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APPLICATION OF THE BESSEL-HYBRID FUNCTIONS FOR THE LINEAR FREDHOLM-VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT: In this paper a collocation method based on the Bessel-hybrid functions is used for approximation of the solution of linear Fredholm-Volterra integro-differential equations (FVIDEs) under mixed conditions. First, we explain the properties of Bessel-hybrid functions, which are combination of block-pulse functions and Bessel functions of first kind. The method is based upon Bessel-hybrid approximations, so that to obtain the operational matrixes and approximation of functions we use the transfer matrix from Bessel-hybrid functions to Taylor polynomials. The matrix equations correspond to a system of linear algebraic equations with the unknown Bessel-hybrid coefficients. Present results and comparisons demonstrate our estimate have good degree of accuracy.

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1. INTRODUCTION

Many analytical and numerical methods have been exited to solve linear and nonlinear integro-differential equations. Now, for solution of these equations we used Bessel-hybrid functions with collocation points. Recently, many of polynomials with block-pulse functions are combined, such as: Legendre, Chebyshev, Taylor and other polynomials. Ordokhani [9] has used Walsh-hybrid functions operational matrix with Newton-Cotes nodes for solving of Fredholm-Hemmerstein integral equations. Maleknejad et al. [8] have solved linear Fredholm and Volterra integral equation of the second by using Legendre wavelets. Hou and Yang in [5], have solved Fredholm integro-differential equations by using hybrid function operational matrix of derivative. Authors in [17] have used hybrid functions for solving Fredholm and Volterra integral equations. Danfu and Xufeng [2] have solved integro-differential equations by using CAS wavelet operational matrix of integration. Razzaghi et al. have used hybrid of block-pulse and Bernoulli polynomials in [14]. Yuzbasi et al. [21], Yuzbasi and Sezer [23], Yuzbasi et al. [20] have worked on the Bessel matrix and collocation methods for the numerical solutions of the neutral delay differential equations, the pantograph equations and the Lane-Emden differential equations. Also, readers who are interested in learning more about this topic, could refer to [1], [3], [4], [6], [12], [13], [15].

Recently, Yazbasi in [22] used Bessel polynomials and Bessel collocation method [23] for solving high-order linear Fredholm-Volterra integro-differential equations.

In this article, by Bessel-hybrid functions and suitable collocation points, we estimate the solution of linear (FVIDEs) of the form:

$$\sum_{k=0}^{r} p_k(x)y^{(k)}(x) = g(x) + \xi_1\int_0^1 k_1(x,t)y(t)dt + \xi_2\int_0^x k_2(x,t)y(t)dt, \qquad 0 \le x,t \le 1, \tag{1}$$

under the mixed conditions

$$\sum_{k=0}^{r-1}[a_{jk}y^{(k)}(0) + b_{jk}y^{(k)}(1)] = \lambda_j, \qquad j = 0,1,\cdots,r-1, \tag{2}$$

where $y(x)$ is an unknown function and $p_k(x)$, $k=0,1,\cdots,r$, $g(x)$, $k_1(x,t)$ and $k_2(x,t)$ are known functions. Also, a_{jk}, b_{jk}, λ_j, ξ_1 and ξ_2 are real or complex constants.

2. PROPERTIES OF BESSEL-HYBRID FUNCTIONS AND TAYLOR POLYNOMIALS

2.1 Bessel-hybrid functions

Bessel-hybrid functions $b(n,m,x)$, $n=1,2,\cdots,N$, $m=0,1,\cdots,M$, have three arguments *n*, *m* and *x*. Respectively, *n* is the order for block-pulse, *m* is the order for Bessel polynomials and *x* is the normalized time, is defined on the interval [0, 1) as

(3)
$$b(n,m,x)=\begin{cases} J_{m,M}(Nx-n+1), & \frac{n-1}{N}\le x<\frac{n}{N}, \\ 0, & \text{otherwise,} \end{cases}$$

where $J_{m,M}(x)$ are Bessel polynomials, obtained as truncated Bessel functions of first kind and of *m* order, defined as in [10]:

(4)
$$J_{m,M}(x)=\sum_{k=0}^{[\frac{M-m}{2}]}\frac{(-1)^k}{k!(k+m)!}\left(\frac{x}{2}\right)^{2k+m}, \quad 0\le x<\infty, \quad m\in \mathrm{N}.$$

In [4] *M* is a positive integer, so that $M\ge m$, $m=0,1,\cdots,M$ and a set of block-pulse functions $\varphi_i(x)$, $i=1,2,\cdots,N$, on the interval $L^2[0,1]$ is defined as [5]:

$$\varphi_i(x)=\begin{cases} 1, & \frac{i-1}{N}\le x<\frac{i}{N}, \\ 0, & \text{otherwise.} \end{cases}$$

The intervals where two distinct block-pulse functions on [0,1) are different from zero are disjoint, for $i=1,2,\cdots,N$. These functions have the property of orthogonality on [0,1).

2.2 Function approximation

Now, we approximate a function $y(x)$ in $L^2[0,1]$ space in Bessel-hybrid functions as

(5)
$$y(x)\simeq A^T B(x)=B^T(x)A,$$

where

(6)
$$A=\left[a_{10},a_{11},\cdots,a_{1M},|\cdots|,a_{N0},a_{N1},\cdots,a_{NM}\right]^T,$$
$$B(x)=\left[b(1,0,x),b(1,1,x),\cdots,b(1,M,x),|\cdots|,b(N,0,x),b(N,1,x)\cdots,b(N,M,x)\right]^T.$$

In this case, for *M* = 2 and *N* = 2 we have

(7)
$$B(x)=[b(1,0,x),b(1,1,x),b(1,2,x)\,|\,b(2,0,x),b(2,1,x),b(2,2,x)]^T,$$

so that

$$\left.\begin{aligned} b(1,0,x)&=1-x^2\\ b(1,1,x)&=x\\ b(1,2,x)&=\frac{x^2}{2}\end{aligned}\right\}, \quad 0\le x<\frac{1}{2}, \qquad \left.\begin{aligned} b(2,0,x)&=\frac{3}{4}+x-x^2\\ b(2,1,x)&=x-\frac{1}{2}\\ b(2,2,x)&=\frac{1}{8}-\frac{x}{2}+\frac{x^2}{2}\end{aligned}\right\}, \quad \frac{1}{2}\le x<1.$$

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We can transfer the Bessel-hybrid functions to M-th degree Taylor basis functions. In matrix form as

(8) $$B(x) = D\hat{X}(x),$$

where

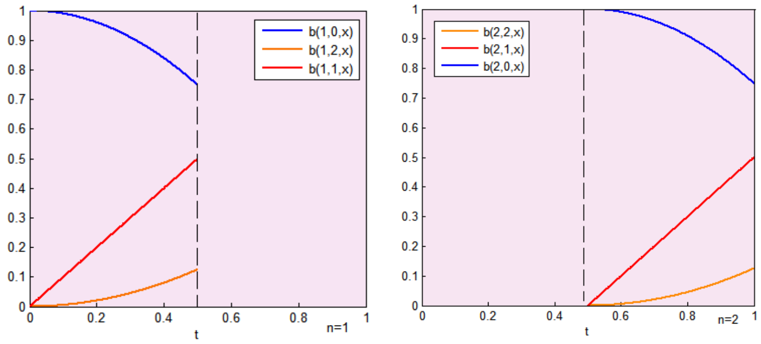

Figure1. Graphs of Bessel-hybrid functions for N=2, M=2.

$$D = \begin{cases} D_1, & 0 \le x < \dfrac{1}{N}, \\ D_2, & \dfrac{1}{N} \le x < \dfrac{2}{N}, \\ \vdots & \vdots \\ D_N, & \dfrac{N-1}{N} \le x < 1, \end{cases}$$

so that

$$D_i = diag(O, \cdots, O, \hat{D}_i, O, \cdots, O),$$

where $\hat{D}_i$ is an $(M+1)\times(M+1)$ matrix defined in the subinterval $[\frac{i-1}{N}, \frac{i}{N}),\ i = 1,2,\cdots,N$ of [0, 1), O is zero matrix with $(M+1)\times(M+1)$ dimension and

$$\hat{X}(x) = \begin{cases} \hat{X}_1(x), & 0 \le x < \dfrac{1}{N}, \\ \hat{X}_2(x), & \dfrac{1}{N} \le x < \dfrac{2}{N}, \\ \vdots & \vdots \\ \hat{X}_N(x), & \dfrac{N-1}{N} \le x < 1, \end{cases}$$

$$\hat{X}_i(x) = [0,0,\cdots,0 | \cdots | \underbrace{1, x, x^2, \cdots, x^M}_{\frac{i-1}{N} \le x < \frac{i}{N}} | \cdots | 0,0,\cdots,0]^T, \qquad i = 1,2,\cdots,N.$$

Also, we can obtain D for different N and M, so for N = 2 and M = 2 we have

$$D = \begin{cases} D_1, & 0 \le x < \dfrac{1}{2}, \\ D_2, & \dfrac{1}{2} \le x < 1, \end{cases}$$

where

$$D_1 = diag(\hat{D}_1, O), \qquad D_2 = diag(O, \hat{D}_2),$$

and

$$\hat{D}_1=\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & 0\\ 0 & 0 & \frac{1}{2}\end{bmatrix},\qquad \hat{D}_2=\begin{bmatrix}\frac{3}{4} & 1 & -1\\ \frac{-1}{2} & 1 & 0\\ \frac{1}{8} & \frac{-1}{2} & \frac{1}{2}\end{bmatrix},$$

so that

$$\hat{X}(x)=\begin{cases}\hat{X}_1(x), & 0\le x<\frac{1}{2},\\ \hat{X}_2(x), & \frac{1}{2}\le x<1,\end{cases},$$

$$\hat{X}_1(x)=[1,x,x^2|0,0,0]^T,\qquad \hat{X}_2(x)=[0,0,0\,|\,1,x,x^2]^T.$$

By substituting Eq. (8) in Eq. (5) we get

(9) $$y(x)\simeq A^T D\hat{X}(x)=\hat{X}^T(x)D^T A.$$

2.3 Operational matrix of Taylor polynomials

We can obtain operational matrix of integration for Taylor polynomials in N subinterval $[\frac{i-1}{N},\frac{i}{N}),\ i=1,2,\cdots,N$ of [0, 1] as

(10) $$\int_0^x \hat{X}(t)dt\simeq \hat{L}\hat{X}(x),$$

where

$$\hat{L}=\begin{cases}diag(L,O,\cdots,O), & 0\le x<\frac{1}{N},\\ diag(O,L,\cdots,O), & \frac{1}{N}\le x<\frac{2}{N},\\ \quad\vdots & \quad\vdots\\ diag(O,O,\cdots,L), & \frac{N-1}{N}\le x<1,\end{cases}$$

and L is operational matrix of integration for Taylor polynomials in [0, 1] as [7]

(11) $$L=\begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & \frac{1}{2} & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & \frac{1}{M}\\ 0 & 0 & 0 & \cdots & 0\end{bmatrix},$$

dimension of $\hat{L}$ is $N(M+1)\times N(M+1)$. We want to present dual operational matrix of $\hat{X}(x)$ with taking the integration of the cross product of two Taylor polynomials function vectors in N subinterval $[\frac{i-1}{N},\frac{i}{N}),\ i=1,2,\cdots,N$ of [0, 1] is obtain

(12) $$\hat{H}_1=\int_0^1 \hat{X}(t)\hat{X}^T(t)dt,$$

or

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$$\hat{H}_1 = \begin{cases} diag(H, O, \cdots, O), & 0 \le x < \frac{1}{N}, \\ diag(O, H, \cdots, O), & \frac{1}{N} \le x < \frac{2}{N}, \\ \vdots & \vdots \\ diag(O, O, \cdots, H), & \frac{N-1}{N} \le x < 1, \end{cases}$$

where H is the present dual operational matrix of Taylor polynomials in [0, 1] with $(M+1)\times(M+1)$ dimension so that [7]

(13) $$H = [h_{ij}], \qquad h_{ij} = \frac{1}{i+j+1}, \qquad i, j = 0,1,\cdots, M.$$

Finally, we obtain operational matrix of derivative for $\hat{X}(x)$ as

(14) $$\hat{X}^{(k)}(x) \simeq (\hat{B})^k \hat{X}(x),$$

where

$$\hat{B} = \begin{cases} diag(B^T, O, \cdots, O), & 0 \le x < \frac{1}{N}, \\ diag(O, B^T, \cdots, O), & \frac{1}{N} \le x < \frac{2}{N}, \\ \vdots & \vdots \\ diag(O, O, \cdots, B^T), & \frac{N-1}{N} \le x < 1, \end{cases}$$

and [19]

$$B^T = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & M \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

B is operational matrix of derivative for Taylor polynomials in [0, 1].

3. FUNDAMENTAL RELATIONS

3.1 Matrix relation for the differential part

In this section we obtain approximation to the k-th derivative of $y(x)$ by using Eqs. (9) and (14). Thus we get

(15) $$y^{(k)}(x) \simeq (\hat{X}^{(k)}(x))^T D^T A = \hat{X}^T(x)(\hat{B}^k)^T D^T A, \qquad k = 0,1,\cdots, r.$$

According to Eq. (15), we have matrix form of differential part as

(16) $$\sum_{k=0}^{r} p_k(x) y^{(k)}(x) \simeq \sum_{k=0}^{r} p_k(x) \hat{X}^T(x)(\hat{B}^k)^T D^T A.$$

3.2 Matrix relation for the Fredholm integral part

We approximate kernel function $k_1(x,t)$ by the truncated Maclaurin series and truncated Bessel-hybrid series, respectively [10]

(17) $$k_1(x,t) = \hat{X}^T(x) k_t^1 \hat{X}(t),$$

(18) $$k_1(x,t) = B^T(x) k_b^1 B(t),$$

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where

$$k_t^1=\begin{cases} diag(k_{t1},O,\cdots,O), & 0\le x<\frac{1}{N},\\ diag(O,k_{t1},\cdots,O), & \frac{1}{N}\le x<\frac{2}{N},\\ \vdots & \vdots\\ diag(O,O,\cdots,k_{t1}), & \frac{N-1}{N}\le x<1,\end{cases}$$

$$k_{t1}=[{}_t k_{mn}^1],\qquad m,n=0,1,\cdots,M,$$

and

$${}_t k_{mn}^1=\frac{1}{m!n!}\frac{\partial^{m+n}k_1(0,0)}{\partial x^m\partial t^n},\qquad m,n=0,1,\cdots,M.$$

By substituting Eq. (8) in Eq. (18) and putting equal to Eq. (17) we obtain:

(19) $$k_t^1=D^T k_b^1 D,\qquad k_b^1=(D^T)^{-1}k_t^1(D)^{-1}.$$

By using the Eqs. (5) and (18) in Fredholm integral part of Eq. (1) we get

(20) $$\int_0^1 k_1(x,t)y(t)dt\simeq\int_0^1 B^T(x)k_b^1 B(t)B^T(t)Adt=B^T(x)k_b^1 Q_1 A,$$

so that

$$Q_1=\int_0^1 B(t)B^T(t)dt\simeq\int_0^1 D\hat{X}(t)\hat{X}^T(t)D^T dt=D\hat{H}_1 D^T,$$

where $\hat{H}_1$, the integration of dual operational matrix of Taylor polynomials is defined in Eq. (12). Finally, by substituting Eq. (8) in Eq. (20) we have matrix form of Fredholm integral part

(21) $$\int_0^1 k_1(x,t)y(t)dt\simeq\hat{X}^T(x)D^T k_b^1 Q_1 A.$$

3.3 Matrix relation for the Volterra integral part

We can write kernel function $k_2(x,t)$ such as $k_1(x,t)$ and we approximate truncated Maclaurin series and truncated Bessel-hybrid series, so matrix form as [10]:

(22) $$k_2(x,t)=\hat{X}^T(x)k_t^2\hat{X}(t),$$

(23) $$k_2(x,t)=B^T(x)k_b^2 B(t),$$

where

$$k_t^2=\begin{cases} diag(k_{t2},O,\cdots,O), & 0\le x<\frac{1}{N},\\ diag(O,k_{t2},\cdots,O), & \frac{1}{N}\le x<\frac{2}{N},\\ \vdots & \vdots\\ diag(O,O,\cdots,k_{t2}), & \frac{N-1}{N}\le x<1,\end{cases}$$

$$k_{t2}=[{}_t k_{mn}^2],\qquad m,n=0,1,\cdots,M,$$

and

$${}_t k_{mn}^2=\frac{1}{m!n!}\frac{\partial^{m+n}k_2(0,0)}{\partial x^m\partial t^n},\qquad m,n=0,1,\cdots,M.$$

By substituting Eq. (8) in Eq. (23) and putting equal to Eq. (22) we obtain

(24) $$k_t^2=D^T k_b^2 D,\qquad k_b^2=(D^T)^{-1}k_t^2(D)^{-1}.$$

By substituting the matrix form of Eqs. (9) and (23) in Volterra integral part of Eq. (1) we have

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(25) $$\int_0^x k_2(x,t)y(t)dt \simeq \int_0^x B^T(x)k_b^2 B(t)B^T(t)Adt = B^T(x)k_b^2 Q_2(x)A,$$

so that

$$Q_2(x) = \int_0^x B(t)B^T(t)dt \simeq \int_0^x D\hat{X}(t)\hat{X}^T(t)D^T dt = D\hat{H}_2(x)D^T,$$

where $\hat{H}_2(x)$, the integration of dual operational matrix of Taylor polynomials in the subinterval $[\frac{i-1}{N},\frac{i}{N}),\ i=1,2,\cdots,N$ of [0, 1], is defined as

$$\hat{H}_2(x) \simeq \int_0^x \hat{X}(t)\hat{X}^T(t)dt,$$

or

$$\hat{H}_2(x) = \begin{cases} diag(H_2(x),O,\cdots,O), & 0 \le x < \frac{1}{N}, \\ diag(O,H_2(x),\cdots,O), & \frac{1}{N} \le x < \frac{2}{N}, \\ \vdots & \vdots \\ diag(O,O,\cdots,H_2(x)), & \frac{N-1}{N} \le x < 1, \end{cases}$$

so that

$$H_2(x) \simeq \int_0^x X(t)X^T(t)dt = [h_{ij}^2(x)], \quad h_{ij}^2(x) = \frac{x^{i+j+1}}{i+j+1}, \quad i,j=0,1,\cdots,M.$$

By substituting Eq. (8) in Eq. (25) we have matrix form of Volterra integral part

(26) $$\int_0^x k_2(x,t)y(t)dt \simeq \hat{X}^T(x)M\hat{H}_2(x)D^T A, \qquad M = D^T k_b^2 D.$$

4. METHOD OF SOLUTION

To solve Eq. (1) with conditions in Eq. (2), we substitute Eqs. (16), (21) and (26) in Eq. (1) as

(27) $$\sum_{k=0}^{r} p_k(x)\hat{X}^T(x)(\hat{B}^k)^T D^T A = g(x) + \xi_1\hat{X}^T(x)D^T k_b^1 Q_1 A + \xi_2\hat{X}^T(x)M\hat{H}_2(x)D^T A.$$

In order to find A, we collocate Eq. (27) in nodal points of Newton-Cotes as [11]

$$x_i = \frac{2i-1}{2N(M+1)}, \qquad i=1,\cdots,N(M+1).$$

We have

(28) $$\sum_{k=0}^{r} p_k(x_i)\hat{X}^T(x_i)(\hat{B}^k)^T D^T A = g(x_i) + \xi_1\hat{X}^T(x_i)D^T k_b^1 Q_1 A + \xi_2\hat{X}^T(x_i)M\hat{H}_2(x_i)D^T A,$$

where $i=1,2,\cdots,N(M+1)$, we can write Eq. (28) as

$$\left(\sum_{k=0}^{r} p_k(x_i)\hat{X}^T(x_i)(\hat{B}^k)^T D^T - \xi_1\hat{X}^T(x_i)D^T k_b^1 Q_1 - \xi_2\hat{X}^T(x_i)M\hat{H}_2(x_i)D^T\right)A = g(x_i),$$

or briefly the fundamental matrix equation as

(29) $$\left(\sum_{k=0}^{r} P_k\hat{X}(\hat{B}^k)^T D^T - \xi_1\hat{X}D^T k_b^1 Q_1 - \xi_2\overline{XMHD}\right)A = G,$$

where

$$P_k = \begin{bmatrix} p_k(x_1) & 0 & \cdots & 0 \\ 0 & p_k(x_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_k(x_{N(M+1)}) \end{bmatrix}, \quad \hat{\mathrm{X}} = \begin{bmatrix} \hat{X}^T(x_1) \\ \hat{X}^T(x_2) \\ \vdots \\ \hat{X}^T(x_{N(M+1)}) \end{bmatrix}, \quad \mathrm{G} = \begin{bmatrix} g(x_1) \\ g(x_2) \\ \vdots \\ g(x_{N(M+1)}) \end{bmatrix},$$

$$\overline{\mathrm{X}} = \begin{bmatrix} \hat{X}^T(x_1) & 0 & \cdots & 0 \\ 0 & \hat{X}^T(x_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \hat{X}^T(x_{N(M+1)}) \end{bmatrix}, \quad \overline{\mathrm{M}} = \begin{bmatrix} \mathrm{M} & 0 & \cdots & 0 \\ 0 & \mathrm{M} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathrm{M} \end{bmatrix}, \quad \overline{\mathrm{D}} = \begin{bmatrix} \mathrm{D}^{\mathrm{T}} \\ \mathrm{D}^{\mathrm{T}} \\ \vdots \\ \mathrm{D}^{\mathrm{T}} \end{bmatrix}$$

and

$$\overline{H} = \begin{bmatrix} \hat{H}_2(x_1) & 0 & \cdots & 0 \\ 0 & \hat{H}_2(x_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \hat{H}_2(x_{N(M+1)}) \end{bmatrix}.$$

Finally, we have fundamental matrix equations as

(30) $$WA = G, \qquad W = \sum_{k=0}^{r} P_k \hat{X}(\hat{B}^k)^T D^T - \xi_1 \hat{X} D^T k_b^1 Q_1 - \xi_2 \overline{X}\,\overline{M}\,\overline{H}\,\overline{D}.$$

Now, we used Eq. (15) to obtain the matrix form of initial conditions, so that

(31) $$\sum_{k=0}^{r-1} [a_{jk} \hat{X}^T(0) + b_{jk} \hat{X}^T(1)](\hat{B}^k)^T D^T A = \lambda_j, \qquad j = 0,1,\cdots,r\text{-}1,$$

or briefly, we can write the fundamental matrix equations for initial conditions as

(32) $$U_j A = [\lambda_j], \quad j = 0,1,\cdots,r\text{-}1,$$

where

$$\begin{aligned} U_j &= \sum_{k=0}^{r-1} [a_{jk} \hat{X}^T(0) + b_{jk} \hat{X}^T(1)](\hat{B}^k)^T D^T \\ &= [u_{j,10}, \cdots, u_{j,1M}, \cdots, u_{j,N0}, \cdots, u_{j,NM}], \qquad j = 0,1,\cdots,r-1. \end{aligned}$$

Ultimately, we are replacing the rows of U_j by last rows of W and the rows of λ_j by last rows of G to obtain the solution of Eq. (1) under conditions

$$\hat{W}A = \hat{G},$$

where

$$[\hat{W} \mid \hat{G}] = \begin{bmatrix} W & \vdots & G \\ \cdots & \cdots & \cdots \\ U_j & \vdots & \lambda_j \end{bmatrix}, \qquad j = 0,1,\cdots,r-1.$$

We can write

(33) $$A = (\hat{W})^{-1}\hat{G}.$$

We obtain A from system of Eq. (33) and with substituting A in Eq. (9), we get approximate solution of Eq. (1).

5. ILLUSTRATIVE EXAMPLES

The aim of this method to obtain an approximate solution to the problem with minimum errors, we report the results of some examples where given in the different papers. In addition, we have expressed absolute error function which are define as $|y(x) - y_N(x)|$,

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where $y(x)$ is the exact solution of Eq. (1) and $y_N(x)$ is the approximate of $y(x)$. All the examples were performed on the computer using a program written in MATLAB.

Example 5.1. Consider the FVIDE [18, 19]

(34) $$xy''(x) - xy'(x) + 2y(x) = g(x) + \int_0^1 (x+t)y(t)dt + \int_0^x (x-t)y(t)dt,$$

with conditions $y(0) = 1$ and $y'(0) - 2y(1) + 2y(0) = 1$. The exact solution to Eq. (34) is $y(x) = -x^2 + x + 1$. where $g(x) = \frac{1}{12}x^4 - \frac{1}{6}x^3 - \frac{1}{2}x^2 - \frac{13}{6}x + \frac{17}{12}$, $k_1(x,t) = x+t$, $k_2(x,t) = x-t$, $\xi_1 = 1$ and $\xi_2 = 1$. Also, the set of collocation points for $N = 2$ and $M = 2$ is

$$\left\{x_0 = \frac{1}{12}, x_1 = \frac{3}{12}, x_2 = \frac{5}{12}, x_3 = \frac{7}{12}, x_4 = \frac{9}{12}, x_5 = \frac{11}{12}\right\},$$

so that

$$\hat{X}(x) = \begin{cases} \hat{X}_1(x), & 0 \le x < \frac{1}{2}, \\ \hat{X}_2(x), & \frac{1}{2} \le x < 1, \end{cases}, \qquad \hat{X}_1(x) = [1, x, x^2 | 0,0,0]^T, \qquad \hat{X}_2(x) = [0,0,0 | 1, x, x^2]^T,$$

$$0 \le x < \frac{1}{2} \Rightarrow G = \begin{bmatrix} g(x_0) \\ g(x_1) \\ g(x_2) \\ 0 \\ 0 \\ 0 \end{bmatrix} \simeq \begin{bmatrix} 1.3225 \\ 0.84147 \\ 0.4175 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad k_t^1 = diag(k_{t1}, O), \qquad k_{t1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$k_b^1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad k_t^2 = diag(k_{t2}, O), \qquad k_{t2} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad k_b^2 = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$H_2 = \begin{bmatrix} x & \frac{x^2}{2} & \frac{x^3}{3} \\ \frac{x^2}{2} & \frac{x^3}{3} & \frac{x^4}{4} \\ \frac{x^3}{3} & \frac{x^4}{4} & \frac{x^5}{5} \end{bmatrix}, \qquad \hat{H}_2 = diag(H_2, O),$$

$$0 \le x < \frac{1}{2} \Rightarrow G = \begin{bmatrix} g(x_0) \\ g(x_1) \\ g(x_2) \\ 0 \\ 0 \\ 0 \end{bmatrix} \simeq \begin{bmatrix} 1.3225 \\ 0.84147 \\ 0.4175 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad k_t^1 = diag(k_{t1}, O), \qquad k_{t1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

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$$k_b^1=\begin{bmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&1&1&2\\0&0&0&1&0&2\\0&0&0&2&2&4\end{bmatrix},\qquad k_t^2=diag(O,k_{t2}),\qquad k_{t2}=\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix},\qquad k_b^2=\begin{bmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&-1&0\\0&0&0&1&0&2\\0&0&0&0&-2&0\end{bmatrix},$$

$$H_2=\begin{bmatrix}x&\frac{x^2}{2}&\frac{x^3}{3}\\\frac{x^2}{2}&\frac{x^3}{3}&\frac{x^4}{4}\\\frac{x^3}{3}&\frac{x^4}{4}&\frac{x^5}{5}\end{bmatrix},\qquad \hat{H}_2=diag(O,H_2),$$

and

$$\overline{X}=diag(\hat{X}^T(x_0),\hat{X}^T(x_1),\hat{X}^T(x_2),\hat{X}^T(x_3),\hat{X}^T(x_4),\hat{X}^T(x_5)),$$
$$\overline{H}=diag(\hat{H}_2(x_0),\hat{H}_2(x_1),\hat{H}_2(x_2),\hat{H}_2(x_3),\hat{H}_2(x_4),\hat{H}_2(x_5)).$$

The matrix form of initial conditions for Eq. (32) is

$$U_jA=\lambda_j,\qquad j=0,1,$$

or

$$U_0=[0.9930\quad 0.08333\quad 0.0034\quad 0\quad 0\quad 0],\qquad \lambda_0=1,$$
$$U_1=[1.8194\quad 1.1667\quad 0.09027\quad -1.6528\quad 0.8833\quad -0.1736],\qquad \lambda_1=1.$$

Hence, by using Eq. (30) and matrices obtained above, we obtain Bessel coefficient matrix as

$$A=[1.0777\quad -1.0181\quad 4.2055\quad 0.4250\quad -0.00698\quad -3.200]^T.$$

Finally, by substituting A in Eq. (8) for N = 2 and M = 2, we have approximate solution of Eq. (34) by Bessel-hybrid functions as

$$y(x)=1+x-x^2,\qquad 0\le x<\frac{1}{2}$$

$$y(x)=1.00000000000+1.00000000000x-1.00000000000x^2,\qquad \frac{1}{2}\le x<1,$$

which is the exact solution of Eq. (34).

Example 5.2. Consider the FVIE [4]

$$y(x)=g(x)+\int_0^1 e^{x+t}y(t)dt+\int_0^x te^x y(t)dt,\qquad 0\le x\le 1,$$

the exact solution to this example is $y(x)=e^{-x}$ and $g(x)=x-2e^x+e^{-x}+1$. The computational result of absolute error for M = 3, N = 3 and N = 3, M = 8,10 with the result of another method are given in Table 1. The values obtained in Table 1 show that if N and M increase, the accuracy will increase.

Table 1. Absolute errors of Example 5.2.

x	Present method N=3, M=3	N=3, M=8	N=3, M=10	Method of [4] K=3, M=3
0.0	4.42×10^{-2}	2.95×10^{-7}	2.21×10^{-9}	8.6×10^{-6}
0.1	2.36×10^{-2}	3.30×10^{-7}	2.46×10^{-9}	5.3×10^{-7}
0.2	1.12×10^{-2}	3.73×10^{-7}	2.77×10^{-9}	2.0×10^{-6}
0.3	5.27×10^{-3}	4.25×10^{-7}	3.16×10^{-9}	3.0×10^{-6}
0.4	4.04×10^{-3}	4.92×10^{-7}	3.67×10^{-9}	1.9×10^{-5}
0.5	5.61×10^{-3}	5.79×10^{-7}	4.34×10^{-9}	9.9×10^{-8}
0.6	8.03×10^{-3}	6.78×10^{-7}	5.19×10^{-9}	9.3×10^{-7}
0.7	9.28×10^{-3}	7.40×10^{-7}	5.98×10^{-9}	4.3×10^{-6}
0.8	7.31×10^{-3}	6.08×10^{-7}	5.53×10^{-9}	5.3×10^{-7}
0.9	2.43×10^{-3}	1.73×10^{-7}	5.99×10^{-10}	3.6×10^{-7}
1.0	1.48×10^{-2}	2.79×10^{-6}	2.65×10^{-8}	-

Example 5.3. Consider the FIDE [1, 6]

$$y'(x)=\int_0^1 e^{xt}y(t)dt+y(x)+\frac{1-e^{(x+1)}}{x+1}, \qquad 0\le x\le 1,$$

with condition $y(0) = 1$ and the exact solution is $y(x) = \exp(x)$. Respectively, the absolute error and maximum absolute error values are given for different values of N and M in Table 2 and Table 3.

Table 2. Absolute errors of Example 5.3.

x	Present method N=3, M=4	N=3, M=6	N=3, M=8	Method of [1]	Method of [6]
0.05	2.86×10^{-3}	1.42×10^{-6}	5.63×10^{-9}	4.89×10^{-5}	8.90×10^{-6}
0.15	2.94×10^{-3}	1.70×10^{-6}	5.64×10^{-9}	1.75×10^{-4}	5.75×10^{-6}
0.25	3.03×10^{-3}	1.84×10^{-6}	5.65×10^{-9}	3.14×10^{-4}	2.45×10^{-5}
0.35	3.05×10^{-3}	1.91×10^{-6}	5.67×10^{-9}	4.92×10^{-4}	1.24×10^{-5}
0.45	3.06×10^{-3}	1.96×10^{-6}	5.53×10^{-9}	6.97×10^{-4}	2.18×10^{-6}
0.55	3.04×10^{-3}	1.92×10^{-6}	5.18×10^{-9}	9.46×10^{-4}	6.98×10^{-6}
0.65	3.01×10^{-3}	1.57×10^{-6}	4.17×10^{-9}	1.23×10^{-4}	9.17×10^{-6}
0.75	2.94×10^{-3}	6.32×10^{-7}	4.71×10^{-10}	1.59×10^{-3}	2.99×10^{-5}
0.85	2.77×10^{-3}	1.28×10^{-6}	1.14×10^{-8}	2.00×10^{-3}	1.31×10^{-5}
0.95	2.39×10^{-3}	5.04×10^{-6}	4.49×10^{-8}	3.40×10^{-7}	1.03×10^{-5}

Table 3. Maximum absolute errors of Example 5.3.

N	M	Maximum absolute error
2	2	5.26×10^{-1}
2	3	5.95×10^{-2}
3	8	4.49×10^{-8}
3	10	2.76×10^{-10}
4	4	1.05×10^{-2}
4	6	5.04×10^{-6}

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Example 5.4. Consider the linear VIE [8]

$$y(x) = (3-x)e^{x} - 2 - x - 4x^{2} + \int_{0}^{x} (x + 6(x-t) - 4(x-t)^{2})y(t)dt, \qquad 0 \le x \le 1,$$

the exact solution to this equation is $y(x) = \exp(x)$. We obtained approximate solution of this equation by Bessel-hybrid functions. The values obtained in Table 4 show that if N and M increase, the accuracy will increase.

Table 4. Absolute errors of Example 5.4.

x	Present method N=3, M=3	Present method N=3, M=8	Present method N=3, M=10	Method of [8] N=2, M=3
0.0	2.15×10^{-3}	1.02×10^{-10}	2.71×10^{-13}	3.48×10^{-2}
0.2	1.15×10^{-3}	1.03×10^{-11}	8.08×10^{-14}	6.78×10^{-3}
0.4	2.46×10^{-3}	2.28×10^{-11}	1.27×10^{-12}	1.79×10^{-2}
0.6	4.84×10^{-3}	1.55×10^{-11}	2.20×10^{-12}	3.57×10^{-2}
0.8	9.48×10^{-3}	1.56×10^{-11}	4.72×10^{-12}	1.98×10^{-2}
1.0	2.04×10^{-3}	5.06×10^{-10}	6.65×10^{-12}	1.06×10^{-2}

Example 5.5. We consider the linear VIE [16]

$$\begin{aligned} y(x) = {} & 2x + 4\sin(x^{2})(\sin(2x)\text{-}\cos(2x)) + (\sin(2x)+\cos(2x))(1+2x\cos(x^{2})) \\ & + \int_{0}^{x} (2x^{2}-8)\sin(xt)y(t)dt, \qquad 0 \le x \le 1, \end{aligned} \tag{35}$$

the exact solution to Eq. (35) is $y(x)=\sin(2x)+\cos(2x)$. The maximum absolute error are given for different values of N and M in Table 5 and we wrote absolute error of this example for (N = 3, M = 8, 10) and (N = 4, M =4, 6) by Bessel-hybrid functions in Table 6.

Table 5. Maximum absolute errors of Example 5.5.

N	M	Maximum absolute error
2	3	6.86×10^{-2}
2	5	1.46×10^{-3}
3	4	5.19×10^{-2}
3	10	3.20×10^{-7}
4	4	4.71×10^{-2}

Table 6. Absolute errors of Example 5.5.

x	Present method N=3, M=8	Present method N=3, M=10	Present method N=4, M=4	Present method N=4, M=6
0.0	5.61×10^{-8}	2.08×10^{-10}	1.70×10^{-3}	1.68×10^{-7}
0.1	3.27×10^{-8}	4.59×10^{-12}	3.53×10^{-5}	1.75×10^{-6}
0.2	9.65×10^{-9}	5.93×10^{-12}	1.25×10^{-4}	4.07×10^{-6}
0.3	9.43×10^{-9}	1.03×10^{-11}	8.87×10^{-5}	4.33×10^{-6}
0.4	2.98×10^{-8}	3.93×10^{-12}	3.83×10^{-4}	5.36×10^{-6}
0.5	5.83×10^{-8}	3.10×10^{-11}	8.87×10^{-4}	6.29×10^{-6}
0.6	5.56×10^{-8}	7.09×10^{-11}	8.17×10^{-4}	8.23×10^{-6}
0.7	7.06×10^{-8}	4.17×10^{-10}	1.22×10^{-3}	6.25×10^{-5}
0.8	1.04×10^{-6}	5.02×10^{-9}	7.49×10^{-3}	1.59×10^{-4}
0.9	6.54×10^{-6}	4.72×10^{-8}	2.12×10^{-2}	5.73×10^{-4}
1.0	2.93×10^{-5}	3.20×10^{-7}	4.71×10^{-2}	1.65×10^{-3}

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6. CONCLUSION

In this paper, we have solved linear FVIDEs by Bessel-hybrid functions of the first kind and collocation points. One significant advantage of this method is that with the increasing values of N and M, approximate solution is convergent and the accuracy is increased sufficiently. Another reason for the increased accuracy of this method, using the transfer matrix from Bessel-hybrid functions to Taylor polynomials and produce sparse matrix. As you have seen, the results of the proposed method are more accurate than the results of Legendre-hybrid, Legendre wavelets and CAS wavelets functions. Also, our comparison with satisfactory results show that the proposed method is efficient.

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