can be defined by the identity

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \alpha_n \frac{t^n}{n!}.$$

The first few Bernoulli numbers are

$$\begin{aligned} &\alpha_0 = 1, \\ &\alpha_1 = \tfrac{-1}{2}, \\ &\alpha_2 = \tfrac{1}{6}, \\ &\alpha_4 = \tfrac{-1}{30}, \end{aligned}$$

with $\alpha_{2k+1} = 0,\ k = 1, 2, 3, \ldots$.

The first few Bernoulli polynomials are

$$\begin{aligned} &\beta_0(t) = 1, \\ &\beta_1(t) = t - \tfrac{1}{2}, \\ &\beta_2(t) = t^2 - t + \tfrac{1}{6}, \\ &\beta_3(t) = t^3 - \tfrac{3}{2}t^2 + \tfrac{1}{2}t. \end{aligned}$$

These polynomials satisfy the following formula [1]

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$$\beta_m(0) = \alpha_m, \qquad m \geq 0, \tag{2}$$

$$\int_a^x \beta_m(t)dt = \frac{\beta_{m+1}(x) - \beta_{m+1}(a)}{m+1}, \tag{3}$$

$$\int_0^1 \beta_n(t)\beta_m(t)dt = (-1)^{n-1}\frac{m!n!}{(m+n)!}\alpha_{n+m}, \qquad m, n \geq 1. \tag{4}$$

According to [14], Bernoulli polynomials, form a complete basis over the interval [0,1].

2.2 Function approximation

Suppose that $H = L^2[0,1]$ and $\{b_{10}(t), b_{20}(t), ..., b_{NM}(t)\} \subset H$ be the set of hybrid of block-pulse and Bernoulli polynomials and

$$Y = span\{b_{10}(t), b_{20}(t), ..., b_{N0}(t), b_{11}(t), b_{21}(t), ..., b_{N1}(t), ..., b_{1M}(t), b_{2M}(t), ..., b_{NM}(t)\},$$

and f be an arbitrary element in H. Since Y is a finite dimensional vector space, f has the unique best approximation out of Y such as $f_0 \in Y$, that is

$$\forall y \in Y, \parallel f - f_0 \parallel \leq \parallel f - y \parallel .$$

Since $f_0 \in Y$, there exists the unique coefficients $c_{10}, c_{20}, ..., c_{NM}$ such that

$$f \simeq f_0 = \sum_{m=0}^{M}\sum_{n=1}^{N} c_{nm}b_{nm}(t) = C^T B(t), \tag{5}$$

where

$$B^T(t) = [b_{10}(t), b_{20}(t), ..., b_{N0}(t), b_{11}(t), b_{21}(t), ..., b_{N1}(t), ..., b_{1M}(t), b_{2M}(t), ..., b_{NM}(t)], \tag{6}$$

and

$$C^T = [c_{10}, c_{20}, ..., c_{N0}, c_{11}, c_{21}, ..., c_{N1}, ..., c_{1M}, c_{2M}, ..., c_{NM}]. \tag{7}$$

2.3 Integration of $B(t)B^T(t)$

Using Eq. (5) we obtain

$$f_{ij} =< \sum_{m=0}^{M}\sum_{n=1}^{N} c_{nm}b_{nm}(t), b_{ij}(t) >= \sum_{m=0}^{M}\sum_{n=1}^{N} c_{nm}d_{nm}^{ij},$$

$$i = 1, 2, ..., N, \quad j = 0, 1, ..., M,$$

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where $f_{ij} =< f, b_{ij}(t) >$, $d^{ij}_{nm} =< b_{nm}(t), b_{ij}(t) >$, and $<, >$ denotes inner product.

Therefore

$$f_{ij} = C^T[d^{ij}_{10}, d^{ij}_{20}, ..., d^{ij}_{N0}, d^{ij}_{11}, d^{ij}_{21}, ..., d^{ij}_{N1}, ..., d^{ij}_{1M}, d^{ij}_{2M}, ..., d^{ij}_{NM}]^T,$$
$$i = 1, 2, ..., N, \quad j = 0, 1, ..., M.$$

So we get

$$\Phi = D^T C,$$

with

$$\Phi = [f_{10}, f_{20}, \ldots, f_{N0}, f_{11}, f_{21}, \ldots, f_{N1}, \ldots, f_{1M}, f_{2M}, \ldots, f_{NM}]^T,$$

and

$$D = [d^{ij}_{nm}],$$

where D is a matrix of order $N(M+1) \times N(M+1)$ and is given by

$$D = \int_0^1 B(t)B^T(t)dt. \tag{8}$$

Using Eq. (4) in each interval $n = 1, 2, ..., N$, we can get matrix D. For example with $N = 2$ and $M = 3$, D is

$$D = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{12} & 0 & 0 & 0 & \frac{-1}{120} & 0 \\ 0 & 0 & 0 & \frac{1}{12} & 0 & 0 & 0 & \frac{-1}{120} \\ 0 & 0 & 0 & 0 & \frac{1}{180} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{180} & 0 & 0 \\ 0 & 0 & \frac{-1}{120} & 0 & 0 & 0 & \frac{1}{840} & 0 \\ 0 & 0 & 0 & \frac{-1}{120} & 0 & 0 & 0 & \frac{1}{840} \end{bmatrix}.$$

It is seen that the matrix D is a sparse matrix. Furthermore, if we choose large values of M and N the non zero elements of D will tend to zero.

2.4 Operational matrix of integration

The integration of the $B(t)$ defined in Eq. (6) is given by

$$\int_0^t B(t')dt' \simeq PB(t), \tag{9}$$

where P is the $N(M+1) \times N(M+1)$ operational matrix of integration and is given by [19]

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$$P = \frac{t_f}{N}\begin{bmatrix} P_0 & I & O & \dots & O \\ \frac{-1}{2}\alpha_2 I & O & \frac{1}{2}I & \dots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{-1}{M}\alpha_M I & O & O & \dots & \frac{1}{M}I \\ \frac{-1}{M+1}\alpha_{M+1} I & O & O & \dots & O \end{bmatrix},$$

where I and O are $N \times N$ identity and zero matrices respectively, and

$$P_0 = \begin{bmatrix} -\alpha_1 & 1 & \dots & 1 & 1 \\ 0 & -\alpha_1 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -\alpha_1 & 1 \\ 0 & 0 & \dots & 0 & -\alpha_1 \end{bmatrix}.$$

It is seen that P is a sparse matrix.

2.5 The operational matrix of product

The following property of the product of two hybrid function vectors will also be used. Let

$$B(t)B^T(t)C \simeq \tilde{C}B(t), \tag{10}$$

where $\tilde{C}$ is a $N(M+1) \times N(M+1)$ product operational matrix. To illustrate the calculation procedure we choose $t_f = 1$, $M = 2$ and $N = 3$. Thus we have

$$C = [c_{10}, c_{20}, c_{30}, c_{11}, c_{21}, c_{31}, c_{12}, c_{22}, c_{32}]^T, \tag{11}$$

$$B(t) = [b_{10}(t), b_{20}(t), b_{30}(t), b_{11}(t), b_{21}(t), b_{31}(t), b_{12}(t), b_{22}(t), b_{32}(t)]^T. \tag{12}$$

In Eq. (12) we have

$$\left.\begin{array}{l} b_{10}(t) = 1 \\ b_{11}(t) = 3t - \frac{1}{2} \\ b_{12}(t) = 9t^2 - 3t + \frac{1}{6} \end{array}\right\} \qquad 0 \le t < \frac{1}{3}, \tag{13}$$

$$\left.\begin{array}{l} b_{20}(t) = 1 \\ b_{21}(t) = (3t-1) - \frac{1}{2} \\ b_{22}(t) = (3t-1)^2 - (3t-1) + \frac{1}{6} \end{array}\right\} \quad \frac{1}{3} \le t < \frac{2}{3}, \tag{14}$$

$$\left.\begin{array}{l} b_{30}(t) = 1 \\ b_{31}(t) = (3t-2) - \frac{1}{2} \\ b_{32}(t) = (3t-2)^2 - (3t-2) + \frac{1}{6} \end{array}\right\} \quad \frac{2}{3} \le t < 1. \tag{15}$$

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We also get

$$B(t)B^T(t) = \begin{bmatrix} b_{10}b_{10} & b_{10}b_{20} & b_{10}b_{30} & \cdots & b_{10}b_{12} & b_{10}b_{22} & b_{10}b_{32} \\ b_{20}b_{10} & b_{20}b_{20} & b_{20}b_{30} & \cdots & b_{20}b_{12} & b_{20}b_{22} & b_{20}b_{32} \\ b_{30}b_{10} & b_{30}b_{20} & b_{30}b_{30} & \cdots & b_{30}b_{12} & b_{30}b_{22} & b_{30}b_{32} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{12}b_{10} & b_{12}b_{20} & b_{12}b_{30} & \cdots & b_{12}b_{12} & b_{12}b_{22} & b_{12}b_{32} \\ b_{22}b_{10} & b_{22}b_{20} & b_{22}b_{30} & \cdots & b_{22}b_{12} & b_{22}b_{22} & b_{22}b_{32} \\ b_{32}b_{10} & b_{32}b_{20} & b_{32}b_{30} & \cdots & b_{32}b_{12} & b_{32}b_{22} & b_{32}b_{32} \end{bmatrix}. \tag{16}$$

From Eqs. (14)-(15) we have

$$\begin{aligned} &b_{ij}b_{kl} = 0, \quad i \neq k, \\ &b_{i0}b_{ij} = b_{ij}, \\ &b_{i1}b_{i1} = \tfrac{1}{12}b_{i0} + b_{i2}, \\ &b_{i1}b_{i2} = \tfrac{1}{6}b_{i1} + b_{i3}, \\ &b_{i2}b_{i2} = \tfrac{1}{180}b_{i0} + \tfrac{1}{3}b_{i2} + b_{i4}. \end{aligned}$$

Using Eq. (16) we get

$$B(t)B^T(t) = \begin{bmatrix} b_{10} & 0 & 0 & \cdots & b_{12} & 0 & 0 \\ 0 & b_{20} & 0 & \cdots & 0 & b_{22} & 0 \\ 0 & 0 & b_{30} & \cdots & 0 & 0 & b_{32} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{12} & 0 & 0 & \cdots & \frac{1}{180}b_{10} + \frac{1}{3}b_{12} & 0 & 0 \\ 0 & b_{22} & 0 & \cdots & 0 & \frac{1}{180}b_{20} + \frac{1}{3}b_{22} & 0 \\ 0 & 0 & b_{32} & \cdots & 0 & 0 & \frac{1}{180}b_{30} + \frac{1}{3}b_{32} \end{bmatrix}.$$

From the vector C in Eq. (11), the 9×9 matrix $\tilde{C}$ in Eq. (10) is given by

$$\tilde{C} = \begin{bmatrix} \tilde{C}_0 & \tilde{C}_1 & \tilde{C}_2 \\ \frac{1}{12}\tilde{C}_1 & \tilde{C}_0 + \frac{1}{6}\tilde{C}_2 & \tilde{C}_1 \\ \frac{1}{180}\tilde{C}_2 & \frac{1}{6}\tilde{C}_1 & \tilde{C}_0 + \frac{1}{3}\tilde{C}_2 \end{bmatrix},$$

where $\tilde{C}_i,\ i = 0, 1, 2$ are 3×3 matrices given by

$$\tilde{C}_i = \begin{bmatrix} c_{1i} & 0 & 0 \\ 0 & c_{2i} & 0 \\ 0 & 0 & c_{3i} \end{bmatrix}.$$

Similarly for other values of M and N, the product operational matrix $\tilde{C}$ in Eq. (10) can be obtained.

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3 Problem statement

Consider the following variational problems:

$$J[x(t)] = \int_0^1 F(t, x(t), \dot{x}(t), \ldots, x^{(n)}(t))dt, \tag{17}$$

with the boundary conditions

$$x(0) = a_0, \;\; \dot{x}(0) = a_1, \ldots, x^{(n-1)}(0) = a_{n-1}, \tag{18}$$

$$x(1) = b_0, \;\; \dot{x}(1) = b_1, \ldots, x^{(n-1)}(1) = b_{n-1}. \tag{19}$$

The problem is to find the extremum of Eq. (17), subject to boundary conditions in Eqs. (18) and (19). The method consists of reducing the variational problems into a set of algebraic equations by first expanding $x^{(n)}(t)$ in terms of a hybrid of block-pulse and Bernoulli polynomials with unknown coefficients.

4 The numerical method

By expanding $x^{(n)}(t)$ in the hybrid of block-pulse and Bernoulli polynomials we have,

$$x^{(n)}(t) = X^T B(t). \tag{20}$$

where X is vector of order $N \times (M+1)$ given by

$$X = [x_{10}, x_{20}, ..., x_{N0}, x_{11}, x_{21}, ..., x_{N1}, ..., x_{1M}, x_{2M}, ..., x_{NM}]^T,$$

By integrating Eq. (20) from 0 to t we get

$$x^{(n-1)}(t) - x^{(n-1)}(0) = \int_0^t X^T B(t')dt' = X^T P B(t),$$

where P is operational matrix of integration given in Eq. (9). By using Eq. (18) we get

$$x^{(n-1)}(t) = a_{n-1} + X^T P B(t). \tag{21}$$

By $n-1$ times integrating Eq. (21) from 0 to t and using boundary conditions given in Eq. (18) we have

$$x^{(n-2)}(t) = a_{n-2} + a_{n-1}t + X^T P^2 B(t), \tag{22}$$

$$\vdots$$

$$\dot{x}(t) = a_1 + a_2 t + \frac{a_3}{2!}t^2 + \cdots + \frac{a_{n-1}}{(n-2)!}t^{n-2} + X^T P^{n-1} B(t), \tag{23}$$

$$x(t) = a_0 + a_1 t + \frac{a_2}{2!}t^2 + \cdots + \frac{a_{n-1}}{(n-1)!}t^{n-1} + X^T P^n B(t). \tag{24}$$

Assume that each of t^i, $i = 1, 2, \ldots, n-1$, and each of a_{n-i}, $i = 1, 2, \ldots, n$, can be written in terms of hybrid functions as

$$t^i = d_i^T B(t), \quad i = 1, 2, \ldots, n-1 \tag{25}$$

$$a_{n-i} = a_{n-i} E^T B(t), \quad i = 1, 2, \ldots, n, \tag{26}$$

where

$$E^T = [\underbrace{1, 1, ..., 1}_{N}, \underbrace{0, 0, ..., 0}_{NM}].$$

Substituting Eqs. (25) and (26) into Eqs. (21)-(24) we obtain

$$\begin{aligned}
&x^{(n)}(t) = X^T B(t), \\
&x^{(n-1)}(t) = (a_{n-1}E^T + X^T P)B(t), \\
&\vdots \\
&\dot{x}(t) = (a_1 E^T + a_2 d_1^T + \frac{a_3}{2!}d_2^T + \cdots + \frac{a_{n-1}}{(n-2)!}d_{n-2}^T + X^T P^{n-1})B(t), \\
&x(t) = (a_0 E^T + a_1 d_1^T + \frac{a_2}{2!}d_2^T + \cdots + \frac{a_{n-1}}{(n-1)!}d_{n-1}^T + X^T P^n)B(t).
\end{aligned}$$

Substituting above equations into Eq. (17) we get

$$J[x(t)] = J[X], \tag{27}$$

the boundary conditions in Eq. (19) can be expressed as

$$q_k = x^{(k)}(1) - b_k = 0, \quad k = 0, ..., n-1. \tag{28}$$

We now find the extremum of Eq. (27) subject to Eq. (28) using the Lagrange multiplier technique. Let

$$J^*[X, \lambda] = J[X] + \lambda Q,$$

where the vector λ represents the unknown Lagrange multipliers and

$$Q = [q_0, q_1, \ldots, q_{n-1}]^T.$$

The necessary conditions are given by

$$\begin{cases} \frac{\partial}{\partial X} J^*[X, \lambda] = 0, \\ \frac{\partial}{\partial \lambda} J^*[X, \lambda] = 0. \end{cases}$$

By solving above equations, we can obtain X.

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5 Illustrative examples

In this section, four examples are given to demonstrate the applicability and accuracy of our method. Examples 1 and 2 are the variational problems for which the boundary conditions are fixed. Example 1 was first considered in [4] and Example 2 was given in [25]. Example 3 is a variational problem that some of its boundary conditions are fixed and others are unspecified, this example was considered in [2] and [26]. For Examples $1-3$ we obtain the exact solutions by using the present method. The exact solutions were not obtained in [4], [2], [25], and [26]. For example 4, we compare our findings with the numerical results obtained by using hybrid of block-pulse functions and Legendre polynomials together with the CPU time and exact values.

5.1 Example 1

Consider the extremization of [4]

$$J=\int_0^1[\frac{1}{2}\dot{x}^2-xg(t)]dt, \tag{29}$$

where

$$g(t)=\begin{cases}-1, & 0\le t\le \frac{1}{4},\quad \frac{1}{2}\le t\le 1,\\ 3, & \frac{1}{4}\le t\le\frac{1}{2},\end{cases} \tag{30}$$

with the boundary conditions

$$\dot{x}(0)=0, \qquad \dot{x}(1)=0. \tag{31}$$

Schechter [27] gave a physical interpretation of this problem by noting an application in heat conduction. The exact solution is

$$x(t)=\begin{cases}\frac{1}{2}t^2, & 0\le t\le\frac{1}{4},\\ \frac{3}{2}t^2+t-\frac{1}{8}, & \frac{1}{4}\le t\le\frac{1}{2},\\ \frac{1}{2}t^2-t+\frac{3}{8}, & \frac{1}{2}\le t\le 1.\end{cases}$$

Here, we solve this problem by using the hybrid of block-pulse functions and Bernoulli polynomials, we assume

$$\dot{x}(t)=C^TB(t),$$

in view of Eq. (30), we write Eq. (29) as

$$J=\frac{1}{2}\int_0^1\dot{x}^2(t)dt+4\int_0^{\frac{1}{4}}x(t)dt-4\int_0^{\frac{1}{2}}x(t)dt+\int_0^1 x(t)dt,$$

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or

$$J = \frac{1}{2}\int_0^1 C^T B(t) B^T(t) C dt + 4C^T P \int_0^{\frac{1}{4}} B(t)dt - 4C^T P \int_0^{\frac{1}{2}} B(t)dt + C^T P \int_0^1 B(t)dt.$$

Let

$$V(t) = \int_0^t B(t')dt', \tag{32}$$

using Eq. (8) we get

$$J = \frac{1}{2}C^T DC + C^T P[4V(\frac{1}{4}) - 4V(\frac{1}{2}) + V(1)], \tag{33}$$

the boundary conditions in Eq. (31) can be expressed in terms of hybrid functions as

$$C^T B(0) = 0, \quad C^T B(1) = 0. \tag{34}$$

We now find the extremum of Eq. (33) subject to Eq. (34) using the Lagrange multiplier technique. Suppose

$$J^* = J + \lambda_1 C^T B(0) + \lambda_2 C^T B(1),$$

where λ_1 and λ_2 are two multipliers. Then the necessary condition is given by

$$\frac{\partial}{\partial C} J^* = DC + P[4V(\frac{1}{4}) - 4V(\frac{1}{2}) + V(1)] + \lambda_1 B(0) + \lambda_2 B(1) = 0. \tag{35}$$

Equations (34) and (35) define a set of simultaneous linear algebraic equations from which the vector C and the multipliers λ_1 and λ_2 can be found. By solving above equations with $M = 2$ and $N = 4$ we get the exact solution.

5.2 Example 2

Consider the problem of finding the extremum of the functional [25]

$$J[x(t)] = \int_0^1 (\dot{x}^2(t) f(t))dt, \tag{36}$$

where

$$f(t) = \begin{cases} -1, & 0 \le t < \frac{1}{4}, \\ 1, & \frac{1}{4} < t \le 1, \end{cases} \tag{37}$$

with the boundary conditions

$$x(0) = 0, \quad x(1) = 1. \tag{38}$$

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Suppose

$$\dot{x}(t) = C^T B(t),$$

in view of Eq. (37), we write Eq. (36) as

$$J = -2\int_0^{\frac{1}{4}} \dot{x}^2(t)dt + \int_0^1 \dot{x}^2(t)dt,$$

or

$$J = -2\int_0^{\frac{1}{4}} C^T B(t)B^T(t)Cdt + \int_0^1 C^T B(t)B^T(t)Cdt.$$

Using Eqs. (8) and (10) we have

$$J = -2\int_0^{\frac{1}{4}} C^T \tilde{C} B(t)dt + C^T DC,$$

applying Eq. (32) we get

$$J = -2C^T \tilde{C} V(\frac{1}{4}) + C^T DC, \tag{39}$$

the boundary condition in Eq. (38) can be expressed in terms of hybrid functions as

$$C^T PB(1) = 1. \tag{40}$$

We now find the extremum of Eq. (39) subject to Eq. (40) using the Lagrange multiplier technique. Suppose

$$J^* = J + \lambda(C^T PB(1) - 1),$$

where λ is multiplier. Then the necessary conditions are given by

$$\begin{cases} \frac{\partial}{\partial C} J^* = 0, \\ \frac{\partial}{\partial \lambda} J^* = 0. \end{cases}$$

By solving above equations with $M = 1$ and $N = 4$, we get

$$x(t) = \begin{cases} -2t, & 0 \leq t \leq \frac{1}{4}, \\ 2t - 1, & \frac{1}{4} \leq t \leq 1, \end{cases}$$

which is the exact solution.

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5.3 Example 3

Find the extremum of the functional [2, 26]

$$J[x(t)] = \int_0^1 [\frac{1}{2}\ddot{x}^2(t) + 4(1-t)\dot{x}(t)]dt = \int_0^1 F(t, x(t), \dot{x}(t), \ddot{x}(t))dt, \tag{41}$$

with the initial conditions

$$x(0) = 0, \quad \dot{x}(0) = 0, \tag{42}$$

and the values of $x(1)$ and $\dot{x}(1)$ are unspecified.
The exact solution via Euler equation is $x(t) = -\frac{1}{6}t^4 + \frac{2}{3}t^3 - t^2$.
The natural boundary conditions are found from following equations [10]

$$F_{\dot{x}} - \tfrac{d}{dt}(F_{\ddot{x}})|_{t=1} = 0,$$

$$F_{\ddot{x}}|_{t=1} = 0,$$

that imply

$$\dddot{x}(1) = 0, \tag{43}$$

$$\ddot{x}(1) = 0. \tag{44}$$

Suppose

$$\dddot{x}(t) = C^T B(t), \tag{45}$$

by 3 times integrating Eq. (45) from 0 to t and using boundary conditions given in Eq. (42) we have

$$\ddot{x}(t) = C^T P B(t) + \ddot{x}(0), \tag{46}$$

$$\dot{x}(t) = C^T P^2 B(t) + \ddot{x}(0)t, \tag{47}$$

$$x(t) = C^T P^3 B(t) + \ddot{x}(0) d_1^T P B(t), \tag{48}$$

where $t \simeq d_1^T B(t)$ and P is operational matrix of integration given in Eq. (9). Using Eqs. (44) and (46) we get

$$\ddot{x}(0) = -C^T P B(1). \tag{49}$$

Suppose

$$4(1-t) = A^T B(t), \tag{50}$$

substituting Eqs. (46), (47) and (50) into Eq. (41) and using Eqs. (8) and (32) we have

$$\begin{aligned} J = {} & \tfrac{1}{2}C^T P D P^T C - C^T P V(1) B^T(1) P^T C + \tfrac{1}{2}C^T P B(1) B^T(1) P^T C \\ & + A^T D P^{2^T} C - A^T D d_1 B^T(1) P^T C. \end{aligned} \tag{51}$$

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By applying Eq. (45)the boundary condition in Eq. (43) can be written as

$$C^T B(1) = 0. \tag{52}$$

We now find the extremum of Eq. (51) subject to Eq. (52) using the Lagrange multiplier technique. Suppose

$$J^* = J + \lambda(C^T B(1)),$$

where λ is multiplier. Then the necessary conditions are given by

$$\begin{cases} \frac{\partial}{\partial C} J^* = 0, \\ \\ \frac{\partial}{\partial \lambda} J^* = 0. \end{cases}$$

By solving above equations with $M = 5$ and $N = 1$, we obtain

$$c_{10} = 2,\ c_{11} = -4\ , c_{12} = 0\ , c_{13} = 0\ , c_{14} = 0\ , c_{15} = 0$$

substituting above values in Eq. (48) the exact solution is obtained.

5.4 Example 4

Find the extremum of the functional

$$J[x(t)] = \int_0^{\frac{\pi}{4}} (x^2(t) - \dot{x}^2(t))dt, \quad x(0) = 1, \quad \dot{x}(\frac{\pi}{4}) = 0. \tag{53}$$

The exact value is $x(t) = \sin(t) + \cos(t)$. In Table 1, the values of $x(t)$ using the hybrid of block-pulse and Legendre polynomials (B-P Legendre), the hybrid of block-pulse and Bernoulli polynomials (B-P Bernoulli), together with CPU time and the exact solution are listed.

Table 1. Estimated and exact values of x(t)

t	B-P Legendre $N = 4, M = 2$	B-P Bernoulli $N = 4, M = 2$	B-P Legendre $N = 4, M = 3$	B-P Bernoulli $N = 4, M = 3$	Exact
0	0.999943	1.000000	0.999999	0.999999	1.000000
0.1	1.094835	1.094834	1.094788	1.094838	1.094838
0.3	1.250852	1.250849	1.250709	1.250857	1.250857
0.5	1.357004	1.357005	1.356769	1.357008	1.357008
0.7	1.409056	1.409056	1.407521	1.409059	1.409059
CPU time	0.732	0.578	0.954	0.818	

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6 Conclusion

In the present work the hybrid of block-pulse functions and Bernoulli polynomials are used to solve variational problems. The variational problems has been reduced to a problem of solving a system of algebraic equations. For constructing matrices D and P in Eqs. (8) and (9) we use Bernoulli numbers α_n which are a sequence of singed rational numbers and $\alpha_{2n+1} = 0, \ n = 1, 2, 3, \ldots$. Thus the matrices P and D have many zero elements and they are sparse, hence the present method is very attractive and reduces the CPU time and the computer memory. Illustrative examples are given to demonstrate the validity and applicability of the proposed method.

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