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INTERVAL-VALUED INTUITIONISTIC FUZZY BI-IDEALS IN TERNARY SEMIRINGS

D. KRISHNASWAMY, J. JAYARAJ AND T. ANITHA

Abstract. In this paper we introduce the notions of interval-valued fuzzy bi-ideal, interval-valued anti fuzzy bi-ideal and interval-valued intuitionistic fuzzy bi-ideal in ternary semirings and some of the basic properties of these ideals are investigated. We also introduce normal interval-valued intuitionistic fuzzy ideals in ternary semirings.

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1. Introduction

The notion of ternary algebraic system was introduced by Lehmer (see [18]) in 1932. He investigated certain ternary algebraic systems called triplexes. In 1971, Lister (see [19]) characterized additive semigroups of rings which are closed under the triple ring product and he called this algebraic system a ternary ring. Dutta and Kar (see [3]) introduced a notion of ternary semirings which is a generalization of ternary rings and semirings, and they studied some properties of ternary semirings (see [3] - [9], [12]).

The theory of fuzzy sets was first studied by Zadeh (see [21]) in 1965. Many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic, set theory, group theory, ring theory, real analysis, topology, measure theory, etc. Interval-valued fuzzy sets were introduced independently by Zadeh (see [22]), Grattan-Guiness (see [11]), Sambuc (see [20]) in the same year 1975 as a generalization of fuzzy set. An interval-valued fuzzy set is a fuzzy set whose membership function is many-valued and forms an interval in the membership scale. This idea gives the simplest method to capture the imprecision of the membership grades for a fuzzy set. Thus, interval-valued fuzzy sets provide a more adequate description of uncertainty than the traditional fuzzy sets. It is therefore important to use interval-valued fuzzy sets in applications. One of the main applications is in fuzzy control and the most computationally intensive part of fuzzy control is defuzzification. Since the transition of interval-valued fuzzy sets usually increases the amount of computations, it is vitally important to design some faster algorithms for the necessarily defuzzification. On the other hand, Atanassov (see [1]) introduced the notion of intuitionistic fuzzy sets as an extension of fuzzy set in which not only a membership degree is given, but also a non-membership degree is involved. Atanassov and Gargov (see [2]) introduced the notion of interval-valued intuionistic fuzzy sets which is a common generalization of intuitionistic fuzzy sets and interval-valued fuzzy sets. Dutta et al. (see [10]) introduced the notion of interval-valued fuzzy prime ideal of a semiring. Kar et al. (see[13]) introduced the notion of interval-valued prime fuzzy ideal of semigroups. Kavikumar et al. (see [14] and [15]) studied fuzzy ideals, fuzzy bi-ideals and fuzzy quasi-ideals in ternary semirings. Krishnaswamy and Anitha (see [16]) and (see [17]) studied the fuzzy prime ideals and (λ, μ)-fuzzy quasi ideals and bi-ideals in ternary semirings. In this paper we first apply the concept of interval-valued intuitionistic fuzzy sets to ternary semirings. Then we introduce the notions of interval-valued fuzzy bi-ideal, interval-valued anti fuzzy bi-ideal and interval-valued intuitionistic fuzzy bi-ideal in ternary semirings and some of the basic properties of these ideals are investigated. We also introduce normal interval-valued intuitionistic fuzzy ideals in ternary semirings.

2. Preliminaries

In this section, we refer to some elementary aspects of the theory of ternary semirings and interval-valued fuzzy algebraic systems that are necessary for this paper.

Definition 2.1. [15] A nonempty set S together with a binary operation called, addition $+$ and a ternary multiplication, denoted by juxtaposition, is said to be a ternary semiring if $(S, +)$ is a commutative semigroup satisfying the following conditions:
(i) $(abc)de = a(bcd)e = ab(cde)$,
(ii) $(a + b)cd = acd + bcd$,
(iii) $a(b + c)d = abd + acd$
and (iv) $ab(c + d) = abc + abd$ for all $a, b, c, d, e \in S$.

Definition 2.2. [15] Let S be a ternary semiring. If there exists an element $0 \in S$ such that $0 + x = x = x + 0$ and $0xy = x0y = xy0 = 0$ for all $x, y \in S$, then 0 is called the zero element or simply the zero of the ternary semiring S. In this case we say that S is a ternary semiring with zero.

Throughout this paper S denotes a ternary semiring with zero.

Definition 2.3. [15] An additive subsemigroup T of S is called a ternary subsemiring of S if $t_1t_2t_3 \in T$ for all $t_1, t_2, t_3 \in T$.

Definition 2.4. [15] An additive subsemigroup I of S is called a left [resp. right, lateral] ideal of S if $s_1s_2i \in I$ [resp. $is_1s_2 \in I, s_1is_2 \in I$] for all $s_1, s_2 \in S$ and $i \in I$. If I is a left, right and lateral ideal of S, then I is called an ideal of S.

It is obvious that every ideal of a ternary semiring with zero contains the zero element.

Definition 2.5. [15] An additive subsemigroup $(B, +)$ of a ternary semiring S is called a bi-ideal of S if $BSBSB \subseteq B$.

An interval number on $[0, 1]$, denoted by $\widetilde{a}$, is defined as the closed sub interval of $[0, 1]$, where $\widetilde{a} = [a^-, a^+]$ satisfying $0 \leq a^- \leq a^+ \leq 1$.

The set of all interval numbers is denoted by $D[0, 1]$. The interval $[a, a]$ is identified with the number $a \in [0, 1]$.

Definition 2.6. [10], [13] Let $\widetilde{a} = [a^-, a^+]$ and $\widetilde{b} = [b^-, b^+]$ be two interval numbers in $D[0, 1]$. Then
i) $\widetilde{a} \leq \widetilde{b}$ if and only if $a^- \leq b^-$ and $a^+ \leq b^+$,
ii) $\widetilde{a} + \widetilde{b} = [a^- + b^-, a^+ + b^+]$,
iii) If $\widetilde{a} \geq \widetilde{b}$ then $\widetilde{a} - \widetilde{b} = [\ min\{a^- - b^-, a^+ - b^+\},\ max\{a^- - b^-, a^+ - b^+\}\ ]$,
iv) $\inf\widetilde{a_i} = [\bigwedge_{i\in I} {a_i}^-, \bigwedge_{i\in I} {a_i}^+]$, $\sup\widetilde{a_i} = [\bigvee_{i\in I} {a_i}^-, \bigvee_{i\in I} {a_i}^+]$ for interval numbers $\widetilde{a_i} = [{a_i}^-, {a_i}^+] \in D[0, 1], i \in I$.

Let $\{\widetilde{a_i}\}, i = 1, 2, \cdots, n$ for some $n \in Z^+$ be a finite number of interval numbers, where $\widetilde{a_i} = [{a_i}^-, {a_i}^+]$. Then we define $Max^i\{\widetilde{a_i}\} = [\ max\{{a_i}^-\},\ max\{{a_i}^+\}\ ]$ and $Min^i\{\widetilde{a_i}\} = [\ min\{{a_i}^-\},\ min\{{a_i}^+\}\ ]$.

In this paper we assume that any two interval numbers in $D[0, 1]$ are comparable. i.e. for any two interval numbers $\widetilde{a}$ and $\widetilde{b}$ in $D[0, 1]$, we have either $\widetilde{a} \leq \widetilde{b}$ or $\widetilde{a} > \widetilde{b}$. It is clear that $(D[0, 1], \leq, \vee, \wedge)$ is a complete lattice with $\widetilde{0} = [0, 0]$ as the least element and $\widetilde{1} = [1, 1]$ as the greatest element.

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Definition 2.7. [10] Let X be a non-empty set. A map $\widetilde{\mu} : X \to D[0,1]$ is called an interval-valued fuzzy subset of X. The complement of an interval-valued fuzzy subset $\widetilde{\mu}$ of a set X is denoted by $\widetilde{\mu}^c$ and defined as $\widetilde{\mu}^c(x) = \widetilde{1} - \widetilde{\mu}(x)$, for all $x \in X$.

Note: We can write $\widetilde{\mu}(x) = [\mu^-(x), \mu^+(x)]$ for all $x \in X$, for any interval-valued fuzzy subset $\widetilde{\mu}$ of a non empty set X, where μ^- and μ^+ are some fuzzy subsets of X.

Definition 2.8. [10] Let $\widetilde{\mu}$ and $\widetilde{\nu}$ be two interval-valued fuzzy subsets of a non-empty set X. Then $\widetilde{\mu}$ is said to be a subset of $\widetilde{\nu}$, denoted by $\widetilde{\mu} \subseteq \widetilde{\nu}$ if $\widetilde{\mu}(x) \leq \widetilde{\nu}(x)$, i.e., $\mu^-(x) \leq \nu^-(x)$ and $\mu^+(x) \leq \nu^+(x)$, for all $x \in X$ where $\widetilde{\mu}(x) = [\mu^-(x), \mu^+(x)]$ and $\widetilde{\nu}(x) = [\nu^-(x), \nu^+(x)]$.

Definition 2.9. [10] An upper level set of an interval-valued fuzzy subset $\widetilde{\mu}$ denoted by $\overline{U}(\widetilde{\mu};\widetilde{t})$ is defined as $\overline{U}(\widetilde{\mu};\widetilde{t}) = \{x \in X/\widetilde{\mu}(x) \geq \widetilde{t}\}$ and a lower level set of an interval-valued fuzzy subset $\widetilde{\mu}$ denoted by $\overline{L}(\widetilde{\mu};\widetilde{t})$ is defined as $\overline{L}(\widetilde{\mu};\widetilde{t}) = \{x \in X/\widetilde{\mu}(x) \leq \widetilde{t}\}$, for all $\widetilde{t} \in D[0,1]$.

Definition 2.10. [10] Let $\widetilde{\mu}$ and $\widetilde{\nu}$ be any two interval-valued fuzzy subsets of a nonempty set X. Then $\widetilde{\mu} \cap \widetilde{\nu}$, $\widetilde{\mu} \cup \widetilde{\nu}$, $\widetilde{\mu} + \widetilde{\nu}$, $\widetilde{\mu} \circ \widetilde{\nu}$ are interval-valued fuzzy subsets of S defined by, for all $x \in S$,

$$(\widetilde{\mu} \cap \widetilde{\nu})(x) = Min^i\{\widetilde{\mu}(x),\ \widetilde{\nu}(x)\},$$
$$(\widetilde{\mu} \cup \widetilde{\nu})(x) = Max^i\{\widetilde{\mu}(x),\ \widetilde{\nu}(x)\},$$

$$(\widetilde{\mu} + \widetilde{\nu})(x) = \begin{cases} sup\{Min^i\{\widetilde{\mu}(y), \widetilde{\nu}(z)\}\}\ if\ x = y + z \\ \widetilde{0} \text{ otherwise,} \end{cases}$$

$$(\widetilde{\mu} \circ \widetilde{\nu})(x) = \begin{cases} sup\{Min^i\{\widetilde{\mu}(u), \widetilde{\nu}(v)\}\}\ if\ x = uv, \\ \widetilde{0} \text{ otherwise.} \end{cases}$$

An interval-valued intuitionistic fuzzy subset (IIFS for short) defined on non-empty set S as objects of the form

$$A = \{< \widetilde{\mu}_A(x), \widetilde{\nu}_A(x) > /x \in S\},$$

where the function $\widetilde{\mu} : S \to D[0,1]$ and $\widetilde{\nu} : S \to D[0,1]$ denote the degree of membership (namely $\widetilde{\mu}_A(x)$) and the degree of non-membership (namely $\widetilde{\nu}_A(x)$) for each element $x \in S$ to the set A, respectively, and $\widetilde{0} \leq \widetilde{\mu}_A(x) + \widetilde{\nu}_A(x) \leq \widetilde{1}$, for each $x \in S$.

For the sake of simplicity, we shall use the symbol $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ for the interval-valued intuitionistic fuzzy subset $A = \{< \widetilde{\mu}_A(x), \widetilde{\nu}_A(x) > /x \in S\}$.

3. Interval-valued intuitionistic fuzzy bi-ideals

Definition 3.1. An interval-valued intuitionistic fuzzy subset $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ in S is called an interval-valued intuitionistic fuzzy right (left, lateral) ideal of S if
1. $\widetilde{\mu}_A(x + y) \geq Min^i\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(y)\}$,
2. $\widetilde{\mu}_A(xyz) \geq \widetilde{\mu}_A(x)$ $(\widetilde{\mu}_A(xyz) \geq \widetilde{\mu}_A(z), \widetilde{\mu}_A(xyz) \geq \widetilde{\mu}_A(y))$,
3. $\widetilde{\nu}_A(x + y) \leq Max^i\{\widetilde{\nu}_A(x), \widetilde{\nu}_A(y)\}$,
4. $\widetilde{\nu}_A(xyz) \leq \widetilde{\nu}_A(x)$ $(\widetilde{\nu}_A(xyz) \leq \widetilde{\nu}_A(z),\ \widetilde{\nu}_A(xyz) \leq \widetilde{\nu}_A(y))$, for all $x, y, z \in S$.

Example 3.2. Consider the ternary semiring $S = Z_0^-$, the set of all non positive integers with the usual addition and ternary multiplication. Let the interval-valued fuzzy subset $\widetilde{\mu}_A$ and $\widetilde{\nu}_A$ of S be defined by

$$\widetilde{\mu}_A(x) = \begin{cases} [0.7, 0.8] \text{ ,if } x \in \langle -3 \rangle \\ [0.1, 0.3], \text{ otherwise,} \end{cases}$$

$$\widetilde{\nu}_A(x) = \begin{cases} [0.1, 0.2], \text{ if } x \in \langle -3 \rangle \\ [0.5, 0.6], \text{ otherwise.} \end{cases}$$

Then $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ is an interval-valued intuitionistic fuzzy right ideal of S.

Definition 3.3. Let $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ be an interval-valued intuitionistic fuzzy subset of S and let $\widetilde{s}, \widetilde{t} \in D[0,1]$. Then the set $\overline{S}_A^{(\widetilde{s},\widetilde{t})} = \{x \in S /\ \widetilde{\mu}_A(x) \geq \widetilde{s}, \widetilde{\nu}_A(x) \leq \widetilde{t}\}$ is called a $(\widetilde{s},\widetilde{t})$-level set of $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$.

The set $\{(\widetilde{s},\widetilde{t}) \in Im(\widetilde{\mu}_A) \times Im(\widetilde{\nu}_A) /\ \widetilde{s} + \widetilde{t} \leq \widetilde{1}\}$ is called image of $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$. Clearly $\overline{S}_A^{(\widetilde{s},\widetilde{t})} = \overline{U}(\widetilde{\mu}_A; \widetilde{s}) \cap \overline{L}(\widetilde{\nu}_A; \widetilde{t})$, where $\overline{U}(\widetilde{\mu}_A; \widetilde{s})$ and $\overline{L}(\widetilde{\nu}_A; \widetilde{t})$ are upper and lower level subsets of $\widetilde{\mu}_A$ and $\widetilde{\nu}_A$ respectively.

Definition 3.4. An interval-valued fuzzy subset $\widetilde{\mu}$ of a ternary semiring S is said to be an interval-valued fuzzy bi-ideal of S if
1. $\widetilde{\mu}_A(x+y) \geq Min^i\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(y)\}$,
2. $\widetilde{\mu}_A(xs_1ys_2z) \geq Min^i\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(y), \widetilde{\mu}_A(z)\}$, for all $x, s_1, y, s_2, z \in S$.

Example 3.5. Consider the ternary semiring $S = Z_0^-$, the set of all non positive integers with the usual addition and ternary multiplication. Let the interval-valued fuzzy subset $\widetilde{\mu}_A$ of S be defined by

$$\widetilde{\mu}_A(x) = \begin{cases} [0.6, 0.7], & \text{if } x \in \langle -2 \rangle \\ [0.3, 0.4], & \text{otherwise.} \end{cases}$$

Then $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ is an interval-valued fuzzy bi-ideal of S.

Definition 3.6. An interval-valued fuzzy subset $\widetilde{\mu}$ of a ternary semiring S is said to be an interval-valued anti fuzzy bi-ideal of S if
1. $\widetilde{\mu}_A(x+y) \leq Max^i\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(y)\}$,
2. $\widetilde{\mu}_A(xs_1ys_2z) \leq Max^i\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(y), \widetilde{\mu}_A(z)\}$, for all $x, s_1, y, s_2, z \in S$.

Example 3.7. Consider the ternary semiring $S = Z_0^-$, the set of all non positive integers with the usual addition and ternary multiplication. Let the interval-valued fuzzy subset $\widetilde{\nu}_A$ of S be defined by

$$\widetilde{\nu}_A(x) = \begin{cases} [0.1, 0.2], & \text{if } x \in \langle -2 \rangle \\ [0.7, 0.9], & \text{otherwise.} \end{cases}$$

Then $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ is an interval-valued anti fuzzy bi-ideal of S.

Definition 3.8. An interval-valued intuitionistic fuzzy subset $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ in S is called an interval-valued intuitionistic fuzzy bi-ideal of S if
1. $\widetilde{\mu}_A(x+y) \geq Min^i\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(y)\}$,
2. $\widetilde{\mu}_A(xs_1ys_2z) \geq Min^i\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(y), \widetilde{\mu}_A(z)\}$,
3. $\widetilde{\nu}_A(x+y) \leq Max^i\{\widetilde{\nu}_A(x), \widetilde{\nu}_A(y)\}$,
4. $\widetilde{\nu}_A(xs_1ys_2z) \leq Max^i\{\widetilde{\nu}_A(x), \widetilde{\nu}_A(y), \widetilde{\nu}_A(z)\}$, for all $x, s_1, y, s_2, z \in S$.

Example 3.9. Consider

$$S = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ a & b & c \\ d & e & h \end{pmatrix} : a, b, c, d, e, h \in Z_0^- \right\}.$$

Then S is a ternary semiring with respect to matrix addition and matrix multiplication. Let

$$B = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & p & q \\ 0 & 0 & 0 \end{pmatrix} : p, q \in Z_0^- \right\}.$$

Let the interval-valued fuzzy subset $\widetilde{\mu}_A$ and $\widetilde{\nu}_A$ of S be defined by

$$\widetilde{\mu}_A(x) = \begin{cases} [0.6, 0.8], & \text{if } x \in B, \\ [0.1, 0.3], & \text{otherwise,} \end{cases}$$

$$\widetilde{\nu}_A(x) = \begin{cases} [0.1, 0.2], & \text{if } x \in B, \\ [0.4, 0.6], & \text{otherwise.} \end{cases}$$

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Then $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ is an interval-valued intuitionistic fuzzy bi-ideal of S, but not an interval-valued intuitionistic fuzzy ideal of S. Since $\widetilde{\mu}_A(ssb) = [0.1, 0.3] < \widetilde{\mu}_A(b)$; $\widetilde{\mu}_A(sbs) = [0.1, 0.3] < \widetilde{\mu}_A(b)$; $\widetilde{\mu}_A(bss) = [0.1, 0.3] < \widetilde{\mu}_A(b)$; $\widetilde{\nu}_A(ssb) = [0.4, 0.6] > \widetilde{\nu}_A(b)$; $\widetilde{\nu}_A(sbs) = [0.4, 0.6] > \widetilde{\nu}_A(b)$ and $\widetilde{\nu}_A(bss) = [0.4, 0.6] > \widetilde{\nu}_A(b)$, where

$$s = \begin{pmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix},\ b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Theorem 3.10. *If an interval-valued fuzzy subset $\widetilde{\mu}$ is an interval-valued fuzzy bi-ideal of a ternary semiring S if and only if $\widetilde{\mu}^c$ is an interval-valued anti fuzzy bi-ideal of S.*

Proof. Let $\widetilde{\mu}$ be an interval-valued fuzzy bi-ideal of a ternary semiring S. Let $x, y, z \in S$. Then

$$\begin{aligned} \widetilde{\mu}(x+y) &\geq Min^i\{\widetilde{\mu}(x), \widetilde{\mu}(y)\} \\ \Rightarrow -\widetilde{\mu}(x+y) &\leq -Min^i\{\widetilde{\mu}(x), \widetilde{\mu}(y)\} \\ \Rightarrow \widetilde{1} - \widetilde{\mu}(x+y) &\leq \widetilde{1} - Min^i\{\widetilde{\mu}(x), \widetilde{\mu}(y)\} \\ \Rightarrow \widetilde{1} - \widetilde{\mu}(x+y) &\leq Max^i\{\widetilde{1} - \widetilde{\mu}(x), \widetilde{1} - \widetilde{\mu}(y)\} \\ \Rightarrow \widetilde{\mu}^c(x+y) &\leq Max^i\{\widetilde{\mu}^c(x), \widetilde{\mu}^c(y)\} \end{aligned}$$

and

$$\begin{aligned} \widetilde{\mu}(xs_1ys_2z) &\geq Min^i\{\widetilde{\mu}(x), \widetilde{\mu}(y), \widetilde{\mu}(z)\} \\ \Rightarrow -\widetilde{\mu}(xs_1ys_2z) &\leq -Min^i\{\widetilde{\mu}(x), \widetilde{\mu}(y), \widetilde{\mu}(z)\} \\ \Rightarrow \widetilde{1} - \widetilde{\mu}(xs_1ys_2z) &\leq \widetilde{1} - Min^i\{\widetilde{\mu}(x), \widetilde{\mu}(y), \widetilde{\mu}(z)\} \\ \Rightarrow \widetilde{1} - \widetilde{\mu}(xs_1ys_2z) &\leq Max^i\{\widetilde{1} - \widetilde{\mu}(x), \widetilde{1} - \widetilde{\mu}(y), \widetilde{1} - \widetilde{\mu}(z)\} \\ \Rightarrow \widetilde{\mu}^c(xs_1ys_2z) &\leq Max^i\{\widetilde{\mu}^c(x), \widetilde{\mu}^c(y), \widetilde{\mu}^c(z)\}. \end{aligned}$$

Thus $\widetilde{\mu}^c$ is an interval-valued anti fuzzy bi-ideal of S. By similar argument, we can prove the converse part. □

Theorem 3.11. *Every interval-valued intuitionistic fuzzy ideal of S is an interval-valued intuitionistic fuzzy bi-ideal of S.*

Proof. Let $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ be an interval-valued intuitionistic fuzzy ideal of S. Then $\widetilde{\mu}(xs_1ys_2z) \geq Min^i\{\widetilde{\mu}(x), \widetilde{\mu}(s_1ys_2), \widetilde{\mu}(z)\} \geq Min^i\{\widetilde{\mu}(x), \widetilde{\mu}(y), \widetilde{\mu}(z)\}$ and $\widetilde{\nu}(xs_1ys_2z) \leq Max^i\{\widetilde{\nu}(x), \widetilde{\nu}(s_1ys_2), \widetilde{\nu}(z)\} \leq Max^i\{\widetilde{\nu}(x), \widetilde{\nu}(y), \widetilde{\nu}(z)\}$. Thus A is an interval-valued intuitionistic fuzzy bi-ideal of S. □

The converse of the above theorem is need not be true as given in Example 3.9.

Theorem 3.12. *An IIFS $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ in S is an interval-valued intuitionistic fuzzy bi-ideal of S if and only if any level set $\overline{S}_A^{(\widetilde{s},\widetilde{t})}$ is a bi-ideal of S for $\widetilde{s}, \widetilde{t} \in D[0,1]$ whenever nonempty.*

Proof. Let $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ be an interval-valued intuitionistic fuzzy bi-ideal of S. Let $x, y, z \in \overline{S}_A^{(\widetilde{s},\widetilde{t})}$ and $u, v \in S$. Then $\widetilde{\mu}_A(x+y) \geq Min^i\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(y)\} \geq \widetilde{s}$ and $\widetilde{\nu}_A(x+y) \leq Max^i\{\widetilde{\nu}_A(x), \widetilde{\nu}_A(y)\} \leq \widetilde{t}$. So $x + y \in \overline{S}_A^{(\widetilde{s},\widetilde{t})}$. Again $\widetilde{\mu}_A(xuyvz) \geq Min^i\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(y), \widetilde{\mu}_A(z)\} \geq \widetilde{s}$ and $\widetilde{\nu}_A(xuyvz) \leq Max^i\{\widetilde{\nu}_A(x), \widetilde{\nu}_A(y), \widetilde{\nu}_A(z)\} \leq \widetilde{t}$ which implies $xuyvz \in \overline{S}_A^{(\widetilde{s},\widetilde{t})}$. Hence $\overline{S}_A^{(\widetilde{s},\widetilde{t})}$ is a bi-ideal. Conversely let $\overline{S}_A^{(\widetilde{s},\widetilde{t})}$ be a bi-ideal of S, for any $\widetilde{s}, \widetilde{t} \in D[0,1]$ with $\widetilde{s} + \widetilde{t} \leq \widetilde{1}$. Let $x, y \in S$ such that $\widetilde{\mu}_A(x) = \widetilde{\alpha}_1, \widetilde{\mu}_A(y) = \widetilde{\alpha}_2$ and $\widetilde{\nu}_A(x) = \widetilde{\beta}_1, \widetilde{\nu}_A(y) = \widetilde{\beta}_2$ where $\widetilde{\alpha}_1, \widetilde{\alpha}_2, \widetilde{\beta}_1, \widetilde{\beta}_2 \in D[0,1]$. Then $\widetilde{\alpha}_1 + \widetilde{\beta}_1 \leq \widetilde{1}, \widetilde{\alpha}_2 + \widetilde{\beta}_2 \leq \widetilde{1}$. Let $\widetilde{\alpha} = Min^i\{\widetilde{\alpha}_1, \widetilde{\alpha}_2\}$ and $\widetilde{\beta} = Max^i\{\widetilde{\beta}_1, \widetilde{\beta}_2\}$ then $x, y \in S_A^{(\widetilde{\alpha},\widetilde{\beta})}$. Since $S_A^{(\widetilde{\alpha},\widetilde{\beta})}$ be a bi-ideal of S then $x + y \in S_A^{(\widetilde{\alpha},\widetilde{\beta})}$ that means $\widetilde{\mu}_A(x+y) \geq \widetilde{\alpha} = Min^i\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(y)\}$, $\widetilde{\nu}_A(x+y) \leq \widetilde{\beta} = Max^i\{\widetilde{\nu}_A(x), \widetilde{\nu}_A(y)\}$. Similarly we prove $\widetilde{\mu}_A(xuyvz) \geq Min^i\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(y), \widetilde{\mu}_A(z)\}$ and $\widetilde{\nu}_A(xuyvz) \leq Max^i\{\widetilde{\nu}_A(x), \widetilde{\nu}_A(y), \widetilde{\nu}_A(z)\}$. Therefore A is an interval-valued intuitionistic fuzzy bi-ideal. □

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Corollary 3.13. *An IIFS $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ in S is an interval-valued intuitionistic fuzzy bi-ideal of S if and only if for every $\widetilde{s}, \widetilde{t} \in D[0,1]$ such that $\widetilde{s} + \widetilde{t} \leq \widetilde{1}$ all non-empty $\overline{U}(\widetilde{\mu}_A; \widetilde{s})$ and $\overline{L}(\widetilde{\nu}_A; \widetilde{t})$ are bi-ideals of S.*

Theorem 3.14. *Let I be a non-empty subset of a ternary semiring S. Then an IIFS $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ defined by*

$$\widetilde{\mu}_A(x) = \begin{cases} \widetilde{s}_2, & \text{if } x \in I, \\ \widetilde{s}_1, & \text{otherwise} \end{cases}$$

$$\widetilde{\nu}_A(x) = \begin{cases} \widetilde{t}_2, & \text{if } x \in I \\ \widetilde{t}_1, & \text{otherwise,} \end{cases}$$

where $\widetilde{0} \leq \widetilde{s}_1 < \widetilde{s}_2 \leq \widetilde{1}$, $\widetilde{0} \leq \widetilde{t}_2 < \widetilde{t}_1 \leq \widetilde{1}$ and $\widetilde{s}_i + \widetilde{t}_i \leq \widetilde{1}$ for each $i = 1, 2$ is an interval-valued intuitionistic fuzzy bi-ideal of S if and only if I is a bi-ideal of S.

Proof. Let I be a bi-ideal of S. Let $x, y, z, u, v \in S$. If $x, y, z \in I$, then $x + y, xuyvz \in I$. Then $\widetilde{\mu}_A(x + y) = \widetilde{s}_2 \geq Min^i\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(y)\}$, $\widetilde{\nu}_A(x + y) = \widetilde{t}_2 \leq Max^i\{\widetilde{\nu}_A(x), \widetilde{\nu}_A(y)\}$, $\widetilde{\mu}_A(xuyvz) = \widetilde{s}_2 \geq Min^i\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(y), \widetilde{\mu}_A(z)\}$ and $\widetilde{\nu}_A(xuyvz) = \widetilde{t}_2 \leq Max^i\{\widetilde{\nu}_A(x), \widetilde{\nu}_A(y), \widetilde{\nu}_A(z)\}$. If either x or y or $z \notin I$, then also $\widetilde{\mu}_A(x + y) \geq \widetilde{s}_1 = Min^i\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(y)\}$, $\widetilde{\nu}_A(x + y) \leq \widetilde{t}_1 = Max^i\{\widetilde{\nu}_A(x), \widetilde{\nu}_A(y)\}$, $\widetilde{\mu}_A(xuyvz) \geq \widetilde{s}_1 = Min^i\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(y), \widetilde{\mu}_A(z)\}$ and $\widetilde{\nu}_A(xuyvz) \leq \widetilde{t}_1 = Max^i\{\widetilde{\nu}_A(x), \widetilde{\nu}_A(y), \widetilde{\nu}_A(z)\}$. Hence $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ is an interval-valued intuitionistic fuzzy bi-ideal of S.

Conversely, let $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ is an interval-valued intuitionistic fuzzy bi-ideal of S. Then $\overline{S}_A^{(\widetilde{s}_2, \widetilde{t}_2)} = I$. So, by Theorem 3.12, I must be a bi-ideal of S. □

Theorem 3.15. *Let $(\widetilde{\mu}_i, \widetilde{\nu}_i)_{i \in I}$ be a family of interval-valued intuitionistic fuzzy bi-ideals of S then $(\cap\widetilde{\mu}_i, \cup\widetilde{\nu}_i)$ is also an interval-valued intuitionistic fuzzy bi-ideal of S.*

Proof. Let $\widetilde{\mu} = \bigcap_{i \in I} \widetilde{\mu}_i$ and $\widetilde{\nu} = \bigcup_{i \in I} \widetilde{\nu}_i$. For any $x, y, z \in S$,

1. $\widetilde{\mu}(x + y) = \bigcap_{i \in I} \widetilde{\mu}_i(x + y) \geq \bigcap_{i \in I} Min^i\{\widetilde{\mu}_i(x), \widetilde{\mu}_i(y)\}$
$= Min^i\{\bigcap_{i \in I} \widetilde{\mu}_i(x), \bigcap_{i \in I} \widetilde{\mu}_i(y)\} = Min^i\{\widetilde{\mu}(x), \widetilde{\mu}(y)\}$.
2. $\widetilde{\mu}(xs_1ys_2z) = \bigcap_{i \in I} \widetilde{\mu}_i(xs_1ys_2z) \geq \bigcap_{i \in I} Min^i\{\widetilde{\mu}_i(x), \widetilde{\mu}_i(y), \widetilde{\mu}_i(z)\}$
$= Min^i\{\bigcap_{i \in I} \widetilde{\mu}_i(x), \bigcap_{i \in I} \widetilde{\mu}_i(y), \bigcap_{i \in I} \widetilde{\mu}_i(z)\} = Min^i\{\widetilde{\mu}(x), \widetilde{\mu}(y), \widetilde{\mu}(z)\}$.
3. $\widetilde{\nu}(x + y) = \bigcup_{i \in I} \widetilde{\nu}_i(x + y) \leq \bigcup_{i \in I} Max^i\{\widetilde{\nu}_i(x), \widetilde{\nu}_i(y)\}$
$= Max^i\{\bigcup_{i \in I} \widetilde{\nu}_i(x), \bigcup_{i \in I} \widetilde{\nu}_i(y)\} = Max^i\{\widetilde{\nu}(x), \widetilde{\nu}(y)\}$.
4. $\widetilde{\nu}(xs_1ys_2z) = \bigcup_{i \in I} \widetilde{\nu}_i(xs_1ys_2z) \leq \bigcup_{i \in I} Max^i\{\widetilde{\nu}_i(x), \widetilde{\nu}_i(y), \widetilde{\nu}_i(z)\}$
$= Max^i\{\bigcup_{i \in I} \widetilde{\nu}_i(x), \bigcup_{i \in I} \widetilde{\nu}_i(y), \bigcup_{i \in I} \widetilde{\nu}_i(z)\} = Max^i\{\widetilde{\nu}(x), \widetilde{\nu}(y), \widetilde{\nu}(z)\}$.

Therefore $(\cap\widetilde{\mu}_i, \cup\widetilde{\nu}_i)$ is an interval-valued intuitionistic fuzzy bi-ideal of S. □

Theorem 3.16. *An IIFS $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ in S is an interval-valued intuitionistic fuzzy bi-ideal of S if and only if the interval-valued fuzzy subsets $\widetilde{\mu}_A$ and $\widetilde{\nu}_A^c$ are interval-valued fuzzy bi-ideals of S.*

Proof. If $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ is an interval-valued intuitionistic fuzzy bi-ideal of S, then clearly $\widetilde{\mu}_A$ is an interval-valued fuzzy bi-ideal of S. For all $x, y, z, s_1, s_2 \in S$, $\widetilde{\nu}_A^c(x+y) = \widetilde{1} - \widetilde{\nu}_A(x+y) \geq \widetilde{1} - Max^i\{\widetilde{\nu}_A(x), \widetilde{\nu}_A(y)\} = Min^i\{\widetilde{1} - \widetilde{\nu}_A(x), \widetilde{1} - \widetilde{\nu}_A(y)\} = Min^i\{\widetilde{\nu}_A^c(x), \widetilde{\nu}_A^c(y)\}$ and $\widetilde{\nu}_A^c(xs_1ys_2z) = \widetilde{1} - \widetilde{\nu}_A(xs_1ys_2z) \geq \widetilde{1} - Max^i\{\widetilde{\nu}_A(x), \widetilde{\nu}_A(y), \widetilde{\nu}_A(z)\} = Min^i\{\widetilde{1} - \widetilde{\nu}_A(x), \widetilde{1} - \widetilde{\nu}_A(y), \widetilde{1} - \widetilde{\nu}_A(z)\} = Min^i\{\widetilde{\nu}_A^c(x), \widetilde{\nu}_A^c(y), \widetilde{\nu}_A^c(z)\}$. Thus $\widetilde{\nu}_A^c$ is an interval-valued fuzzy bi-ideal of S. Conversely assume that $\widetilde{\mu}_A$ and $\widetilde{\nu}_A^c$ are interval-valued fuzzy bi-ideals of S, then clearly the conditions 1) and 2) of Definition 3.8 are satisfied. Now for all $x, y, z, s_1, s_2 \in S$, $\widetilde{1} - \widetilde{\nu}_A(x + y) = \widetilde{\nu}_A^c(x + y) \geq Min^i\{\widetilde{\nu}_A^c(x), \widetilde{\nu}_A^c(y)\} = Min^i\{\widetilde{1} -$

$\widetilde{\nu}_A(x), \widetilde{1} - \widetilde{\nu}_A(y)\} = \widetilde{1} - Max^i\{\widetilde{\nu}_A(x), \widetilde{\nu}_A(y)\}$ which implies $-\widetilde{\nu}_A(x+y) \geq -Max^i\{\widetilde{\nu}_A(x), \widetilde{\nu}_A(y)\}$ implies $\widetilde{\nu}_A(x+y) \leq Max^i\{\widetilde{\nu}_A(x), \widetilde{\nu}_A(y)\}$ and $\widetilde{1} - \widetilde{\nu}_A(xs_1ys_2z) = \widetilde{\nu}^c_A(xs_1ys_2z) \geq Min^i\{\widetilde{\nu}^c_A(x), \widetilde{\nu}^c_A(y), \widetilde{\nu}^c_A(z)\} = Min^i\{\widetilde{1} - \widetilde{\nu}_A(x), \widetilde{1} - \widetilde{\nu}_A(y), \widetilde{1} - \widetilde{\nu}_A(z)\} = \widetilde{1} - Max^i\{\widetilde{\nu}_A(x), \widetilde{\nu}_A(y), \widetilde{\nu}_A(z)\}$ which implies $-\widetilde{\nu}_A(xs_1ys_2z) \geq -Max^i\{\widetilde{\nu}_A(x), \widetilde{\nu}_A(y), \widetilde{\nu}_A(z)\}$ implies $\widetilde{\nu}_A(xs_1ys_2z) \leq Max^i\{\widetilde{\nu}_A(x), \widetilde{\nu}_A(y), \widetilde{\nu}_A(z)\}$. □

Corollary 3.17. *If an IIFS $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ in S is an interval-valued intuitionistic fuzzy bi-ideal of S if and only if IIFS $A_1 = (\widetilde{\mu}_A, \widetilde{\mu}^c_A)$ and IIFS $A_2 = (\widetilde{\nu}^c_A, \widetilde{\nu}_A)$ are interval-valued intuitionistic fuzzy bi-ideals of S.*

Proof. It is straightforward by Theorem 3.10 and Theorem 3.16. □

Definition 3.18. Let S_1 and S_2 be ternary semirings. A mapping $f : S_1 \to S_2$ is said to be a homomorphism if $f(x+y) = f(x) + f(y)$ and $f(xyz) = f(x)f(y)f(z)$ for all $x, y, z \in S_1$.

Let $f : S_1 \to S_2$ be an onto homomorphism of ternary semirings. Note that if I is an ideal of S_1, then $f(I)$ is an ideal of S_2. If S_1 and S_2 are ternary semirings with zero 0, then $f(0) = 0$.

Theorem 3.19. *Let S_1, S_2 be ternary semirings and let $\Phi : S_1 \to S_2$ be an onto homomorphism and let $B = (\widetilde{\mu}_B, \widetilde{\nu}_B)$ be an interval-valued intuitionistic fuzzy bi-ideal of S_2. Then $B = (\widetilde{\mu}_B, \widetilde{\nu}_B)$ is an interval-valued intuitionistic fuzzy bi-ideal of S_2 if and only if $\Phi^{-1}(B) = (\Phi^{-1}(\widetilde{\mu}_B), \Phi^{-1}(\widetilde{\nu}_B))$, where $\Phi^{-1}(\widetilde{\mu}_B)(x) = \widetilde{\mu}_B(\Phi(x))$ and $\Phi^{-1}(\widetilde{\nu}_B)(x) = \widetilde{\nu}_B(\Phi(x))$, for all $x \in S_1$, is an interval-valued intuitionistic fuzzy bi-ideal of S_1.*

Proof. Assume $B = (\widetilde{\mu}_B, \widetilde{\nu}_B)$ is an interval-valued intuitionistic fuzzy bi-ideal of S_2, and let $x, y, z, u, v \in S_1$. Then
1. $\Phi^{-1}(\widetilde{\mu}_B)(x+y) = \widetilde{\mu}_B(\Phi(x+y)) = \widetilde{\mu}_B(\Phi(x) + \Phi(y))$
$\geq Min^i\{\widetilde{\mu}_B(\Phi(x)), \widetilde{\mu}_B(\Phi(y))\} = Min^i\{\Phi^{-1}(\widetilde{\mu}_B)(x), \Phi^{-1}(\widetilde{\mu}_B)(y)\}$.
2. $\Phi^{-1}(\widetilde{\mu}_B)(xuyvz) = \widetilde{\mu}_B(\Phi(xuyvz)) = \widetilde{\mu}_B(\Phi(x)\Phi(u)\Phi(y)\Phi(v)\Phi(z))$
$\geq Min^i\{\widetilde{\mu}_B(\Phi(x)), \widetilde{\mu}_B(\Phi(y)), \widetilde{\mu}_B(\Phi(z))\}$
$= Min^i\{\Phi^{-1}(\widetilde{\mu}_B)(x), \Phi^{-1}(\widetilde{\mu}_B)(y), \Phi^{-1}(\widetilde{\mu}_B)(z)\}$.
3. $\Phi^{-1}(\widetilde{\nu}_B)(x+y) = \widetilde{\nu}_B(\Phi(x+y)) = \widetilde{\nu}_B(\Phi(x) + \Phi(y))$
$\leq Max^i\{\widetilde{\nu}_B(\Phi(x)), \widetilde{\nu}_B(\Phi(y))\} = Max^i\{\Phi^{-1}(\widetilde{\nu}_B)(x), \Phi^{-1}(\widetilde{\nu}_B)(y)\}$.
4. $\Phi^{-1}(\widetilde{\nu}_B)(xuyvz) = \widetilde{\nu}_B(\Phi(xuyvz)) = \widetilde{\nu}_B(\Phi(x)\Phi(u)\Phi(y)\Phi(v)\Phi(z))$
$\leq Max^i\{\widetilde{\nu}_B(\Phi(x)), \widetilde{\nu}_B(\Phi(y)), \widetilde{\nu}_B(\Phi(z))\}$
$= Max^i\{\Phi^{-1}(\widetilde{\nu}_B)(x), \Phi^{-1}(\widetilde{\nu}_B)(y), \Phi^{-1}(\widetilde{\nu}_B)(z)\}$.
Therefore $\Phi^{-1}(B) = (\Phi^{-1}(\widetilde{\mu}_B), \Phi^{-1}(\widetilde{\nu}_B))$ is an interval-valued intuitionistic fuzzy bi-ideal of S_1.

Conversely, assume that $\Phi^{-1}(B) = (\Phi^{-1}(\widetilde{\mu}_B), \Phi^{-1}(\widetilde{\nu}_B))$ is an interval-valued intuitionistic fuzzy bi-ideal of S_1. Let $y_1, y_2, y_3, y_4, y_5 \in S_2$ such that $\Phi(x_1) = y_1, \Phi(x_2) = y_2, \Phi(x_3) = y_3, \Phi(x_4) = y_4, \Phi(x_5) = y_5$ where $x_1, x_2, x_3, x_4, x_5 \in S_1$.
1. $\widetilde{\mu}_B(y_1 + y_2)) = \widetilde{\mu}_B(\Phi(x_1) + \Phi(x_2)) = \widetilde{\mu}_B(\Phi(x_1 + x_2)) = \Phi^{-1}(\widetilde{\mu}_B)(x_1 + x_2)$
$\geq Min^i\{\Phi^{-1}(\widetilde{\mu}_B)(x_1), \Phi^{-1}(\widetilde{\mu}_B)(x_2)\} = Min^i\{\widetilde{\mu}_B(\Phi(x_1)), \widetilde{\mu}_B(\Phi(x_2))\}$.
2. $\widetilde{\mu}_B(y_1y_2y_3y_4y_5)) = \widetilde{\mu}_B(\Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4)\Phi(x_5))$
$= \widetilde{\mu}_B(\Phi(x_1x_2x_3x_4x_5)) = \Phi^{-1}(\widetilde{\mu}_B)(x_1x_2x_3x_4x_5)$
$\geq Min^i\{\Phi^{-1}(\widetilde{\mu}_B)(x_1), \Phi^{-1}(\widetilde{\mu}_B)(x_3), \Phi^{-1}(\widetilde{\mu}_B)(x_5)\}$
$= Min^i\{\widetilde{\mu}_B(\Phi(x_1)), \widetilde{\mu}_B(\Phi(x_3)), \widetilde{\mu}_B(\Phi(x_5))\}$.
3. $\widetilde{\nu}_B(y_1 + y_2)) = \widetilde{\nu}_B(\Phi(x_1) + \Phi(x_2)) = \widetilde{\nu}_B(\Phi(x_1 + x_2)) = \Phi^{-1}(\widetilde{\nu}_B)(x_1 + x_2)$
$\leq Max^i\{\Phi^{-1}(\widetilde{\nu}_B)(x_1), \Phi^{-1}(\widetilde{\nu}_B)(x_2)\} = Max^i\{\widetilde{\nu}_B(\Phi(x_1)), \widetilde{\nu}_B(\Phi(x_2))\}$.
4. $\widetilde{\nu}_B(y_1y_2y_3y_4y_5)) = \widetilde{\nu}_B(\Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4)\Phi(x_5))$
$= \widetilde{\nu}_B(\Phi(x_1x_2x_3x_4x_5)) = \Phi^{-1}(\widetilde{\nu}_B)(x_1x_2x_3x_4x_5)$
$\leq Max^i\{\Phi^{-1}(\widetilde{\nu}_B)(x_1), \Phi^{-1}(\widetilde{\nu}_B)(x_3), \Phi^{-1}(\widetilde{\nu}_B)(x_5)\}$
$= Max^i\{\widetilde{\nu}_B(\Phi(x_1)), \widetilde{\nu}_B(\Phi(x_3)), \widetilde{\nu}_B(\Phi(x_5))\}$. Thus $B = (\widetilde{\mu}_B, \widetilde{\nu}_B)$ is an interval-valued intuitionistic fuzzy bi-ideal of S_2. □

4. Normal interval-valued intuitionistic fuzzy right ideals

Definition 4.1. An interval-valued intuitionistic fuzzy right (left, lateral) ideal $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ of a ternary semiring S is said to be normal if $A(0) = (\widetilde{1}, \widetilde{0})$, that means $\widetilde{\mu}_A(0) = \widetilde{1}$, $\widetilde{\nu}_A(0) = \widetilde{0}$. Denote by NIIFRI(S) (NIIFLI(S), NIIFMI(S)) the set of all normal interval-valued intuitionistic fuzzy right (left, lateral) ideals of S. Note that NIIFRI(S) (NIIFLI(S), NIIFMI(S)) is a poset under set inclusion.

Example 4.2. Consider the ternary semiring $S = Z_0^-$, the set of all non positive integers with usual addition and ternary multiplication. Let the interval-valued fuzzy subset $\widetilde{\mu}_A$ and $\widetilde{\nu}_A$ of S be defined by

$$\widetilde{\mu}_A(x) = \begin{cases} \widetilde{1}, \text{ if } x = 0 \\ [0.5, 0.6], \text{ if } x \in \langle -2 \rangle \setminus \{0\} \\ [0.2, 0.3], \text{ otherwise,} \end{cases}$$

$$\widetilde{\nu}_A(x) = \begin{cases} \widetilde{0}, \text{ if } x = 0 \\ [0.1, 0.2], \text{ if } x \in \langle -2 \rangle \setminus \{0\} \\ [0.4, 0.7], \text{ otherwise.} \end{cases}$$

Then $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ is a normal interval-valued intuitionistic fuzzy ideal of S.

Theorem 4.3. *Given an interval-valued intuitionistic fuzzy right (left, lateral) ideal $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ of a ternary semiring S. Let $\widetilde{\mu}_A^+(x) = \widetilde{\mu}_A(x) + \widetilde{1} - \widetilde{\mu}_A(0)$ and $\widetilde{\nu}_A^+(x) = \widetilde{\nu}_A(x) - \widetilde{\nu}_A(0)$, for all $x \in S$. Then $A^+ = (\widetilde{\mu}_A^+, \widetilde{\nu}_A^+)$ is a normal interval-valued intuitionistic fuzzy right (left, lateral) ideal containing $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ of S.*

Proof. For any $x, y, z \in S$
1. $\widetilde{\mu}_A^+(x+y) = \widetilde{\mu}_A(x+y) + \widetilde{1} - \widetilde{\mu}_A(0) \geq Min^i\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(y)\} + \widetilde{1} - \widetilde{\mu}_A(0)$
$= Min^i\{\widetilde{\mu}_A(x) + \widetilde{1} - \widetilde{\mu}_A(0), \widetilde{\mu}_A(y) + \widetilde{1} - \widetilde{\mu}_A(0)\} = Min^i\{\widetilde{\mu}_A^+(x), \widetilde{\mu}_A^+(y)\}$.
2. $\widetilde{\mu}_A^+(xyz) = \widetilde{\mu}_A(xyz) + \widetilde{1} - \widetilde{\mu}_A(0) \geq \widetilde{\mu}_A(x) + \widetilde{1} - \widetilde{\mu}_A(0) = \widetilde{\mu}_A^+(x)$.
3. $\widetilde{\nu}_A^+(x+y) = \widetilde{\nu}_A(x+y) - \widetilde{\nu}_A(0) \leq Max^i\{\widetilde{\nu}_A(x), \widetilde{\nu}_A(y)\} - \widetilde{\nu}_A(0)$
$= Max^i\{\widetilde{\nu}_A(x) - \widetilde{\nu}_A(0), \widetilde{\nu}_A(y) - \widetilde{\nu}_A(0)\} = Max^i\{\widetilde{\nu}_A^+(x), \widetilde{\nu}_A^+(y)\}$.
4. $\widetilde{\nu}_A^+(xyz) = \widetilde{\nu}_A(xyz) - \widetilde{\nu}_A(0) \leq \widetilde{\nu}_A(x) - \widetilde{\nu}_A(0) = \widetilde{\nu}_A^+(x)$.

Hence A^+ is an interval-valued intuitionistic fuzzy right ideal of S. Again we have $\widetilde{\mu}_A^+(0) = \widetilde{\mu}_A(0) + \widetilde{1} - \widetilde{\mu}_A(0) = \widetilde{1}$ and $\widetilde{\nu}_A^+(0) = \widetilde{\nu}_A(0) - \widetilde{\nu}_A(0) = \widetilde{0}$. Hence A^+ is a normal interval-valued intuitionistic fuzzy right ideal of S and by definition $A \subseteq A^+$. □

Corollary 4.4. *Let A and A^+ be as in the Theorem 4.3. A is a normal interval-valued intuitionistic fuzzy right ideal of S if and only if $A^+ = A$.*

Remark 4.5. If $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ is an interval-valued intuitionistic fuzzy right (left, lateral) ideal of S, then $(A^+)^+ = A^+$. In particular, if A is normal, then $(A^+)^+ = A^+ = A$.

Theorem 4.6. *Let $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ be an interval-valued intuitionistic fuzzy right (left, lateral) ideal of a ternary semiring S and let $\Phi : D[0,1] \to D[0,1]$ be an increasing function. Then an IIFS $A_\Phi = ((\widetilde{\mu}_A)_\Phi, (\widetilde{\nu}_A)_\Phi)$ where $(\widetilde{\mu}_A)_\Phi(x) = \Phi(\widetilde{\mu}_A(x))$ and $(\widetilde{\nu}_A)_\Phi(x) = \Phi(\widetilde{\nu}_A(x))$ for all $x \in S$ is an interval-valued intuitionistic fuzzy right (left, lateral) ideal of S. Moreover, if $\Phi(\widetilde{\mu}_A(0)) = \widetilde{1}$ and $\Phi(\widetilde{\nu}_A(0)) = \widetilde{0}$, then A_Φ is normal.*

Proof. Let $x, y, z \in S$.
1. $(\widetilde{\mu}_A)_\Phi(x+y) = \Phi(\widetilde{\mu}_A(x+y)) \geq \Phi(Min^i\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(y)\})$
$= Min^i\{\Phi(\widetilde{\mu}_A(x)), \Phi(\widetilde{\mu}_A(y))\} = Min^i\{(\widetilde{\mu}_A)_\Phi(x), (\widetilde{\mu}_A)_\Phi(y)\}$
2. $(\widetilde{\mu}_A)_\Phi(xyz) = \Phi(\widetilde{\mu}_A(xyz)) \geq \Phi(\widetilde{\mu}_A(x)) = (\widetilde{\mu}_A)_\Phi(x)$
3. $(\widetilde{\nu}_A)_\Phi(x+y) = \Phi(\widetilde{\nu}_A(x+y)) \leq \Phi(Max^i\{\widetilde{\nu}_A(x), \widetilde{\nu}_A(y)\})$
$= Max^i\{\Phi(\widetilde{\nu}_A(x)), \Phi(\widetilde{\nu}_A(y))\} = Max^i\{(\widetilde{\nu}_A)_\Phi(x), (\widetilde{\nu}_A)_\Phi(y)\}$

4. $(\widetilde{\nu}_A)_\Phi(xyz) = \Phi(\widetilde{\nu}_A(xyz)) \leq \Phi(\widetilde{\nu}_A(x)) = (\widetilde{\nu}_A)_\Phi(x)$.
Hence A_Φ is an interval-valued intuitionistic fuzzy right ideal of S. If $\Phi(\widetilde{\mu}_A(0)) = \widetilde{1}$, $\Phi(\widetilde{\nu}_A(0)) = \widetilde{0}$ then $(\widetilde{\mu}_A)_\Phi(0) = \widetilde{1}$ and $(\widetilde{\nu}_A)_\Phi(0) = \widetilde{0}$ and hence $A_\Phi = ((\widetilde{\mu}_A)_\Phi, (\widetilde{\nu}_A)_\Phi)$ is a normal interval-valued intuitionistic fuzzy right ideal of S. □

Definition 4.7. An interval-valued intuitionistic fuzzy ideal $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ of a ternary semiring S is said to be an interval-valued intuitionistic fuzzy maximal if it satisfies:
i) A is non-constant.
ii) A^+ is a maximal element of NIIFI(S), where NIIFI(S) denotes the set of all normal interval-valued intuitionistic fuzzy ideal of S.

Example 4.8. Consider the ternary semiring $S = Z_0^-$, the set of all non positive integers with the usual addition and ternary multiplication. Let the interval-valued fuzzy subset $\widetilde{\mu}_A$ and $\widetilde{\nu}_A$ of S be defined by

$$\widetilde{\mu}_A(x) = \begin{cases} \widetilde{1}, & \text{if } x \in \langle -2 \rangle \\ \widetilde{0}, & \text{otherwise,} \end{cases}$$

$$\widetilde{\nu}_A(x) = \begin{cases} \widetilde{0}, & \text{if } x \in \langle -2 \rangle \\ \widetilde{1}, & \text{otherwise.} \end{cases}$$

Then $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ is an interval-valued intuitionistic fuzzy maximal ideal of S.

Theorem 4.9. *Let $A = (\widetilde{\mu}_A, \widetilde{\nu}_A) \in NIIFRI(S)$ be non-constant such that it is maximal in the poset of NIIFRI(S) under set inclusion. Then both $\widetilde{\mu}_A$ and $\widetilde{\nu}_A$ takes only the values $(\widetilde{1}, \widetilde{0})$ and $(\widetilde{0}, \widetilde{1})$ respectively.*

Proof. Since A is normal interval-valued intuitionistic fuzzy right ideal, so $A(0) = (\widetilde{1}, \widetilde{0})$. Let $x_0\ (\neq 0) \in S$ be arbitrary with $\widetilde{\mu}_A(x_0) \neq \widetilde{1}$. We claim that $\widetilde{\mu}_A(x_0) = \widetilde{0}$. If not then there exists an element $c \in S$ such that $\widetilde{0} < \widetilde{\mu}_A(c) < \widetilde{1}$. Let $A_c = (\widetilde{\sigma}_A, \widetilde{\eta}_A)$ be an interval-valued intuitionistic fuzzy subset of S defined by $\widetilde{\sigma}_A(x) = \frac{1}{2}[\widetilde{\mu}_A(x) + \widetilde{\mu}_A(c)]$, $\widetilde{\eta}_A(x) = \frac{1}{2}[\widetilde{\nu}_A(x) + \widetilde{\nu}_A(c)]$. Clearly A_c is well-defined. Now,

$$\widetilde{\sigma}_A(0) = \frac{1}{2}[\widetilde{\mu}_A(0) + \widetilde{\mu}_A(c)] \geq \frac{1}{2}[\widetilde{\mu}_A(x) + \widetilde{\mu}_A(c)] = \widetilde{\sigma}_A(x),$$

$$\widetilde{\eta}_A(0) = \frac{1}{2}[\widetilde{\nu}_A(0) + \widetilde{\nu}_A(c)] \leq \frac{1}{2}[\widetilde{\nu}_A(x) + \widetilde{\nu}_A(c)] = \widetilde{\eta}_A(x),$$

for any $x \in S$. Again, for any $x, y, z \in S$,
1. $\widetilde{\sigma}_A(x + y) = \frac{1}{2}[\widetilde{\mu}_A(x + y) + \widetilde{\mu}_A(c)] \geq \frac{1}{2}[Min^i\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(y)\} + \widetilde{\mu}_A(c)]$
$= Min^i\{\frac{1}{2}[\widetilde{\mu}_A(x) + \widetilde{\mu}_A(c)], \frac{1}{2}[\widetilde{\mu}_A(y) + \widetilde{\mu}_A(c)]\} = Min^i\{\widetilde{\sigma}_A(x), \widetilde{\sigma}_A(y)\}$.
2. $\widetilde{\sigma}_A(xyz) = \frac{1}{2}[\widetilde{\mu}_A(xyz) + \widetilde{\mu}_A(c)] \geq \frac{1}{2}[\widetilde{\mu}_A(x) + \widetilde{\mu}_A(c)] = \widetilde{\sigma}_A(x)$.
3. $\widetilde{\eta}_A(x + y) = \frac{1}{2}[\widetilde{\nu}_A(x + y) + \widetilde{\nu}_A(c)] \leq \frac{1}{2}[Max^i\{\widetilde{\nu}_A(x), \widetilde{\nu}_A(y)\} + \widetilde{\nu}_A(c)]$
$= Max^i\{\frac{1}{2}[\widetilde{\nu}_A(x) + \widetilde{\nu}_A(c)], \frac{1}{2}[\widetilde{\nu}_A(y) + \widetilde{\nu}_A(c)]\} = Max^i\{\widetilde{\eta}_A(x), \widetilde{\eta}_A(y)\}$.
4. $\widetilde{\eta}_A(xyz) = \frac{1}{2}[\widetilde{\nu}_A(xyz) + \widetilde{\nu}_A(c)] \leq \frac{1}{2}[\widetilde{\nu}_A(x) + \widetilde{\nu}_A(c)] = \widetilde{\eta}_A(x)$.
Hence A_c is an interval-valued intuitionistic fuzzy right ideal of S. Define $A_c^+ = (\widetilde{\sigma}_A^+, \widetilde{\eta}_A^+)$. Then by Theorem 4.3, A_c^+ is a normal interval-valued intuitionistic fuzzy right ideal of S, where

$$\widetilde{\sigma}_A^+(x) = \widetilde{\sigma}_A(x) + \widetilde{1} - \widetilde{\sigma}_A(0) = \frac{1}{2}[\widetilde{\mu}_A(x) + \widetilde{\mu}_A(c)] + \widetilde{1} - \frac{1}{2}[\widetilde{\mu}_A(0) + \widetilde{\mu}_A(c)] = \frac{1}{2}[1 + \widetilde{\mu}_A(x)]$$

and

$$\widetilde{\eta}_A^+(x) = \widetilde{\eta}_A(x) - \widetilde{\eta}_A(0) = \frac{1}{2}[\widetilde{\nu}_A(x) + \widetilde{\nu}_A(c)] - \frac{1}{2}[\widetilde{\nu}_A(0) + \widetilde{\nu}_A(c)] = \frac{1}{2}[\widetilde{\nu}_A(x)].$$

Clearly $A \subseteq A_c^+$. Since $\widetilde{\sigma}_A^+(x) = \frac{1}{2}[1 + \widetilde{\mu}_A(x)] > \widetilde{\mu}_A(x)$ and $\widetilde{\eta}_A^+(x) = \frac{1}{2}[\widetilde{\nu}_A(x)] \leq \widetilde{\nu}_A(x)$, A is a proper subset of A_c^+. Again since $\widetilde{\sigma}_A^+(c) = \frac{1}{2}[1 + \widetilde{\mu}_A(c)] < \widetilde{1} = \widetilde{\sigma}_A^+(0)$. Hence A_c^+ is non-constant and A is not a maximal element of NIIFRI(S). This is a contradiction. Therefore $\widetilde{\mu}_A$ takes only two values $\widetilde{1}$ and $\widetilde{0}$. Hence $\widetilde{\nu}_A$ takes the values $\widetilde{0}$ and $\widetilde{1}$. This completes the proof. □

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Department of Mathematics, Annamalai University,Annamalai Nagar,Tamilnadu, India.
E-mail address: krishna_swamy2004@yahoo.co.in

Mathematics Wing, D.D.E , Annamalai University,Annamalai Nagar,Tamilnadu, India.
E-mail address: joe.jayaraj@gmail.com

Mathematics Wing, D.D.E , Annamalai University,Annamalai Nagar,Tamilnadu, India.
E-mail address: anitha81t@gmail.com