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MARKOV GRAPHS OF ONE–DIMENSIONAL DYNAMICAL SYSTEMS AND THEIR DISCRETE ANALOGUES

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Abstract. One feature of the famous Sharkovsky's theorem is that it can be proved using digraphs of a special type (the so–called Markov graphs). The most general definition assigns a Markov graph to every continuous map from the topological graph to itself. We show that this definition is too broad, i.e. every finite digraph can be viewed as a Markov graph of some one–dimensional dynamical system on a tree. We therefore consider discrete analogues of Markov graphs for vertex maps on combinatorial trees and characterize all maps on trees whose discrete Markov graphs are of the following types: complete, complete bipartite, the disjoint union of cycles, with every arc being a loop.

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1. Introduction

Let $f : [0,1] \to [0,1]$ be a continuous map of the closed unit interval to itself which has a periodic point $x \in [0,1]$ with period $n \in \mathbb{N}$. Denote the restriction of f to $orb(x) = \{x, f(x), \dots, f^{n-1}(x)\}$ as σ. Thus σ is a cyclic permutation of $orb(x)$. Also, let $orb(x) = \{x_1 < x_2 < ... < x_n\}$ be the natural ordering of $orb(x)$.

Since f is continuous, an f–image of every minimal closed interval $[x_i, x_{i+1}]$ covers all minimal closed intervals from $[f(x_i), f(x_{i+1})]$ (or $[f(x_{i+1}), f(x_i)]$).

The corresponding discrete Markov graph (or A–graph [4], or B–graph [12], or I–graph [6], or Straffin digraph [7],[8], or periodic graph [10]) is a digraph with the vertex set $\{1, \dots, n-1\}$ and there is an arc $i \to j$ if and only if

$$\min\{f(x_i), f(x_{i+1})\} \leq x_j < \max\{f(x_i), f(x_{i+1})\}.$$

Such discretization of a dynamical system is useful and also interesting. As it can be seen from [8] Markov graphs lie at the heart of an elegant and purely combinatorial proof of the famous Sharkovsky's theorem. Moreover, one can define Markov graphs for the special classes of continuous maps on arbitrary topological graphs and then prove an analogue of Sharkovsky's theorem for them [1]–[3].

Such a crucial role that Markov graphs play in one–dimensional dynamics stimulating us to study their properties from graph–theoretical point of view. There is a couple of papers [1],[9]–[11] which follow this line. In particular, Pavlenko [10] calculated the number of non–isomorphic periodic graphs with given number of vertices and gave a criterion for arbitrary finite digraph being periodic graph [11]. In [1] Bernhardt proved that the correspondence which assigns to every permutation of the vertex set of a finite tree the adjacency matrix of its discrete Markov graph is an injective homomorphism (i.e. a representation) from S_n into $Gl(n-1, 2)$. One can also show that this representation is irreducible.

The present paper continues this research. We will show that any finite digraph Γ is isomorphic to a Markov graph $\Gamma(X, f)$, where X is a topological tree and then present some bounds on indegrees and outdegrees of vertices in Markov graphs (these bounds become equalities for discrete Markov graphs). We also prove that in fact the correspondence $\sigma \to M_{\Gamma(X,\sigma)}$ (here $M_{\Gamma(X,\sigma)}$ denotes the adjacency matrix of the discrete Markov graph $\Gamma(X, \sigma)$) establishes "almost injective" homomorphism from the semigroup of all vertex maps on X into $Mat\,(n-1, 2)$. Moreover, we describe all maps on trees whose discrete Markov graphs are of the following types: complete, complete bipartite, the disjoint union of cycles, with every arc being a loop.

2. Definitions and some results

This paper deals with finite topological graphs. The closed interval $[x, y]$ is called an *edge*. Its two end–points x and y are called *vertices*. Thus, a *graph* X is a finite collection of edges with the property that the intersection of two of them is empty or consists of one vertex. By $V(X)$ and $E(X)$ we denote the vertex set and the edge set of X, respectively. Further, for every $A \subset V(X)$ the symbol $X[A]$ denotes the subgraph of X induced by A.

Two graphs are *isomorphic* if there exists a bijection between their vertex sets which preserves adjacency in both ways. Every such bijection is called an *isomorphism*. An *automorphism* is a graph isomorphism to itself.

A graph is called *connected* if for every pair of vertices there exists a path joining them. If X is a connected graph, then $d_X(u, v)$ denotes the *distance* (which is the number of edges on a shortest $u - v$ path) between two vertices u and v. The diameter $diam\,X$ of a given connected graph X is the maximal distance between the pairs of its vertices.

The *degree* $d_X(u)$ of the vertex $u \in V(X)$ in a graph X is the number of edges incident to u, i.e. $d_X(u) = |\{v \in V(X) : uv \in E(X)\}|$.

A *tree* is a connected graph without cycles. If X is a tree, then the vertex $u \in V(X)$ is called a *leaf* provided $d_X(u) = 1$. The set of all leaf vertices of X will be denoted as $L(X)$. A *star* is a tree with a unique non-leaf vertex which is called the *center* of a star.

Let X be a graph, $u \in V(X)$ and $e \in E(X)$. A graph $X - \{u\}$ is obtained from X by deletion the vertex u with all interiors of edges incident to u. Similarly, a graph $X - \{e\}$ is obtained from X by deletion only the interior of the edge e.

A *digraph* Γ is a pair (V, A), where $A \subset V \times V$. The sets $V = V(\Gamma)$ and $A = A(\Gamma)$ are called the *vertex set* and the *arc set* of Γ. If $(u, u) \in A(\Gamma)$, then we say that the vertex u has a *loop*. A digraph Γ is called *empty* if $A(\Gamma) = \emptyset$. Similarly, a digraph Γ is called *complete* if $A(\Gamma) = V \times V$.

Now let $a, b \geq 1$ be two integers. A *complete bipartite* digraph $K_{a,b}$ is a digraph such that the vertex set $V(K_{a,b})$ can be partitioned into two disjoint subsets $V(K_{a,b}) = A \cup B$ with $|A| = a$, $|B| = b$ and $A(K_{a,b}) = (A \times B) \cup (B \times A)$.

For each vertex $u \in V(\Gamma)$ in a digraph Γ consider the next sets $N^+_\Gamma(u) = \{v \in V(\Gamma) : (u, v) \in A(\Gamma)\}$ and $N^-_\Gamma(u) = \{v \in V(\Gamma) : (v, u) \in A(\Gamma)\}$. The cardinalities $d^+_\Gamma(u) := |N^+_\Gamma(u)|$ and $d^-_\Gamma(u) := |N^-_\Gamma(u)|$ are called the *outdegree* and the *indegree* of the vertex u, respectively.

Recall also some basic definitions from topological dynamics. A *dynamical system* is a pair (X, f), where X is a topological space and $f : X \to X$ is a continuous map from X to itself. An element $x \in X$ is called *periodic point* if there exists $n \geq 1$ such that $f^n(x) = x$. The smallest n satisfying the condition above is called the *period* of x. If $f(x) = x$ then x is called a *fixed point*.

Now we move to the one–dimensional dynamical systems that is dynamical systems on topological graphs.

Definition 2.1. Let X be a graph and $f : X \to X$ be a continuous map.

We will call f a

- *vertex map* if $f(V(X)) \subset V(X)$;
- *permutation map* if $f(V(X)) = V(X)$ (i.e. if restriction $f|_{V(X)}$ is a permutation of $V(X)$).

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Suppose X is a graph and f is a vertex map on X.

Definition 2.2. The *Markov graph* $\Gamma = \Gamma(X, f)$ is a directed graph with the vertex set $V(\Gamma) := \{v_e : e \in E(X)\}$ and there exists an arc $v_{e_1} \to v_{e_2}$ if e_1 f–covers e_2, i.e. if $e_2 \subset f(e_1)$.

The following result shows why Markov graphs provide a useful tool in combinatorial dynamics.

Lemma 2.3. [5] *Let X be a graph and $f : X \to X$ be a permutation map on X. Also, let $v_{e_0} \to \ldots \to v_{e_n} \to v_{e_0}$ be some cycle in $\Gamma(X, f)$. Then there exists a periodic point $x \in X$ such that $f^{n+1}(x) = x$ and $f^k(x) \in e_k$ for every $k \in \overline{0, n}$. Moreover, if the cycle $v_{e_1} \to \ldots \to v_{e_n} \to v_{e_1}$ is primitive (i.e. it does not contain a proper subcycle), then period of x equals to $n+1$. Conversely, every periodic point $x \in X$ of period $n+1$ corresponds to some primitive cycle of length $n+1$ in $\Gamma(X, f)$.*

Now we show that every finite digraph can be viewed as a Markov graph of some one–dimensional dynamical system.

Theorem 2.4. *Let Γ be finite digraph. Then there exists a graph X and a vertex map $f : X \to X$ such that $\Gamma(X, f)$ is isomorphic to Γ.*

Proof. Let $|V(\Gamma)| = n$. The construction of X depends only on n.

Consider the unit circle on a complex plane. Define

$$V(X) := \{\exp(\frac{2\pi i}{n}k) : 0 \le k \le n-1\} \cup \{0\}$$

and

$$E(X) := \{[0, \exp(\frac{2\pi i}{n}k)] : 0 \le k \le n-1\},$$

where $[z_1, z_2]$ denotes the closed line segment between $z_1, z_2 \in \mathbb{C}$. The obtained graph X is just a plane embedding of a star.

At first, put $f(0) := 0$. Further, let $V(\Gamma) = \{u_0, \ldots, u_{n-1}\}$. For any $k \in \overline{0, n-1}$ consider the set $N^+_\Gamma(u_k)$. If $N^+_\Gamma(u_k)$ is empty, then put $f(z) := 0$ for all $z \in [0, \exp(\frac{2\pi i}{n}k)]$.

Now suppose that $N^+_\Gamma(u_k) = \{u_{k_1}, \ldots, u_{k_m}\}$.

Fix some point $z \in [0, \exp(\frac{2\pi i}{n}k)]$. Then $z = r\exp(\frac{2\pi i}{n}k)$, where $0 \le r \le 1$.

Put

$$f(z) := \begin{cases} \{2mr\}\exp(\frac{2\pi i}{n}k_{\frac{\lfloor 2mr\rfloor+2}{2}}), & \text{if } \lfloor 2mr\rfloor \text{ is even,} \\ (1-\{2mr\})\exp(\frac{2\pi i}{n}k_{\frac{\lfloor 2mr\rfloor+1}{2}}), & \text{if } \lfloor 2mr\rfloor \text{ is odd,} \end{cases}$$

where the symbols $\{x\}$ and $\lfloor x\rfloor$ denote the fractional and integer parts of $x \in \mathbb{R}$, respectively.

It is easy to see that f is a vertex map on X.

From the construction of f it follows that

$$f([0, \exp(\frac{2\pi i}{n}k)]) = \bigcup_{i=1}^{m}[0, \exp(\frac{2\pi i}{n}k_i)]$$

for all $k \in \overline{0, n-1}$. In other words, $f([0, \exp(\frac{2\pi i}{n}k)])$ covers exactly those edges $[0, \exp(\frac{2\pi i}{n}l)]$ for which there exists an arc $u_k \to u_l$ in Γ. Therefore, the map

$$\phi : V(\Gamma) \to V(\Gamma(X, f)),\ \phi(u_k) := v_{[0, \exp(\frac{2\pi i}{n}k)]}$$

is an isomorphism between Γ and $\Gamma(X, f)$. $\square$

Denote as $k(X)$ the number of connected components in X. Recall that an edge $e \in E(X)$ is a *bridge* if $k(X - \{e\}) > k(X)$.

Proposition 2.5. *Let X be a connected graph, $f : X \to X$ be a vertex map on X and $\Gamma = \Gamma(X, f)$ be the corresponding Markov graph. Then*

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(1) *for every edge* $e = uv \in E(X)$ *we have*

$$d^+_\Gamma(v_e) \geq d_X(f(u), f(v));$$

(2) *if the edge* $e \in E(X)$ *is a bridge and* X_1, X_2 *are components of* $X - \{e\}$*, then*
a) $d^-_\Gamma(v_e) \geq k(f^{-1}(X_1)) + k(f^{-1}(X_2)) - 1$;
b) if also v_e *doesn't have a loop in* Γ*, then*

$$d^+_\Gamma(v_e) \leq \max\{|E(X_1)|, |E(X_2)|\}.$$

Proof. The inequalities 1 and 2.b) are clearly follow from continuity of f. Now consider the new graph X' whose vertices are in one–to–one correspondence with connected components of $f^{-1}(X_1)$ and $f^{-1}(X_2)$. Two vertices from X' are adjacent if the corresponding components share a common edge in X.

It is easy to see that edges in X' correspond precisely to those edges in X which f–cover e. Therefore, $d^-_\Gamma(v_e) = |E(X')|$. But since X is connected, X' is also connected. Thus,

$$d^-_\Gamma(v_e) = |E(X')| \geq |V(X')| - 1 = k(f^{-1}(X_1)) + k(f^{-1}(X_2)) - 1.$$

This proves the inequality 2.a). □

3. Only for trees

3.1. **Basics.** Let X be a tree and $f : X \to X$ be a vertex map on X. Denote the unique shortest path between each pair of vertices $u, v \in V(X)$ as $[u, v]_X$.

Since f is continuous, then for every edge $e = uv \in E(X)$ we have

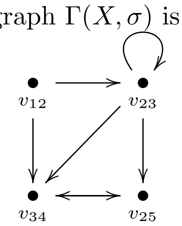
$$[f(u), f(v)]_X \subseteq f(e).$$

We will call f a *minimal map* if the equality $[f(u), f(v)]_X = f(e)$ holds for every $e = uv \in E(X)$.

Definition 3.1. Let X be a tree and $\sigma : V(X) \to V(X)$ be some map. The *discrete Markov graph* $\Gamma(X, \sigma)$ is a digraph $\Gamma(X, f)$, where f is some minimal map on X with $f|_{V(X)} = \sigma$.

Note that this definition is correct since for every two minimal maps f_1 and f_2 on X with $f_1|_{V(X)} = f_2|_{V(X)}$ we necessarily have $\Gamma(X, f_1) = \Gamma(X, f_2)$. Therefore $\Gamma(X, \sigma)$ is a purely combinatorial object and it can be defined for combinatorial trees and their vertex maps.

Example 3.2. Consider a tree X with $V(X) = \{1, 2, 3, 4, 5\}, E(X) = \{12, 23, 25, 34\}$ and a map $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 2 & 3 \end{pmatrix}$. The discrete Markov graph $\Gamma(X, \sigma)$ is following



In Proposition 2.5 we obtained several bounds on indegrees and outdegrees of vertices in arbitrary Markov graphs. For discrete Markov graphs those bounds become equalities. To show this define for each edge $e = uv \in E(X)$ in a tree X the next "half–space"

$$A_X(u, v) := \{x \in V(X) : d_X(x, u) \leq d_X(x, v)\}.$$

It is easy to see that $X[A_X(u, v)]$ and $X[A_X(v, u)]$ are two connected components in $X - \{e\}$.

Proposition 3.3. *Let* X *be a tree,* $\sigma : V(X) \to V(X)$ *and* $\Gamma = \Gamma(X, \sigma)$*. For every edge* $e = uv \in E(X)$ *the next equalities hold*

(1) $d^+_\Gamma(v_e) = d_X(\sigma(u), \sigma(v))$;
(2) $d^-_\Gamma(v_e) = k(X[\sigma^{-1}(A_X(u, v))]) + k(X[\sigma^{-1}(A_X(v, u))]) - 1$.

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Proof. These are follow from the fact that each tree is a connected graph with every edge being a bridge, Proposition 2.5 and definition of discrete Markov graph. □

Corollary 3.4. *Let X be a tree, $\sigma : V(X) \to V(X)$ and $\Gamma = \Gamma(X, \sigma)$. Then $|\sigma(V(X))| - 1 \leq |A(\Gamma)| \leq (n-1) \cdot diam\, \sigma(V(X))$.*

Proof. On one hand, $|A(\Gamma)| = \sum_{v \in V(\Gamma)} d^+_\Gamma(v)$, so the upper bound clearly follows from the claim 1 of Proposition 3.3.

On the other hand, $|A(\Gamma)| = \sum_{v \in V(\Gamma)} d^-_\Gamma(v)$. From claim 2 of Proposition 3.3 we obtain $d^-_\Gamma(v_e) \geq 1$ for all $e \in E(X[\sigma(V(X))])$. Therefore, $|A(\Gamma)| \geq |\sigma(V(X))| - 1$. □

In what follows $fix\, \sigma$ denotes the set of all fixed points of σ.

Proposition 3.5. [9] *Let X be a tree and σ be a permutation of $V(X)$ with $fix\, \sigma = \emptyset$. Then $\Gamma(X, \sigma)$ has a vertex with a loop.*

Note that Proposition 3.5 fails if $fix\, \sigma \neq \emptyset$. Just consider a path with three vertices and its unique nontrivial automorphism.

Let S_n be the symmetric group of all permutations of n–element set. For every permutation $\sigma \in S_n$ one can construct its permutation matrix P_σ which is a square $(0,1)$–matrix of order n. A well–known fact from group theory is that the correspondence $\sigma \to P_\sigma$ gives an injective homomorphism (i.e. a faithful representation) from S_n into the general linear group $Gl(n, 2)$.

It is rather surprising that using discrete Markov graphs one can construct a faithful irreducible representation of S_n into $Gl(n-1, 2)$. By M_Γ we denote the adjacency matrix of Γ.

Theorem 3.6. [1] *Let X be a tree with n vertices. Suppose that some linear ordering of $E(X)$ is fixed. Then $M : S_n \to Gl(n-1, 2)$, where $M(\sigma) := M_{\Gamma(X,\sigma)}$ is a faithful irreducible representation of S_n.*

The correspondence $\sigma \to M_{\Gamma(X,\sigma)}$ in fact establishes a homomorphism from the semigroup of all maps $V(X)^{V(X)}$ to the matrix semigroup $Mat\,(n-1, 2)$. To present this result, we need the definition of the projection map and the following simple lemma which can be proved by induction.

Definition 3.7. Let X be a tree and $H \subset X$ be its subtree. Since X is connected and acyclic, for every vertex $u \in V(X)$ there exists a unique vertex $v \in V(H)$ such that $d_X(u, v) \leq d_X(u, x)$ for every $x \in V(H)$. The map $pr_H : V(X) \to V(X)$, $pr_H(u) := v$ is called a *projection* on H.

Note that every constant map is a projection on a singleton subtree. Also, for all edges $e = uv \in E(X)$ we have $pr_e^{-1}(u) = A_X(u, v)$.

Recall that for a given (vertex) coloring of a graph its edge is called *properly colored* if the corresponding vertices receive different colors. The following lemma (which can be easily proved by induction) tells that for any 2–coloring of a path its two leaf vertices belong to different color classes if and nonly if there is an odd number of properly colored edges.

Lemma 3.8. *Let $X = \{u_1 - ... - u_n\}$ be a path with $n \geq 2$ vertices, S be a set with two elements and $f : V(X) \to S$. Then $f(u_1) \neq f(u_n)$ if and only if the number of indices $k \in \overline{1, n-1}$ with $f(u_k) \neq f(u_{k+1})$ is odd.*

Theorem 3.9. *Let X be a tree with n vertices. Suppose that some linear ordering of $E(X)$ is fixed. Then the function $M : V(X)^{V(X)} \to Mat\,(n-1, 2)$, $M(\sigma) := M_{\Gamma(X,\sigma)}$ is a semigroup homomorphism.*

Proof. Let $E(X) = \{e_1, \ldots, e_{n-1}\}$ be a linear ordering of the edges. Fix two maps $\sigma_{1,2} \in V(X)^{V(X)}$ and put $M_{\Gamma(X,\sigma_1)} = (\alpha_{ij})$, $M_{\Gamma(X,\sigma_2)} = (\beta_{ij})$, $M_{\Gamma(X,\sigma_1)} M_{\Gamma(X,\sigma_2)} = (a_{ij})$, $M_{\Gamma(X,\sigma_2 \circ \sigma_1)} = (b_{ij})$.

Observe that if $a_{ij} = 1$, then $\sum_{k=1}^{n-1} \alpha_{ik} \beta_{kj} = 1 \bmod 2$. This means that the number K_{ij} of indices $k \in \overline{1, n-1}$ with $v_{e_i} \to v_{e_k}$ in $\Gamma(X, \sigma_1)$ and $v_{e_k} \to v_{e_j}$ in $\Gamma(X, \sigma_2)$ is odd.

On the other hand, for all trees X and maps $\sigma : V(X) \to V(X)$ there exists an arc $v_{e_1} \to v_{e_2}$ in $\Gamma(X, \sigma)$ if and only if $pr_{e_2}(\sigma(u_1)) \neq pr_{e_2}(\sigma(v_1))$, where $e_1 = u_1 v_1, e_2 \in E(X)$. This implies that K_{ij}

equals to the number of edges $e = uv$ from $[\sigma_1(u_i), \sigma_1(v_i)]_X$ with $pr_{e_j}(\sigma_2(u)) \neq pr_{e_j}(\sigma_2(v))$. Since K_{ij} is odd, from Lemma 3.8 it follows that $pr_{e_j}(\sigma_2(\sigma_1(u_i))) \neq pr_{e_j}(\sigma_2(\sigma_1(v_i)))$ and therefore $v_{e_i} \to v_{e_j}$ in $\Gamma(X, \sigma_2 \circ \sigma_1)$. This implies $b_{ij} = 1$.

Moreover, Lemma 3.8 is a criterion, so $a_{ij} = 0$ also implies $b_{ij} = 0$ and we can conclude that $M_{\Gamma(X,\sigma_1)} M_{\Gamma(X,\sigma_2)} = M_{\Gamma(X,\sigma_2 \circ \sigma_1)}$. □

It is evident that each constant map has the empty discrete Markov graph. Therefore, generally speaking, the correspondence $\sigma \to M_{\Gamma(X,\sigma)}$ doesn't provide an injective homomorphism from $V(X)^{V(X)}$ into $Mat\,(n-1, 2)$. Nevertheless, one can show that arbitrary tree map almost always can be reconstructed from its labelled discrete Markov graph, i.e. that the correspondence $\sigma \to M_{\Gamma(X,\sigma)}$ is "almost injective".

Proposition 3.10. *Let X be a tree with $|V(X)| \geq 2$ and $\sigma : V(X) \to V(X)$. Then there exists $\sigma' : V(X) \to V(X)$, $\sigma' \neq \sigma$ such that $\Gamma(X, \sigma) = \Gamma(X, \sigma')$ if and only if $|\sigma(V(X))| \leq 2$.*

Proof. Let $|\sigma(V(X))| = 1$. Then $\sigma = const$ on $V(X)$. Choose a vertex $v \in V(X) - \sigma(V(X))$ and define a constant map $\sigma'(x) := v$ for all $x \in V(X)$. Obviously $\Gamma(X, \sigma)$ and $\Gamma(X, \sigma')$ are empty digraphs.

Now let $\sigma(V(X)) = \{u_1, u_2\}$. For all $x \in V(X)$ set

$$\sigma^*(x) := \begin{cases} u_1, & \text{if } \sigma(x) = u_2, \\ u_2, & \text{if } \sigma(x) = u_1. \end{cases}$$

It is easy to see that $\Gamma(X, \sigma) = \Gamma(X, \sigma^*)$. Thus, the above condition is sufficient.

Now we show the necessity. Since $\sigma' \neq \sigma$, then $\sigma(u_0) \neq \sigma'(u_0)$ for some vertex $u_0 \in V(X)$. Let us use the induction on $d_X(u_0, v)$ to show that $\sigma(v) \neq \sigma'(v)$ and $\sigma(v), \sigma'(v) \in \{\sigma'(u), \sigma(u)\}$ for all vertices $v \in V(X)$. At first, let $d_X(u_0, v) = 1$. Since $\Gamma(X, \sigma) = \Gamma(X, \sigma')$, we have $[\sigma(u_0), \sigma(v)]_X = [\sigma'(u_0), \sigma'(v)]_X$. Suppose now that $\sigma(v) = \sigma(u_0)$. Then $\sigma'(v) = \sigma'(u_0)$, which means $\sigma(v) \neq \sigma'(v)$ and $\sigma(v), \sigma'(v) \in \{\sigma'(u), \sigma(u)\}$. Otherwise, let $\sigma(v) \neq \sigma(u_0)$. Since $\sigma(u_0) \neq \sigma'(u_0)$, it holds $\sigma(u_0) = \sigma'(v)$. Therefore $\sigma(v) = \sigma'(u_0)$. Again, $\sigma(v) \neq \sigma'(v)$ and $\sigma(v), \sigma'(v) \in \{\sigma'(u), \sigma(u)\}$.

Now let $d_X(u_0, v) = n + 1$. Choose a vertex $x \in [u_0, v]_X$ adjacent to v. By induction hypothesis $\sigma(x) \neq \sigma'(x)$ and $\sigma(x), \sigma'(x) \in \{\sigma'(u), \sigma(u)\}$. The induction step now can be proven in full analogy with induction base by considering the edge xv instead of $u_0 v$. □

Corollary 3.11. *Let X be a tree and $\sigma_1, \sigma_2 : V(X) \to V(X)$. Then $\Gamma(X, \sigma_1) = \Gamma(X, \sigma_2)$ if and only if $\sigma_1 = \sigma_2$, or σ_1, σ_2 are two constant maps, or $|\sigma_1(V(X))| = 2$ and $\sigma_1^* = \sigma_2$.*

Corollary 3.12. *The number of labelled discrete Markov graphs for any n–vertex tree equals to $n^n - (2^{n-2}n - \frac{n}{2} + 1)(n - 1)$.*

Proof. First, we have n constant maps with empty discrete Markov graphs. The number of maps σ with $|\sigma(V(X)| = 2$ equals to $(2^n - 2)\frac{n(n-1)}{2}$. Such maps generate exactly $(2^n - 2)\frac{n(n-1)}{4}$ different labelled discrete Markov graphs. Other maps are in one–to–one correspondence with their labelled discrete Markov graphs.

Therefore, we have $1 + (2^n - 2)\frac{n(n-1)}{4} + (n^n - n - (2^n - 2)\frac{n(n-1)}{2}) = n^n - (2^{n-2}n - \frac{n}{2} + 1)(n - 1)$ different labelled Markov graphs of maps for every n–vertex tree. □

Obviously, the number of non–isomorphic discrete Markov graphs on a given n–vertex tree is much harder to compute (it will be different for non–isomorphic trees). Some related results can be found in [10], where the author found an explicit formula for the number of non–isomorphic $\Gamma(X, \sigma)$ for n–vertex paths X and cyclic permutations σ.

3.2. **Structural results.** In this subsection we present characterizations of maps on trees for several given (up to isomorphism) discrete Markov graphs.

Proposition 3.13. *Let X be a tree with $n \geq 2$ vertices and $\sigma : V(X) \to V(X)$. Then $\Gamma = \Gamma(X, \sigma)$ is complete if and only if $X = \{u_1 - ... - u_n\}$ is a path and*

$$\sigma(u_k) = \begin{cases} u_1, & \textit{if } k \textit{ is odd,} \\ u_n, & \textit{if } k \textit{ is even} \end{cases} \quad \textit{or } \sigma(u_k) = \begin{cases} u_1, & \textit{if } k \textit{ is even,} \\ u_n, & \textit{if } k \textit{ is odd} \end{cases} \quad \textit{for } k \in \overline{1,n}.$$

Proof. Sufficiency of this condition is obvious, so we prove only its necessity.

Let Γ be a complete digraph. Then for all $v_{e_1}, v_{e_2} \in V(\Gamma)$ the discrete Markov graph Γ contains the arc $v_{e_1} \to v_{e_2}$. It means that $u_2, v_2 \in [\sigma(u_1), \sigma(v_1)]_X$, where $e_i = u_i v_i$ for $i = 1, 2$. Thus, all vertices from X lie on a common path. This is possible only in the case when X is a path itself.

Thus let $X = \{u_1 - ... - u_n\}$. We have $\sigma(u_k) \in L(X) = \{u_1, u_n\}$ for all $k \in \overline{1,n}$. But since for every edge $e = u_k u_{k+1} \in E(X)$ it holds $\sigma(u_k) \neq \sigma(u_{k+1})$, the values $\sigma(u_k)$ are alternating. $\square$

Another class of discrete Markov graphs that realizable only by maps of paths consists of complete bipartite digraphs.

Proposition 3.14. *Let X be a tree with $n \geq 3$ vertices and $\sigma : V(X) \to V(X)$. Then $\Gamma = \Gamma(X, \sigma)$ is complete bipartite if and only if $X = \{u_1 - u_2 - ... - u_n\}$ is a path and there exists $i \in \overline{2, n-1}$ such that*

$$\sigma(u_k) = \begin{cases} u_i, & \textit{if } \ k - i \textit{ is even,} \\ u_1, & \textit{if } \ k - i \geq 0 \textit{ is odd,} \\ u_n, & \textit{if } \ i - k \geq 0 \textit{ is odd} \end{cases} \quad \textit{for } k \in \overline{1,n}.$$

Proof. Let us first prove the sufficiency of this condition. Put $A := \{v_e : e = u_k u_{k+1}, \ k \in \overline{1, i-1}\}$ and $B := \{v_e : e = u_k u_{k+1}, \ k \in \overline{i, n-1}\}$. Clearly $V(\Gamma) = A \cup B$ is a partition. Moreover, from the construction of σ it follows that $A(\Gamma) = (A \times B) \cup (B \times A)$. Thus Γ is a complete bipartite digraph.

Conversely, suppose that $V(\Gamma) = A \cup B$ is a partition and $A(\Gamma) = (A \times B) \cup (B \times A)$. Note that from the definition of discrete Markov graph it follows that the edges corresponding to the vertices of A form a path A' in X. The same holds for B.

We now prove that X is a path itself. Assume the contrary. Thus, let there exists $u \in V(X)$ with $d_X(u) \geq 3$. Fix three vertices $u_1, u_2, u_3 \in V(X)$ with $e_k = uu_k \in E(X)$, $k = \overline{1,3}$. Without loss of generality, we can assume that $v_{e_1}, v_{e_2} \in A$ and $v_{e_3} \in B$. Therefore, since $A = N^+_\Gamma(v_{e_3})$ then again, without loss of generality, assume that $\sigma(u) \in A_X(u_1, u)$ and $\sigma(u_3) \in A_X(u_2, u)$. But since $B = N^+_\Gamma(v_{e_1})$, then $\sigma(u_1) \in A_X(u_3, u) \subset A_X(u, u_1)$. This means that v_{e_1} has a loop in Γ which is a contradiction. Thus $X = \{u_1 - ... - u_n\}$ is a path.

Since $E(X) = A' \cup B'$ and A' and B' are two edge disjoint paths, A' and B' share a common vertex u_i in X. Also, $A' \neq \emptyset$ and $B' \neq \emptyset$ implies $i \in \overline{2, n-1}$. We have $A' = \{u_1 - ... - u_i\}$ and $B' = \{u_i - ... - u_n\}$.

Further, since $v_{u_k u_{k+1}} \in N^+_\Gamma(v_{u_l u_{l+1}})$ for all $k \in \overline{i,n}$ and $l \in \overline{1,i}$, then $\sigma(u_l) \in \{u_i, u_n\}$ for all $l \in \overline{1,i}$. Similarly, $\sigma(u_k) \in \{u_1, u_i\}$ for all $k \in \overline{i,n}$. In particular, $\sigma(u_i) = u_i$. Finally, it is easy to see that the values $\sigma(u_l)$, $l = \overline{1,i}$ are alternating. The same holds for the values $\sigma(u_k)$, $k = \overline{i,n}$. $\square$

Now we show that automorphisms of trees X are precisely those maps σ for which $\Gamma(X, \sigma)$ splits into disjoint union of cycles (in full analogy with their functional graphs).

Proposition 3.15. *Let X be a tree and $\sigma : V(X) \to V(X)$. Then $\Gamma = \Gamma(X, \sigma)$ is a disjoint union of cycles if and only if σ is automorphism of X.*

Proof. Again, the sufficiency of the condition is obvious. Now let Γ be the union of cycles. Since $d^+_\Gamma(v_e) = 1$ for all $e \in E(X)$, it follows that σ is a homomorphism. Suppose there exist two vertices $u, v \in V(X)$, $u \neq v$ such that $\sigma(u) = \sigma(v)$. Without loss of generality, we can assume that for all $x, y \in [u, v)_X$ it holds $\sigma(x) \neq \sigma(y)$.

If $d_X(u, v) = 1$, then $d^+_\Gamma(v_e) = 0$, where $e = uv \in E(X)$ which is a contradiction. Similarly, if $d_X(u, v) = 2$, then $(u, v)_X = \{x\}$ for some $x \in V(X)$. Now $N^+_\Gamma(v_{e_1}) = N^+_\Gamma(v_{e_2})$, where $e_1 = ux$ and $e_2 = xv$. This means that there exists an edge $e \in E(X)$ with $d^-_\Gamma(v_e) \geq 2$. Thus a contradiction again.

Suppose now that $d_X(u, v) \geq 3$. Consider the shortest path

$$[u, v]_X = \{u = u_1 - u_2 - ... - u_m = v\},$$

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where $m \geq 4$. Then $\sigma(u) - \sigma(u_2) - ... - \sigma(v) = \sigma(u)$ is a cycle in X which is a contradiction again. This means that σ is a bijective homomorphism from X to itself, i.e. an automorphism of X. $\square$

Corollary 3.16. *Let X be a tree with $n \geq 3$ vertices and $\sigma : V(X) \to V(X)$. Then $\Gamma = \Gamma(X, \sigma)$ is a cycle if and only if X is a star, the restriction $\sigma|_{L(X)}$ is a cyclic permutation and $\sigma(u_0) = u_0$, where u_0 is a center of X.*

Proof. Let Γ be a cycle. It follows from Proposition 3.15 that σ is an automorphism of X.

Consider two arbitrary edges $e_i = u_i v_i \in E(X)$, $i = 1, 2$. Define $h := d_\Gamma(v_{e_1}, v_{e_2})$ and $\pi := \sigma^{h-1}$. Obviously, π is an automorphism of X. Moreover, it is easy to see that $\pi(\{u_1, v_2\}) = \{u_2, v_2\}$.

This means that X is edge–transitive. But since X is a tree, each edge must be incident to a leaf. Thus, X is a star.

Further, the equality $\sigma(u_0) = u_0$ trivially holds since σ is an automorphism of X. At last, since Γ is a cycle, σ is cyclically permutes the set of all leaf vertices in X. This proves the necessity of this condition. The sufficiency again is obvious. $\square$

Lemma 3.17. *Let X be a tree, $\sigma : V(X) \to V(X)$ and $\Gamma = \Gamma(X, \sigma)$. Then for every pair of vertices $u, v \in V(X)$ and an edge $e = xy \in E(X)$, where $x, y \in [\sigma(u), \sigma(v)]_X$ there exists an edge $wz \in E(X)$, where $w, z \in [u, v]_X$ such that $v_{wz} \to v_{xy}$ in Γ.*

Proof. Apply Lemma 3.8 to the path $[u, v]_X$, the set $S = \{x, y\}$ and the function $f : [u, v]_X \to S$, $f = pr_{xy} \circ \sigma$. Since, $f(u) \neq f(v)$, there exists an edge $wz \in E(X)$, where $w, z \in [u, v]_X$ such that $f(w) \neq f(z)$ which means that $v_{wz} \to v_{xy}$ in Γ. $\square$

For every tree X, its subtree $H \subset X$ and a vertex $u \in V(X)$ define the distance from u to H as $d_X(u, H) := d_X(u, pr_H(u))$.

Theorem 3.18. *Let X be a tree, $\sigma : V(X) \to V(X)$, $\Gamma = \Gamma(X, \sigma)$ and $A(\Gamma) \neq \emptyset$. Then every arc in Γ is a loop if and only if σ is a projection on some subtree in X, or there exists an edge $e \in E(X)$ such that $\sigma = pr^*_e$.*

Proof. Let us prove the necessity of this condition. At first, let $|A(\Gamma)| = 1$. Then there exists a unique edge $e = uv \in E(X)$ such that $(v_e, v_e) \in A(\Gamma)$ is a loop. This implies $\sigma(V(X)) = \{u, v\}$. Therefore, for all $x \in A_X(u, v), y \in A_X(v, u)$ we have $\sigma(x) = \sigma(u)$ and $\sigma(y) = \sigma(v)$.

Further, since v_e has a loop, then $\sigma(u) \neq \sigma(v)$. If $\sigma(u) = u$, then $\sigma(v) = v$ and so σ is a projection on the edge e. Otherwise $\sigma(u) = v$ and $\sigma(v) = u$. Then $\sigma(x) = \begin{cases} u, & \text{if } x \in A_X(v, u), \\ v, & \text{if } x \in A_X(u, v) \end{cases}$ for all $x \in V(X)$. In other words, $\sigma = pr^*_e$.

Now let $|A(\Gamma)| \geq 2$. Consider the subgraph of X induced by the image of σ: $H = X[\sigma(V(X))]$.

Claim 1. H is a subtree.

From Lemma 3.17 it follows that for every pair of vertices $u, v \in V(X)$ and an edge $e = xy \in E(X)$, where $x, y \in [\sigma(u), \sigma(v)]_X$ there exists an edge $wz \in E(X)$, where $w, z \in [u, v]_X$ such that $v_{wz} \to v_{xy}$ in Γ. Since each arc in Γ is a loop, we obtain that $\{x, y\} = \{w, z\} = \{\sigma(w), \sigma(z)\}$. This implies $x \in \sigma(V(X))$. Therefore, for each pair of vertices $u, v \in V(X)$ it holds $[\sigma(u), \sigma(v)]_X \subset \sigma(V(X))$. In other words, H is a connected subgraph of X.

Since $|A(\Gamma)| \geq 2$, then $|V(H)| \geq 3$. Hence, there exists a vertex $u_0 \in V(H)$ adjacent to at least two vertices in $V(H)$. But every arc in Γ is a loop. From this one can conclude that $\sigma(u_0) = u_0$.

Claim 2. $\sigma|_{V(H)} = \mathrm{id}|_{\mathrm{V(H)}}$.

Let us use the induction on $d_X(u_0, x)$ to show that $\sigma(x) = x$ for any vertex $x \in V(H)$. Suppose $d_X(u_0, x) = 1$. Since v_{u_0x} has a loop in Γ, we have $\sigma(x) \in A_X(x, u_0)$. But every arc in Γ is a loop. Thus $\sigma(x) = x$. Suppose now that $d_X(u_0, x) = n + 1$. Choose a vertex $y \in [u_0, x]_X$ adjacent to x. By Claim 1 we have $y \in V(H)$ and $d_X(u_0, y) = n$, so by induction hypothesis $\sigma(y) = y$. Similarly, considering the edge xy we can conclude that $\sigma(x) = x$.

Finally, by induction on $d_X(x,H)$ we will show that for every $x \in V(X) - V(H)$ it holds $\sigma(x) = pr_H(x)$. If $d_X(x,H) = 1$, then there exists a vertex $u \in V(H)$ adjacent to x. Since v_{ux} has a zero outdegree in Γ, we have $\sigma(x) = \sigma(u)$. From Claim 2 it follows $\sigma(u) = u$. Therefore $\sigma(x) = u = pr_H(x)$.

Now let $d_X(x,H) = n+1$. Then there exists a vertex $u \in V(X)$ adjacent to x with $d_X(u,H) = n$. By the induction hypothesis $\sigma(u) = pr_H(u)$. Again, since v_{ux} has a zero outdegree in Γ, we have $\sigma(x) = \sigma(u) = pr_H(u)$. It is evident that $pr_H(u) = pr_H(x)$. Therefore $\sigma(x) = pr_H(x)$ for all $x \in V(X) - V(H)$. Combining this fact with Claim 2, we obtain $\sigma = pr_H$.

Sufficiency of this condition can be easily checked by direct computation of $\Gamma(X,\sigma)$, where σ is a projection on a subtree in X. $\square$

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