ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2017, VOLUME 7, ISSUE 2, p.110-122

SOME NEW ASSOCIATED CURVES OF AN ADMISSIBLE FRENET CURVE IN 3-DIMENSIONAL AND 4-DIMENSIONAL GALILEAN SPACES

N. MACİT , M.AKBIYIK, S. YÜCE

Abstract. In this paper, we introduce principal direction curve and binormal direction curve of a given Frenet curve by using integral curves of the Frenet vector fields 3-dimensional Galilean space $\mathbb{G}^3$. Besides, we define W-direction curve and W-rectifying curve of a Frenet curve in $\mathbb{G}^3$ by using the unit Darboux vector field W of the Frenet curve and give some characterizations together with the relationships between the curvatures of each associated curve. Then, we classify the curves in $\mathbb{G}^4$ such as $\mathbb{G}^3$ and we introduce slant helix and B_2-slant helix in $\mathbb{G}^4$. In addition to this, some new associated curves of a Frenet curve are defined in $\mathbb{G}^4$.
Mathematics Subject Classification (2010):53A04, 53A35.
Key words: Galilean space, associated curve, integral curve, general helices, slant helices.

Article history:
Received 13 February 2017
Accepted 20 October 2017

1. Introduction

In differential geometry, the theory of curves is one of the main study area. In the theory of curves, helices, slant helices and rectifying curves as in [13], [14], [1], [6], are the most fascinating curves. Besides, associated curves which is called curves that found in a differential and mathematical relationship between two or more curves are widely studied. Among these curves the most studied ones are Bertrand curve couple, Mannheim partner curves, spherical indicatrices and involute-evolute curve couple, as in [2], [3], [10].

Non-Euclidean geometries have an important place in the history of humanity. By constrast with known, it has used also in architecture from far in the past. For instance Hagia Sophia in Istanbul is included elliptic geometry. Structures are prospered with non-Euclidean geometry in modern architecture. Conton tower in China was constructed via hiperbolic geometry when Tote Restaurant in Mumbai was built by fractal geometry. These are some of the example of non-Euclidean works [9].

On the other hand, in many field of science, one can run across with non-Euclidean geometry types. In some new developments on physical science Galilean geometry that is one of the non-Euclidean geometries is in use. In this study, we work on Galilean geometry which has been developed over the last two centuries and some properties of curves and surfaces are more emphasized in currently developed non-Euclidean geometries than in the Euclidean.I. M. Yaglom have explained basics of Galilean geometry in [19]. Differential geometry of the Galilean space $\mathbb{G}^3$ has been largely developed in O. Röschel's paper [18].

The Darboux vector field $\omega = \tau \mathsf{T} + \kappa \mathsf{B}$ which can be interpreted kinematically as a shear along the absolute line in Galilean space has an important place for the space curves in differential geometry, [18].

A. O. Ogrenmis, M. Ergut and M. Bektas have obtained characterizations for a curve to be a helix with respect to the Frenet frame in 3-dimensional Galilean space $\mathbb{G}^3$ in [16]. Moreover, characterizations of slant helix in Galilean and Pseudo-Galilean spaces are studied by H. Oztekin et al. and M. K. Karacan at al. in [4], and [11], respectively. Later, a characterization for position vector of rectifying curves are expressed in $\mathbb{G}^3$ by S. Yilmaz, U. Z. Savci, and A. Magden [20]. Construction of the Frenet-Serret frame of a curve in $\mathbb{G}^4$ are introduced by S.Yılmaz, [21]. Then, researchers have studied some special curves in 4-dimensional Galilean Space such as Inclined Curves, Bertrand Curves, Mannheim curves in [22], [17], [5].

In a recent paper, Choi and Kim introduce principal (binormal)-direction curve, principal (binormal)-donor curve and PD-rectifying curves in $\mathbb{E}^3$. They give handy characterizations for the general and slant helices via their associated curves and give a useful method to obtain general helix and slant helix from a planar curve. Also, they give a new characterization for Bertrand curves by using the PD-rectifying curve, [7]. Later, Choi et al. introduce the notion of the principal (binormal)-direction curve and the principal (binormal)-donor curve of the Frenet curve in the Minkowski space $\mathbb{E}^3_1$,[8], and Körpınar et al. give new associated curves by using Bishop frame in $\mathbb{E}^3$ [12]. Then, N.Macit and M.Düldül, [15]; defined some new associated curves of a Frenet curve in $\mathbb{E}^3$ and $\mathbb{E}^4$. In the light of these studies we introduce some associated curves of a given curve in Galilean 3-space and Galilean 4-space. In this study, we define principal-direction curve, binormal-direction curve, W-direction curve, W-rectifying curve in $\mathbb{G}^3$ and principal-direction curve, B_1-direction curve, B_2-direction curve in $\mathbb{G}^4$.

All these new associated curves are defined as the integral curves of vector fields taken from the Frenet frame along a curve in 3-dimensional Galilean space. Some characterizations of these new curves are also studied.

2. **Preliminaries**

2.1. **3-Dimensional Galilean Geometry.** 3-dimensional Galilean geometry can be described as the study of properties of 3-dimensional space with coordinates that are invariant under general Galilean transformations

$$\begin{aligned} x' &= x + a \\ y' &= v\cos\alpha x + (\cos\varphi)\, y + (\sin\varphi)\, z + b \\ z' &= v\sin\alpha x + (-\sin\varphi)\, y + (\cos\varphi)\, z + d. \end{aligned}$$

Let $\mathbf{a} = (x, y, z)$ and $\mathbf{b} = (x_1, y_1, z_1)$ be vectors in the Galilean space. The scalar product is defined by

$$< \mathbf{a}, \mathbf{b} >_{\mathbb{G}} = \begin{cases} xx_1, & \text{if} \quad x \neq 0 \text{ or } x_1 \neq 0 \\ yy_1 + zz_1, & \text{if} \quad x = x_1 = 0. \end{cases}$$

A vector $\mathbf{a} = (x, y, z)$ is said to be *non-isotropic* if $x \neq 0$. On the otherhand, a vector $\mathbf{a} = (x, y, z)$ is said to be *isotropic* if $x = 0$. All unit non-isotropic vectors and isotropic vectors are of the form $\mathbf{a} = (x, y, z)$ and $\mathbf{p} = (0, y, z)$, respectively. The orthogonality of vectors in Galilean Space, $\mathbf{a} \perp_{\mathbb{G}} \mathbf{b}$, means that

$$< \mathbf{a}, \mathbf{b} >_{\mathbb{G}} = 0.$$

The norm of a non-isotropic vector $\mathbf{a}$ is defined by

$$\| \mathbf{a} \|_{\mathbb{G}} = |x|,$$

and $\mathbf{a}$ is called a unit vector if

$$\| \mathbf{a} \|_{\mathbb{G}} = 1.$$

The norm of an isotropic vector $\mathbf{p}$ defined by

$$\| \mathbf{p} \|_{\mathbb{G}} = \sqrt{y^2 + z^2}$$

and $\mathbf{p}$ is called a unit isotropic vector if

$$\| \mathbf{p} \|_{\mathbb{G}} = 1,$$

If $\mathbf{a} = (x, y, z)$ and $\mathbf{b} = (x_1, y_1, z_1)$ are vectors in Galilean space, we define the vector product of $\mathbf{a}$ and $\mathbf{b}$ as the following:

$$\mathbf{a} \times_{\mathbb{G}} \mathbf{b} = \begin{cases} \begin{vmatrix} \mathbf{0} & \mathbf{e}_2 & \mathbf{e}_3 \\ x & y & z \\ x_1 & y_1 & z_1 \end{vmatrix}, & \text{if } x \neq 0 \text{ or } x_1 \neq 0 \\ \\ \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 0 & y & z \\ 0 & y_1 & z_1 \end{vmatrix}, & \text{if } x = x_1 = 0. \end{cases}$$

[19]. Let α be a spatial curve given by

$$\alpha(t) = (x(t), y(t), z(t))$$

where $x(t), y(t), z(t) \in C^3$(the set of three-times continuously differentiable functions) and t run through a real interval. If $x'(t) = 0$, then curve α is called a *non-admissible* curve. On the other hand, if $x'(t) \neq 0$, then curve α is called an *admissible curve*. A non-admissible curve α is given by the parametrization

$$\alpha(t) = (c, y(t), z(t))$$

where c is a constant, *i.e.*, a non-admissible curve α is on Euclidean Plane $x = c$. Hence,

$$\alpha'(t) = (0, y'(t), z'(t)).$$

If $\|\alpha'(t)\|_{\mathbb{G}} = 1$ then a non-admissible curve α is an isotropic unit speed curve. So, the tangent vector T of α is defined as the isotropic unit vector

$$T(t) = (0, y'(t), z'(t)).$$

the normal vector N of α is defined as the isotropic unit vector

$$N(t) = (0, -z'(t), y'(t)),$$

and binormal vector B of α is defined as the non-isotropic unit vector

$$B(t) = T(t) \times_{\mathbb{G}} N(t).$$

Finally, curvature of a non-admissible curve α

$$\kappa = \langle T', N \rangle_{\mathbb{G}} = z''(t)y'(t) - y''(t)z'(t),$$

and torsion of a non-admissible curve α

$$\tau = \langle N', B \rangle_{\mathbb{G}} = 0.$$

Frenet formulas of a non-admissible curve can be written as:

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

Let an admissible curve α parameterized by arclength s be

$$\alpha(s) = (s, y(s), z(s)).$$

So, the associated invariant moving trihedron of an admissible curve α is given by

$$\begin{aligned} T(s) &= (1, y'(s), z'(s)) \\ N(s) &= \frac{1}{\kappa(s)} (0, y''(s), z''(s)) \\ B(s) &= \frac{1}{\kappa(s)} (0, -z''(s), y''(s)) \end{aligned}$$

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2017, VOLUME 7, ISSUE 2, p.110-122

where curvature is given by

$$\kappa(s)=\sqrt{y''(s)^2+z''(s)^2}$$

and torsion is obtained as

$$\tau(s)=\frac{1}{\kappa^2(s)}\det\left(\alpha'(s),\alpha''(s),\alpha'''(s)\right).$$

So, Frenet derivative formulas can be written as:

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix}=\begin{bmatrix} 0 & \kappa & 0 \\ 0 & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}\begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

[18].

Theorem 2.1. *Let γ be a curve in 3-dimensional Galilean space $\mathbb{G}^3$, and $\{T,N,B\}$ be the Frenet frame in 3-dimensional Galilean space $\mathbb{G}^3$ along γ. A curve γ such that*

$$\frac{\kappa}{\tau}=constant$$

is called a general helix (where κ and τ are curvature and torsion of γ, respectively) , [16].

A curve γ is called a *slant helix* if there exists a constant vector field $\mathbf{u}$ in $\mathbb{G}^3$ such that the function $\langle N(s),\mathbf{u}\rangle_{\mathbb{G}}$ is constant, [4], [11].

Theorem 2.2. *Let γ be a curve parameterized by the arc length s in $\mathbb{G}^3$ .Then γ is a slant helix if and only if the function*

$$\frac{\kappa^2}{\tau^3}\left(\frac{\tau}{\kappa}\right)'$$

is constant everywhere τ does not vanish, [4], [11].

Let γ be a curve in $\mathbb{G}^3$. γ is called a *rectifying curve* if the position vector of γ always lies in its rectifying plane, [20]. For an admissible Frenet curve $\gamma: I\subset\mathbb{R}\to\mathbb{G}^3$ with the Frenet frame $\{T,N,B\}$, consider a vector field V given by

$$V(s)=u(s)T(s)+v(s)N(s)+w(s)B(s),$$

where u,v,w are functions on I. If $u(s)\neq 0$ then, an integral curve $\overline{\gamma}(s)$ of V defined on I satisfying $u^2(s)=1$ is a unit speed admissible curve in $\mathbb{G}^3$. If $u(s)=0$ then an integral curve $\overline{\gamma}(s)$ of V defined on I satisfying $\sqrt{v^2(s)+w^2(s)}=1$ is a unit speed non-admissible curve in $\mathbb{G}^3$. Also, the arc-length parameter $\overline{s}$ of an integral curve $\overline{\gamma}$ of $V(s)$ is obtained as $\overline{s}=s+c$ for some constant c. Thus, without loss of generality, one can assume $\overline{s}=s$. The integral curve $\overline{\gamma}$ is unique up to translation of $\mathbb{G}^3$. In fact, $\overline{\gamma}$ is determined by the initial point.

2.2. **4-Dimensional Galilean Geometry.** Now, let's talk about some 4-dimensional Galilean Geometry $\mathbb{G}^4$. Four-dimensional Galilean geometry can be defined as the study of properties of four-dimensional space with coordinates that are invariant under general Galilean transformations

$$\begin{aligned}
x' &= (\cos\beta\cos\alpha-\cos\gamma\sin\beta\sin\alpha)\,x-(\sin\beta\cos\alpha-\cos\gamma\sin\beta\sin\alpha)\,y \\
&\quad+(\sin\gamma\sin\alpha)\,z+(v\cos\delta_1)\,t+a \\
y' &= (\cos\beta\sin\alpha+\cos\gamma\sin\beta\cos\alpha)\,x+(-\sin\beta\sin\alpha-\cos\gamma\cos\beta\cos\alpha)\,y \\
&\quad+(\sin\gamma\cos\alpha)\,z+(v\cos\delta_2)\,t+b \\
z' &= (\sin\gamma\sin\beta)\,x-(\sin\gamma\cos\beta)\,y+(\cos\gamma)\,z+(v\cos\delta_3)\,t+c \\
t' &= t+d
\end{aligned}$$

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2017, VOLUME 7, ISSUE 2, p.110-122

where $\cos^2\delta_1+\cos^2\delta_2+\cos^2\delta_3+\cos^2\delta_4=1$, [19]. Let $\mathbf{a}=(x,y,z,w)$ and $\mathbf{b}=(x_1,y_1,z_1,w_1)$ be vectors in the 4-dimensional Galilean space $\mathbb{G}^4$. The scalar product in the 4-dimensional Galilean space $\mathbb{G}^4$ is defined by

$$<\mathbf{a},\mathbf{b}>_{\mathbb{G}}=\begin{cases} xx_1, & \text{if} \quad x\neq 0 \text{ or } x_1\neq 0\\ yy_1+zz_1+ww_1, & \text{if} \quad x=x_1=0.\end{cases}$$

A vector $\mathbf{a}=(x,y,z,w)$ is said to be *non-isotropic* if $x\neq 0$. Otherwise, a vector $\mathbf{a}=(x,y,z,w)$ is said to be *isotropic* if $x=0$. All unit non-isotropic vectors and isotropic vectors are of the form $\mathbf{a}=(x,y,z,w)$ and $\mathbf{p}=(0,y,z,w)$, respectively. The orthogonality of vectors in Galilean Space, $\mathbf{a}\perp_{\mathbb{G}}\mathbf{b}$, means that

$$<\mathbf{a},\mathbf{b}>_{\mathbb{G}}=0.$$

The norm of a non-isotropic vector $\mathbf{a}$ defined by

$$\|\mathbf{a}\|_{\mathbb{G}}=|x|,$$

and $\mathbf{a}$ is called a unit vector if

$$\|\mathbf{a}\|_{\mathbb{G}}=1.$$

The norm of an isotropic vector $\mathbf{p}$ is defined by

$$\|\mathbf{p}\|_{\mathbb{G}}=\sqrt{y^2+z^2+w^2}$$

and $\mathbf{p}$ is called an unit isotropic vector if

$$\|\mathbf{p}\|_{\mathbb{G}}=1.$$

If $\mathbf{a}=(x,y,z,w)$, $\mathbf{b}=(x_1,y_1,z_1,w_1)$ and $\mathbf{c}=(x_2,y_2,z_2,w_2)$ are vectors in the Galilean space $\mathbb{G}^4$, we introduce the vector product of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ as the following:

$$\mathbf{a}\times_{\mathbb{G}}\mathbf{b}\times_{\mathbb{G}}\mathbf{c}=\begin{cases}\begin{vmatrix}\mathbf{0} & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4\\ x & y & z & w\\ x_1 & y_1 & z_1 & w_1\\ x_2 & y_2 & z_2 & z_2\end{vmatrix}, & \text{if} \quad x\neq 0 \text{ or } x_1\neq 0 \text{ or } x_2\neq 0\\ \\ \begin{vmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4\\ 0 & y & z & w\\ 0 & y_1 & z_1 & w_1\\ 0 & y_2 & z_2 & z_2\end{vmatrix}, & \text{if } x=x_1=x_2=0.\end{cases}$$

Let α be a curve in $\mathbb{G}^4$ given by

$$\alpha(t)=(x(t),y(t),z(t),w(t))$$

where $x(t),y(t),z(t),w(t)\in C^4$(the set of four-times continuously differentiable functions) and t run through a real interval. If $x'(t)=0$, then curve α is called a *non-admissible*. If $x'(t)\neq 0$, then curve α is called an *admissible curve*. A non-admissible curve α is given by the parametrization

$$\alpha(t)=(c,y(t),z(t),w(t))$$

where c is a constant, *i.e.*, a non-admissible curve α is on $3-$dimensional Euclidean Space $x=c$. Hence,

$$\alpha'(t)=(0,y'(t),z'(t),w'(t)).$$

If $\|\alpha'(t)\|_{\mathbb{G}}=1$ then a non-admissible curve α is a curve with an isotropic unit velocity. So, the tangent vector T of α is defined as the isotropic unit vector

$$T(t)=\alpha'(t),$$

the normal vector N of α is defined as the isotropic unit vector

$$N(t)=\frac{\alpha''(t)}{\|\alpha''(t)\|_{\mathbb{G}}},$$

the second binormal vector B_2 of α is defined as the non-isotropic unit vector

$$B_2(t)= -\frac{\alpha'(t) \times_{\mathbb{G}} \alpha''(t) \times_{\mathbb{G}} \alpha'''(t)}{\|\alpha'(t) \times_{\mathbb{G}} \alpha''(t) \times_{\mathbb{G}} \alpha'''(t)\|_{\mathbb{G}}}$$

and the first binormal vector B_1 of α is defined as the isotropic unit vector

$$B_1(t)=B_2 \times_{\mathbb{G}} T \times_{\mathbb{G}} N.$$

Finally, first curvature of a non-admissible curve α obtained as

$$k_1 = \langle T', N \rangle_{\mathbb{G}},$$

second curvature of a non-admissible curve α obtained as

$$k_2 = \langle N', B_1 \rangle_{\mathbb{G}},$$

and third-curvature of a non-admissible curve α obtained as

$$k_3 = 0.$$

Frenet formulas of a non-admissible curve can be written as:

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ k_1 & 0 & k_2 & 0 \\ 0 & -k_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}$$

Let an admissible curve α parametrized by the arc length s be

$$\alpha(s) = (s, y(s), z(s), w(s)).$$

The associated invariant moving tetrahedron of an admissible curve is given by

$$\begin{aligned} T(s) &= (1, y'(s), z'(s), w'(s)) \\ N(s) &= \frac{1}{k_1(s)} (0, y''(s), z''(s), w''(s)) \\ B_1(s) &= \frac{1}{k_2(s)} \left(0, \left(\frac{y''(s)}{k_1(s)}\right)', \left(\frac{z''(s)}{k_1(s)}\right)', \left(\frac{w''(s)}{k_1(s)}\right)'\right) \\ B_2(s) &= \mu T \times_{\mathbb{G}} N \times_{\mathbb{G}} B_1 \end{aligned}$$

where $\mu = \pm 1$,

$$k_1 = \|T'\|_{\mathbb{G}}$$

is first curvature,

$$k_2 = \|N'\|_{\mathbb{G}}$$

is second curvature, and third curvature is

$$k_3 = \langle B_1', B_2 \rangle_{\mathbb{G}}.$$

Frenet formulas can be written as:

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ 0 & 0 & k_2 & 0 \\ 0 & -k_2 & 0 & k_3 \\ 0 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

[21]. Let γ be a curve in 4-dimensional Galilean space $\mathbb{G}^4$ and $\{T, N, B_1, B_2\}$ be the Frenet frame along γ in 4-dimensional Galilean space $\mathbb{G}^4$. A curve γ is called a *general helix* if there exists a constant vector field $\mathbf{u}$ in $\mathbb{G}^4$ such that the function $\langle T(s), \mathbf{u} \rangle_{\mathbb{G}}$ is constant, [22].

Definition 2.3. *A curve γ is called a slant helix if there exists a constant vector field $\mathbf{u}$ in $\mathbb{G}^4$ such that the function $\langle N(s), \mathbf{u} \rangle_{\mathbb{G}}$ is constant.*

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2017, VOLUME 7, ISSUE 2, p.110-122

Definition 2.4. *A curve γ is called a B_2-slant helix if there exists a constant vector field* $\mathbf{u}$ *in* $\mathbb{G}^4$ *such that the function* $\langle B_2(s), \mathbf{u}\rangle_{\mathbb{G}}$ *is constant.*

For an admissible Frenet curve $\gamma : I \subset \mathbb{R} \to \mathbb{G}^4$ with the Frenet frame $\{T, N, B_1, B_2\}$, consider a vector field V given by

$$V(s) = u(s)T(s) + v(s)N(s) + w(s)B_1(s) + z(s)B_2(s),$$

where u, v, w, z are functions on I. If $u(s) \neq 0$, an integral curve $\overline{\gamma}(s)$ of V defined on I satisfying $u^2(s) = 1$ is a unit speed admissible curve in $\mathbb{G}^4$. If $u(s) = 0$, an integral curve $\overline{\gamma}(s)$ of V defined on I satisfying $\sqrt{v^2(s) + w^2(s) + z^2(s)} = 1$ is a unit speed non-admissible curve in $\mathbb{G}^4$. Besides, the arc-length parameter $\overline{s}$ of an integral curve $\overline{\gamma}$ of $V(s)$ is obtained as $\overline{s} = s + c$ for some constant c. Thus, without loss of generality, one can assume $\overline{s} = s$. The integral curve $\overline{\gamma}$ is unique up to translation of $\mathbb{G}^4$. In fact, $\overline{\gamma}$ is determined by the initial point.

3. Associated curves of a Frenet curve in $\mathbb{G}^3$

In this section, we define principal-direction curve and binormal-direction curve in $\mathbb{G}^3$.

Definition 3.1. *Let γ be an admissible Frenet curve and $\{T,N,B\}$ be its Frenet frame in $\mathbb{G}^3$.*
An integral curve of the principal normal vector field of γ is called the principal-direction curve of γ.
An integral curve of the binormal vector field of γ is called the binormal-direction curve of γ.

Theorem 3.2. *Let γ be an admissible Frenet curve $\mathbb{G}^3$ whose curvatures are κ, τ and $\overline{\gamma}$ be the principal-direction curve of γ. The curvature of $\overline{\gamma}$ is given*

$$\overline{\kappa}(s) = \tau(s).$$

Proof *Let $\left\{\overline{T}, \overline{N}, \overline{B}, \overline{\kappa}, \overline{\tau}\right\}$ be the Frenet apparatus of $\overline{\gamma}$. By the definition of the principal direction curve, we may write*

$$N(s)|_{\overline{\gamma}(s)} = \overline{\gamma}'(s).$$

Then,

$$\overline{T}(s) = \left(0, \frac{y''(s)}{\kappa(s)}, \frac{z''(s)}{\kappa(s)}\right)$$

Hence

$$\overline{N}(s) = \left(0, \frac{-z''(s)}{\kappa(s)}, \frac{y''(s)}{\kappa(s)}\right) = B(s).$$

Then, the curvature of $\overline{\gamma}$ is given by

$$\overline{\kappa}(s) = \left\langle \overline{T}'(s), \overline{N}(s)\right\rangle_{\mathbb{G}} = \tau(s).$$

Corollary 3.3. *If γ is a plane curve then principal-direction curve of γ is a straight line.*

Theorem 3.4. *Let γ be an admissible Frenet curve $\mathbb{G}^3$ whose curvatures are κ, τ and $\widehat{\gamma}$ be the binormal-direction curve of γ. The curvatures of $\widehat{\gamma}$ are given*

$$\widehat{\kappa}(s) = \tau(s).$$

Proof *Let $\left\{\widehat{T}, \widehat{N}, \widehat{B}, \widehat{\kappa}, \widehat{\tau}\right\}$ be the Frenet apparatus of $\widehat{\gamma}$. By the definition of the binormal-direction curve, we may write*

$$B(s)|_{\widehat{\gamma}(s)} = \widehat{\gamma}'(s)$$

Hence,

$$\widehat{T}(s) = \left(0, \frac{-z''(s)}{\kappa(s)}, \frac{y''(s)}{\kappa(s)}\right)$$

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2017, VOLUME 7, ISSUE 2, p.110-122

So,

$$\widehat{N}(s)=\left(0,\frac{-y''(s)}{\kappa(s)},\frac{-z''(s)}{\kappa(s)}\right)=-N(s)$$

Then, the curvature of $\widehat{\gamma}$ is given by

$$\widehat{\kappa}(s)=\left\langle \widehat{T}'(s),\widehat{N}(s)\right\rangle_{\mathbb{G}}=\tau(s).$$

Corollary 3.5. *If γ is a plane curve then binormal-direction curve of γ is a straight line.*

Corollary 3.6. *If γ is an admissible curve, easily seen that the principal direction curve and the binormal direction curve of γ is a non-admissible curve.*

4. W-direction curves in $\mathbb{G}^3$

In this section we introduce W-direction curve, second W-direction curve and W-rectifying curve in $\mathbb{G}^3$ and give some characterizations. It is obvious that if γ is an admissible curve then W-direction curve of γ is also an admissible curve.

Definition 4.1 (W-direction curves)**.** *Let γ be an admissible Frenet curve in $\mathbb{G}^3$ and W be the unit Darboux vector field of γ. We call an integral curve of $W(s)$ as* W-direction curve *of γ.Namely, if $\overline{\gamma}(s)$ is W-direction curve of γ, then $W(s)=\overline{\gamma}'(s)$, where $W=\frac{1}{|\tau|}(\tau\mathsf{T}+\kappa\mathsf{B})$.*

Theorem 4.2. *Let $\overline{\gamma}$ be the W-direction curve of a nonplanar admissible curve γ. Then, γ is a general helix if and only if $\overline{\gamma}$ is a straight line.*
Proof ($\Rightarrow$) *Let γ be a general helix. Then $\frac{\kappa}{\tau}=c(constant)$. Since $\overline{\gamma}$ is the W-direction curve of γ, we have*

$$\overline{\gamma}'(s)=W(s)=\frac{1}{|\tau|}(\tau\mathsf{T}+\kappa\mathsf{B}).$$

Differentiating gives $\overline{\gamma}''(s)=\mathbf{0}$, i.e. $\overline{\kappa}=0$. Thus, $\overline{\gamma}$ is a straight line.

($\Leftarrow$) *Let $\overline{\gamma}$ be a straight line. Then the velocity $\overline{\gamma}'(s)=W(s)$ is constant. Hence,*

$$\overline{\gamma}''(s)=W'(s)=\left(\frac{1}{|\tau|}(\tau\mathsf{T}+\kappa\mathsf{B})\right)'=\mathbf{0}.$$

Since $\kappa\neq0$ and $\tau\neq0$, we obtain $\frac{\kappa}{|\tau|}=0$, i.e. $\frac{\kappa}{\tau}=constant$. This means γ is a general helix.

Theorem 4.3. *Let γ be an admissible Frenet curve in $\mathbb{G}^3$ with the curvature κ and the torsion τ, and $\overline{\gamma}$ be W-direction curve of γ. If γ is not a general helix, then the curvature $\overline{\kappa}$ and the torsion $\overline{\tau}$ of $\overline{\gamma}$ are given by*

$$\overline{\kappa}=\frac{\kappa'\tau-\kappa\tau'}{\tau^2},\quad \overline{\tau}=|\,\tau\,|.$$

Proof *We can use the same arc-lenght parameter s for γ and $\overline{\gamma}$. By the definition of W-direction curve, we have $W(s)=\overline{\gamma}'(s)=\overline{\mathsf{T}}(s)$. Then, we have*

$$\overline{\mathsf{T}}=\frac{1}{|\tau|}(\tau\mathsf{T}+\kappa\mathsf{B})$$

and the curvature of $\overline{\gamma}$ is given by

$$\overline{\kappa}=\|\,\overline{\mathsf{T}}'\,\|=\|_{\mathbb{G}}\left(\frac{\kappa}{\tau}\right)'B\,\|_{\mathbb{G}}=\left(\frac{\kappa}{\tau}\right)'$$

or

$$\overline{\kappa}=\frac{\tau\kappa'-\tau'\kappa}{\tau^2}.$$

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2017, VOLUME 7, ISSUE 2, p.110-122

The torsion $\overline{\tau}$ *of* $\overline{\gamma}$ *is calculated by*

$$\overline{\tau} = \frac{det(\overline{\gamma}', \overline{\gamma}'', \overline{\gamma}''')}{\overline{\kappa}^2}.$$

Because of fact that

$$\begin{aligned} \overline{\gamma}'(s) &= \overline{\mathsf{T}}(s) = W(s) = \frac{1}{|\tau|}(\tau\mathsf{T} + \kappa\mathsf{B}) \\ \overline{\gamma}''(s) &= W'(s) = \left(\frac{\kappa}{\tau}\right)'\mathsf{B} \end{aligned}$$

and

$$\overline{\gamma}'''(s) = W''(s) = -\tau\left(\frac{\kappa}{\tau}\right)'\mathsf{N} + \left(\frac{\kappa}{\tau}\right)''\mathsf{B},$$

we get $\overline{\tau} =| \tau |$.

Theorem 4.4. *Let* $\overline{\gamma}$ *be the* W*-direction curve of* γ *which is not a general helix. Then,* $\overline{\gamma}$ *is a general helix if and only if* γ *is a slant helix.*

Proof ($\Rightarrow$) *Let* $\overline{\gamma}$ *be a general helix. Then, we have* $\frac{\overline{\tau}}{\overline{\kappa}} = c(constant)$*. Using Theorem 6, we find*

$$\frac{\overline{\tau}}{\overline{\kappa}} = \frac{|\tau|}{\left(\frac{\kappa}{\tau}\right)'} = \frac{\tau^3}{\tau\kappa' - \tau'\kappa} = c \quad \Rightarrow \quad \frac{\kappa^2}{\tau^3}\left(\frac{\tau}{\kappa}\right)' = \frac{1}{c}(constant).$$

This means that γ *is a slant helix.*

($\Leftarrow$) *Let* γ *be a slant helix. In this case, from Theorem 2 we have* $\frac{\kappa^2}{\tau^3}\left(\frac{\tau}{\kappa}\right)' = c$ *or* $\frac{\tau^3}{\tau'\kappa - \tau\kappa'} = \frac{1}{c}(constant)$*, that is,* $\frac{\overline{\tau}}{\overline{\kappa}} = constant$*. This means that* $\overline{\gamma}$ *is a general helix.*

Definition 4.5 (Second W-direction curve)**.** *Let* $\overline{\gamma}$ *be* W*-direction curve of* γ *and* $\overline{\overline{\gamma}}$ *be* W*-direction curve of* $\overline{\gamma}$ *in* $\mathbb{G}^3$*. In this case we call* $\overline{\overline{\gamma}}$ *as second* W*-direction curve of* γ*.*

Corollary 4.6. *If* γ *is a slant helix, then the second* W*-direction curve of* γ *is a straight line.*

Definition 4.7 (W-rectifying curve)**.** *Let* γ *be an admissible Frenet curve and* $\overline{\gamma}$ *be its* W*-direction curve. The curve* $\overline{\gamma}$ *is called* W*-rectifying curve if the position vector of* $\overline{\gamma}$ *always lies in rectifying plane of* γ*.*

Theorem 4.8. *Let* γ *be an admissible Frenet curve and* $\overline{\gamma}$ *be its* W*-direction curve. If* $\overline{\gamma}$ *is a* W*-rectifying curve, then* γ *is a general helix.*

Proof *Using the definition of* W*-rectifying curve, we can write*

(4.1) $$\overline{\gamma} = \lambda(s)T(s) + \mu(s)B(s),$$

where $\lambda(s)$ *and* $\mu(s)$ *are non-zero functions and* $\{T, N, B\}$ *is the Frenet frame along* γ*. By differentiating this equation we get*

(4.2) $$\overline{T} = \lambda'T + (\lambda\kappa - \mu\tau)N + \mu'B.$$

On the other hand, we also have $W = \overline{\gamma}' = \overline{T}$*. So, from 4.2 we obtain*

$$\frac{1}{|\tau|}(\tau T + \kappa B) = \lambda'T + (\lambda\kappa - \mu\tau)N + \mu'B$$

or

$$\begin{cases} \lambda\kappa - \mu\tau = 0, \\ \lambda' = \pm 1, \\ \mu' = \frac{\kappa}{|\tau|}. \end{cases}$$

Using these equations we obtain $\lambda'\mu - \lambda\mu' = 0$*. It means* $\frac{\lambda}{\mu} = c(constant)$*. Then* $\frac{\lambda}{\mu} = \frac{\tau}{\kappa} = c$*, i.e.* γ *is a general helix.*

Example 4.9. *The W-direction curve of the circular helix* $\gamma(s) = (s, -3\cos(s), 3\sin(s))$ *is*

$$\overline{\gamma}(s) = (-s + c_1, c_2, c_3), \quad c_1, c_2, c_3 = constants$$

which is a straight line.

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2017, VOLUME 7, ISSUE 2, p.110-122

We obtain tangent vector field

$$T(s) = (1, 3\sin(s), 3\cos(s)),$$

we get curvature as

$$\kappa(s) = \|\gamma''(s)\|_{\mathbb{G}} = 3.$$

$$N(s) = \frac{1}{\kappa(s)}\gamma''(s) = (0, \cos(s), -\sin(s)),$$

we obtain torsion as $\tau(s) = -1$.

$$B(s) = (0, \sin(s), \cos(s)),$$

Hence W-direction curve is obtained as

$$\overline{\gamma}(s) = (-s + c_1, c_2, c_3), \quad c_1, c_2, c_3 = constants.$$

5. Associated curves of a Frenet curve in $\mathbb{G}^4$

In this section, we define new associated curves in $\mathbb{G}^4$.

Definition 5.1. *Let γ be an admissible Frenet curve and $\{T, N, B_1, B_2\}$ be its Frenet frame in $\mathbb{G}^4$. An integral curve of the principal normal vector field of γ is called the principal-direction curve of γ. An integral curve of the first binormal vector field of γ is called the B_1- direction curve of γ. An integral curve of the second binormal vector field of γ the B_2- direction curve of γ.*

Theorem 5.2. *Let γ be an admissible Frenet curve whose curvatures are k_1, k_2, k_3 and $\overline{\gamma}$ be the principal-direction curve of γ. The curvatures of $\overline{\gamma}$ are given*

$$\begin{aligned}\overline{k}_1(s) &= k_2(s), \\ \overline{k}_2(s) &= sgn(k_3(s))\, k_3(s).\end{aligned}$$

Proof *Let $\{\overline{T}, \overline{N}, \overline{B}_1, \overline{B}_2, \overline{k}_1, \overline{k}_2, \overline{k}_3\}$ be the Frenet apparatus of $\overline{\gamma}$. By the definition of the principal direction curve, we may write*

$$N(s)|_{\overline{\gamma}(s)} = \overline{\gamma}'(s) = \overline{T}(s).$$

Then,

$$\overline{N}(s) = \frac{\overline{\gamma}''(s)}{\|\overline{\gamma}''(s)\|_{\mathbb{G}}} = \frac{(k_2(s)B_1)}{|k_2(s)|} = B_1(s)$$

$$\begin{aligned}\overline{B}_2(s) &= \frac{\overline{\gamma}'(s)\times_{\mathbb{G}}\overline{\gamma}''(s)\times_{\mathbb{G}}\overline{\gamma}'''(s)}{\left\|\overline{\gamma}'(s)\times_{\mathbb{G}}\overline{\gamma}''(s)\times_{\mathbb{G}}\overline{\gamma}'''(s)\right\|_{\mathbb{G}}} \\ &= \frac{\left(k_2^2(s)\, k_3(s)\, T(s)\right)}{|k_2^2(s)\, k_3(s)|} = sgn(k_3(s))\, T(s)\end{aligned}$$

and finally,

$$\overline{B}_1(s) = \overline{B}_2(s)\times_{\mathbb{G}}\overline{T}(s)\times_{\mathbb{G}}\overline{N}(s) = sgn(k_3(s))\, B_2(s)$$

Then, the first curvature of $\overline{\gamma}$ is given by

$$\overline{k}_1(s) = \left\langle \overline{T}', \overline{N}\right\rangle_{\mathbb{G}} = k_2(s).$$

and the second curvature of $\overline{\gamma}$ is given by

$$\overline{k}_2(s) = \left\langle \overline{N}', \overline{B}_1\right\rangle_{\mathbb{G}} = \langle B_1'(s), sgn(k_3(s))\, B_2(s)\rangle = sgn(k_3(s))\, k_3(s).$$

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2017, VOLUME 7, ISSUE 2, p.110-122

Theorem 5.3. *Let γ be an admissible Frenet curve whose curvatures are k_1, k_2, k_3 and $\widehat{\gamma}$ be the B_1-direction curve of γ. The curvatures of $\widehat{\gamma}$ are given*

$$\begin{aligned} \widehat{k}_1(s) &= \sqrt{{k_2}^2+{k_3}^2}, \\ \widehat{k}_2(s) &= \frac{{k_2}'k_3-{k_3}'k_2+{k_3}^2k_2}{{k_2}^2+{k_3}^2}. \end{aligned}$$

Theorem 5.4. *Let γ be an admissible Frenet curve whose curvatures are k_1, k_2, k_3 and $\widetilde{\gamma}$ be the B_2-direction curve of γ. The curvatures of $\widetilde{\gamma}$ are given*

$$\begin{aligned} \widetilde{k}_1(s) &= sgn\left(k_3\left(s\right)\right)k_3\left(s\right), \\ \widetilde{k}_2(s) &= k_2\left(s\right). \end{aligned}$$

Proof *Let $\left\{\widetilde{T},\widetilde{N},\widetilde{B}_1,\widetilde{B}_2,\widetilde{k}_1,\widetilde{k}_2,\widetilde{k}_3\right\}$ be the Frenet apparatus of $\widetilde{\gamma}$. By the definition of the principal direction curve, we may write*

$$B_2\left(s\right)|_{\widetilde{\gamma}'(s)} = \widetilde{\gamma}'\left(s\right) = \widetilde{T}\left(s\right).$$

If we use the definition of Frenet vector fields, then we get

$$\begin{aligned} \widetilde{N}\left(s\right) &= -sgn\left(k_3\left(s\right)\right)B_1\left(s\right) \\ \widetilde{B}_2(s) &= -T\left(s\right) \\ \widetilde{B}_1(s) &= sgn\left(k_3\left(s\right)\right)N\left(s\right). \end{aligned}$$

So, the first curvature of $\widetilde{\gamma}$ is given by

$$\widetilde{k}_1(s) = sgn\left(k_3\left(s\right)\right)k_3\left(s\right)$$

and the second curvature of $\widetilde{\gamma}$ is given by

$$\widetilde{k}_2(s) = k_2\left(s\right).$$

Theorem 5.5. *Let γ be an admissible Frenet curve in $\mathbb{G}^4$ and $\overline{\gamma}$ be the principal-direction curve of γ. Then γ is a slant helix if and only if $\overline{\gamma}$ is a general helix.*

Proof *Let $\{T,N,B_1,B_2\}$ denotes the Frenet frame of γ. By the definition of the principal-direction curve, we have*

$$N\left(s\right) = \overline{\gamma}'\left(s\right) = \overline{T}\left(s\right).$$

Hence,

$$\begin{aligned} \gamma \text{ is a slant helix} &\iff \langle N,\mathbf{u}\rangle_{\mathbb{G}} = c \quad \text{where } \mathbf{u} \text{ is a constant vector and } c = \text{constant} \\ &\iff \left\langle \overline{T},\mathbf{u}\right\rangle_{\mathbb{G}} = c \quad \text{where } \mathbf{u} \text{ is a constant vector and } c = \text{constant} \\ &\iff \overline{\gamma} \text{ is a general helix.} \end{aligned}$$

Theorem 5.6. *Let γ be an admissible Frenet curve in $\mathbb{G}^4$ and $\widetilde{\gamma}$ be the B_2-direction curve of γ. Then γ is a B_2-slant helix if and only if $\widetilde{\gamma}$ is a general helix.*

Proof *Let $\{T,N,B_1,B_2\}$ denotes the Frenet frame of γ. By the definition of the B_2-direction curve, we have*

$$\widetilde{T}\left(s\right) = \widetilde{\gamma}'\left(s\right) = B_2\left(s\right)$$

Hence,

$$\begin{aligned} \gamma \text{ is a } B_2-\text{slant helix} &\iff \langle B_2,\mathbf{v}\rangle_{\mathbb{G}} = c \quad \text{where } \mathbf{v} \text{ is a constant vector and } c = \text{constant} \\ &\iff \left\langle \widetilde{T},\mathbf{v}\right\rangle_{\mathbb{G}} = c \quad \text{where } \mathbf{v} \text{ is a constant vector and } c = \text{constant} \\ &\iff \widetilde{\gamma} \text{ is a general helix.} \end{aligned}$$

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2017, VOLUME 7, ISSUE 2, p.110-122

Conclusion As a result,firstly we introduce the notion of the principal direction curve and binormal direction curve in $\mathbb{G}^3$. We obtain that if γ is a planar curve then both its principal-direction and binormal direction curve is a straight line. Secondly, we give the notion of the $W-$ direction curve in $\mathbb{G}^3$. Also, we see that if γ is a slant helix then $W-$direction curve of γ is a general helix, besides if γ is a general helix then $W-$ direction curve of γ is a straight line. In addition, by giving $W-$ rectifying curve, we obtain that $W-$rectifying curves are associated curves of general helices in $\mathbb{G}^3$.

Moreover, we classify the curves in $\mathbb{G}^4$ such as $\mathbb{G}^3$,then we define slant helix and B_2-slant helix in $\mathbb{G}^4$. After defining principal-direction, B_1- direction and B_2-direction curve in $\mathbb{G}^4$ we obtain the curvatures of this associated curves in terms of the main curve's curvatures. Finally, we get that if γ is a Frenet curve in $\mathbb{G}^4$, $\widetilde{\gamma}$ is the principal direction curve of γ and γ is a slant helix then $\widetilde{\gamma}$ is a general helix and also if $\widetilde{\gamma}$ is the B_2- direction curve of γ and γ is a B_2-slant helix, then $\widetilde{\gamma}$ is a general helix. So this conclusions and characterizations give us a useful method to investigate some curves by the help of other curves.

References

[1] Ali A.T. Position vectors of slant helices in Euclidean Space. *Journal of the Egyptian Mathematical Society*, 20: 1–6, 2012.

[2] Ali A.T. New special curves and their spherical indicatrices. *Global Journal of Advanced Research on Classical and Modern Geometries*, 1: 28–38, 2012.

[3] Babaarslan M., Tandoğan Y.A., Yaylı Y. A note on Bertrand curves and constant slope surfaces according to Darboux frame. *Journal of Advanced Mathematical Studies* ,5: 87–96, 2012.

[4] Balgetir Öztekin, H. & Tatlipinar, S. On some curves in Galilean plane and 3-Dimensional Galilean space. *Journal of Dynamical Systems and Geometric Theories*, 10(2): 189–196, 2012.

[5] Bektaş, M., Ergüt, M., Ogrenmiş, A. Special curves of 4D Galilean space. *International Journal of Mathematical Engineering and Science* ,2(3): Issn:2277–6982, 2013.

[6] Chen B.Y. When does the position vector of a space curve always lie in its Rectifying plane?. *American Mathematical Monthly*, 110: 147–152, 2003.

[7] Choi J.H., Kim Y.H. Associated curves of a Frenet curve and their applications. *Applied Mathematics and Computation* 218: 9116–9124, 2012.

[8] Choi J.H., Kim Y.H., Ali A.T. Some associated curves of Frenet non-lightlike curves in E_1^3. *Journal of Mathematical Analysis and Applications*,394: 712–723, 2012.

[9] Gawell, E. Non-Euclidean geometry in the modeling of contemporary architectural forms. *The Journal of Polish Society for Geometry and Engineering Graphics*, 24: 35 – 43, 2013.

[10] Izumiya S., Takeuchi N. Generic properties of helices and Bertrand curves. *Journal of Geometry* 74: 97–109, 2002.

[11] Karacan M.K. and Tuncer Y. Characterization of Slant helix in Galilean and Pseudo Galilean space. *SAÜ Fen Edebiyat Fak. Dergisi*, 1: 43–53, 2010.

[12] Körpınar T., Sarıaydın M.T., Turhan E. Associated curves according to Bishop frame in Euclidean 3-space. *Advanced Modelling and Optimization*, 15: 713–717, 2013.

[13] Kula L., Yaylı Y. On slant helix and its spherical indicatrix. *Applied Mathematics and Computation*, 169: 600–607, 2005.

[14] Kula L., Ekmekci N., Yaylı Y, İlarslan K. Characterizations of slant helices in Euclidean 3-Space, *Turkish Journal of Mathematics*, 34: 261–273, 2010.

[15] Macit, N., Düldül, M. Some new associated curves of a Frenet curve in E^3 and E^4. *Turkish Journal of Mathematics*, 38: 1023 – 1037, 2014.

[16] Ogrenmiş, A., Ergüt, M., Bektaş, M., On the helices in the Galilean Space G_3. *Iranian Journal of Science Technology*, Transaction A, 31: A2, 2007.

[17] Öztekin, H. Special Bertrand curves in 4D Galilean space. *Mathematical Problems in Engineering*, ID 318458, 7 pages, 2014.

[18] Röschel,O. *Die Geometrie des Galileischen Raumes*, Habilitationsschrift, Institut für Math. und Angew. Geometrie, Leoben, 1984.

[19] Yaglom, I.M. *A simple non-Euclidean Geometry and its Physical Basis*, Springer-Verlag, New York, 1979.

[20] Yılmaz, S., Savci, Ü., Z., and Magden, A. Position vector of some special curves in Galilean 3-Space G^3. *Global Journal of Advanced Research on Classical and Modern Geometries*, 3(1): 7–11, 2014.

[21] Yılmaz, S. Construction of the Frenet-Serret frame of a curve in 4D Galilean space and some applications, *Int. Journal of the Physical Sciences*, 5(8): 1284–1289, 2010.

[22] Yoon, D.W. *On the Inclined curves in Galilean 4-space. Applied Mathematical Sciences, 7(44): 2193–2199, 2013.*

Department of Mathematics, Faculty of Arts And Sciences, Yildiz Technical University, Istanbul, Turkey,
E-mail address: ngurhan@yildiz.edu.tr

Department of Mathematics, Faculty of Arts And Sciences, Yildiz Technical University, Istanbul, Turkey,
E-mail address: makbiyik@yildiz.edu.tr

Department of Mathematics, Faculty of Arts And Sciences, Yildiz Technical University, Istanbul, Turkey,
E-mail address: sayuce@yildiz.edu.tr

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2017, VOLUME 7, ISSUE 2, p.110-122