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THE STRUCTURE OF A RING OF FORMAL SERIES

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If K is a field, by means of a sequence S of elements of K is defined a K-algebra $K_S[[X]]$ of formal series called Newton interpolating series which generalize the formal power series. We study algebraic properties of this algebra, and in the case when S has a finite number of distinct elements we prove that it is isomorphic to a direct sum of a finite number of algebras which are either equal to $K[[X]]$ or to factor rings of $K[X]$.

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1. Introduction

The K-algebra $K[[X]]$ of formal power series is a basic structure in commutative algebra with applications in algebraic geometry. Also formal power series are useful tools in combinatorics as generating functions or in number theory.

In the interpolation theory particular Newton series, with coefficients in a field K, constructed by means of Newton interpolating polynomials is a powerful tool either in archimedean or in non-archimedean analysis. Thus for $K = \mathbb{C}$ the convergence of a family of Newton series is the subject of [6]. Also a proof of a well-known result of Lindemann on the transcendency of e^{γ}, when γ is an algebraic number (see [8], Theorem 6, Ch. 2, Sec. 3) is based on a Newton series with complex coefficients. In the non-archimedean case the Mahler series which are convergent Newton series are used for representation of continuous functions on $\mathbb{Z}_p$ (see for example [1] or [7]). Applications of these series to approximate solutions of boundary value problems for differential equations are presented in [3] and [4]. All these convergent series belong to some subalgebras of a K-algebra of formal series called (formal)

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Newton interpolating series defined below. So the structure of this algebra becomes an important goal for problems of previous types.

Let K be a field and $S = \{\alpha_n\}_{n\geq 1}$ a fixed sequence of elements of K. We consider the polynomials

$$u_0 = 1,\ u_i = \prod_{j=1}^{i} (X - \alpha_j)\,,\ i \geq 1 \tag{1}$$

and the set of formal sums

$$K_S[[X]] = \{f = \sum_{i=0}^{\infty} a_i u_i \mid a_i \in K\}, \tag{2}$$

two such expressions being regarded as equal if and only if they have the same coefficients. We call an element f from $K_S[[X]]$ a (formal) *Newton interpolating series* with coefficients in K defined by the sequence S. If $K_S[[X]]$ is endowed with a suitable addition and multiplication it becomes a K- algebra (see Section 2) which for $\alpha_k = 0$, for all k, is equal to $K[[X]]$.

The main goal of this paper is to give a structure theorem of $K_S[[X]]$ for the case when the sequence S has a finite number of distinct elements.

In order to prepare the proof of the main result, in Section 3, Theorem 1 we describe the units of $K_S[[X]]$, we prove an Isomorphism Theorem (Theorem 2) and we find the spectrum of maximal ideals of $K_S[[X]]$ (see Theorem 3). In Section 4 we prove (see Theorem 5) that it is isomorphic to a direct sum of a finite number of algebras which are either equal to $K[[X]]$ or to factor rings of $K[X]$. Moreover its algebraic structure is uniquely determined by a finite number of non-negative integers, so-called the invariants of the sequence S.

2. Basic notations and definitions

The following lemma is needed to give a suitable form of well-known Newton interpolating polynomial.

Lemma 1. *Let K be a field and let $S = \{\alpha_n\}_{n\geq 1}$ be a sequence of elements of K. Then every polynomial $P \in K[X]$ can be written uniquely in the form $P = \sum_{i=0}^{p} a_i u_i$ with $a_i \in K$, where p is the degree of P and every u_i is defined by* (1).

Proof. If $P = \sum_{j=0}^{p} b_j X^j$, then we may uniquely write $P = a_p u_p + Q_q$, where $a_p = b_p$, $Q_q \in K[X]$ such that $\deg Q_q < p$. Now the lemma follows by induction on p. □

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If $u_i,\ u_j$ are given by (1), we obtain that for every k, $\max\{i,j\} \leq k \leq i+j$ there exist the elements $d_k(i,j)$ in K uniquely determined such that

$$u_i u_j = \sum_{k=\max\{i,j\}}^{i+j} d_k(i,j) u_k. \tag{3}$$

The following relationships are easily checked:

$$d_k(i,j) = d_k(j,i), \tag{4}$$

$$d_{i+j}(i,j) = 1. \tag{5}$$

We define addition and multiplication of two elements $f = \sum_{i=0}^{\infty} a_i u_i$, $g = \sum_{i=0}^{\infty} b_i u_i, \in K_S[[X]]$ as follows

$$f + g = \sum_{i=0}^{\infty} (a_i + b_i) u_i \tag{6}$$

and

$$fg = \sum_{k=0}^{\infty} c_k u_k \tag{7}$$

with

$$c_k = \sum_{(\alpha,\beta)\in I(k)} d_k(\alpha,\beta) a_\alpha b_\beta, \tag{8}$$

where

$$I(k) = \{(\alpha,\beta) \in \mathbb{N}\times\mathbb{N} \mid \max\{\alpha,\beta\} \leq k,\ \alpha+\beta \geq k\} \tag{9}$$

and $d_k(\alpha,\beta)$ are given in (3). It is easily seen that with these definitions of addition and multiplication $K_S[[X]]$ becomes a commutative K-algebra which contains $K[X]$.

If $f = \sum_{i=0}^{\infty} a_i u_i \in K_S[[X]]$, the smallest index i for which the coefficient a_i is different from zero will be called the *order* of f and will be denoted by $o(f)$. We agree to attach the order $+\infty$ to the element 0 from $K_S[[X]]$. Then it is easy to prove that for $f, g \in K_S[[X]]$

$$o(f+g) \geq \min\{o(f), o(g)\}, \tag{10}$$

$$o(fg) \geq \max\{o(f), o(g)\}. \tag{11}$$

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If we fix a positive real number $\delta < 1$ and define the norm $\|f\|$ of an element f of $K_S[[X]]$ by the formula

$$\|f\| = \delta^{o(f)}, \tag{12}$$

$K_S[[X]]$ becomes a $K-$ultrametric normed vector space, where the norm on K is trivial. Moreover for $f, g \in K_S[[X]]$

$$\|fg\| \leq \min\{\|f\|, \|g\|\} \tag{13}$$

holds and $K_S[[X]]$ is a topological K-algebra. Consider a Cauchy sequence $\{f_n = \sum_{i=0}^{\infty} a_{i,n}u_i\}_{n\geq 0}$ of elements from $K_S[[X]]$. Then $f = \sum_{i=0}^{\infty} a_{i,i}u_i \in K_S[[X]]$ and $f = \lim_{n\to\infty} f_n$. Hence it follows that $K_S[[X]]$ is a complete $K-$ultrametric normed vector space and it is a completion of $K[X]$ with respect to the topology defined by the norm (12). This implies as in the classical case (see for example [9], Ch.VII, §1) that the distributive law holds also for infinite sums and in particular we obtain that

$$u_j \sum_{i=0}^{\infty} a_i u_i = \sum_{i=0}^{\infty} a_i u_i u_j \tag{14}$$

For a fix $f \in K_S[[X]]$ we denote $L_f : K_S[[X]] \to K_S[[X]]$ the K-linear application defined by

$$L_f(g) = fg \tag{15}$$

Then, by (3) and (14)

$$L_f(u_i) = fu_i = \sum_{j=i}^{\infty} f_{j,i}u_j \tag{16}$$

where $f_{j,i} \in K$ are uniquely determined by f. Since

$$L_f(u_{i+1}) = \sum_{j=i+1}^{\infty} f_{j,i+1}u_j = fu_i(X - \alpha_{i+1}) = \sum_{j=i}^{\infty} f_{j,i}u_j(X - \alpha_{i+1})$$

$$= \sum_{j=i}^{\infty} f_{j,i}(u_{j+1} + (\alpha_{j+1} - \alpha_{i+1})u_j) = \sum_{j=i+1}^{\infty} (f_{j-1,i} + (\alpha_{j+1} - \alpha_{i+1})f_{j,i})u_j$$

it follows that

$$f_{j,i+1} = f_{j-1,i} + (\alpha_{j+1} - \alpha_{i+1})f_{j,i}, \text{ for } j = i+1, i+2, \ldots . \tag{17}$$

If $g = \sum_{j=0}^{\infty} b_j u_j$, then by (14) and (16) we obtain

$$fg = \sum_{j=0}^{\infty} b_j(fu_j) = \sum_{j=0}^{\infty} b_j(\sum_{i=j}^{\infty} f_{i,j}u_i) = \sum_{j=0}^{\infty} (\sum_{i=0}^{j} b_i f_{j,i})u_j. \tag{18}$$

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For $f = \sum_{j=0}^{\infty} a_j u_j$ if we take $i = 0$ in (16), we get for every j

$$f_{j,0} = a_j. \tag{19}$$

Now for a fix j, by (17) we obtain that

$$f_{j,j} = f_{j-1,j-1} + (\alpha_{j+1} - \alpha_j)f_{j,j-1} = f_{j-2,j-2} + (\alpha_j - \alpha_{j-1})f_{j-1,j-2}+ (\alpha_{j+1}-\alpha_j)(f_{j-1,j-2}+(\alpha_{j+1}-\alpha_{j-1})f_{j,j-2}) = f_{j-2,j-2}+(\alpha_{j+1}-\alpha_{j-1})f_{j-1,j-2}+ (\alpha_{j+1} - \alpha_j)(\alpha_{j+1} - \alpha_{j-1})f_{j,j-2},$$

and by recurrence for all k, $1 \le k \le j$ it follows that

$$f_{j,j} = \sum_{r=0}^{k} f_{j-r,j-k} \prod_{s=1}^{k-r}(\alpha_{j+1} - \alpha_{j-k+s}). \tag{20}$$

In particular for $k = j$ we obtain

$$f_{j,j} = \sum_{r=0}^{j} f_{j-r,0} \prod_{s=1}^{j-r}(\alpha_{j+1} - \alpha_s) = \sum_{r=0}^{j} a_{j-r} \prod_{s=1}^{j-r}(\alpha_{j+1} - \alpha_s), \tag{21}$$

where $\prod_{s=1}^{0}(\alpha_{j+1} - \alpha_s) = 1$.

Similarly for i, j such that $1 \le k \le i < j$,

$$f_{j,i} = f_{j-1,i-1} + (\alpha_{j+1} - \alpha_i)f_{j,i-1} = f_{j-2,i-2} + (\alpha_j - \alpha_{i-1})f_{j-1,i-2} +(\alpha_{j+1} - \alpha_i)(f_{j-1,i-2} + (\alpha_{j+1} - \alpha_{i-1})f_{j,i-2}) = f_{j-2,i-2}+ (\alpha_j + \alpha_{j+1} - \alpha_{i-1} - \alpha_i)f_{j-1,i-2} + (\alpha_{j+1} - \alpha_i)(\alpha_{j+1} - \alpha_{i-1})f_{j,i-2}$$

and generally

$$f_{j,i} = \sum_{r=0}^{k} f_{j-r,i-k}P_{j-r,i-k}, \tag{22}$$

where $P_{j-r,i-k}$ are polynomials in $\alpha_1, ..., \alpha_{j+1}$ with integer coefficients and

$$P_{j,i-k} = (\alpha_{j+1} - \alpha_i)(\alpha_{j+1} - \alpha_{i-1})...(\alpha_{j+1} - \alpha_{i-k+1}). \tag{23}$$

For any sequence $S = \{\alpha_n\}_{n\ge1}$ we define the set

$$I_S = \{i \mid \alpha_i \ne \alpha_j \text{ for all } j < i\} \tag{24}$$

and for any $\gamma \in K$ we put

$$I_S(\gamma) = \{i \mid \alpha_i = \gamma\}. \tag{25}$$

We call the sequence $S = \{\alpha_n\}_{n\ge1}$ *purely periodic* if I_S is finite set having m elements and, for each positive integer i less or equal to m, $\alpha_i = \alpha_{i+jm}$, $j = 1, 2, ...$.

Now if $S = \{\alpha_k\}_{k\ge1}$ is a sequence of elements of K such that I_S

is a finite set, we say that S has *canonical form* if there exist the positive integers $n_1 \leq ... \leq n_s$ such that $\alpha_1 = \alpha_2 = ... = \alpha_{n_1} = \gamma_1$, $\alpha_{n_1+1} = ... = \alpha_{n_1+n_2} = \gamma_2$, ..., $\alpha_{n_1+...+n_{s-1}+1} = ... = \alpha_{n_1+...+n_s} = \gamma_s$ with $\gamma_i \neq \gamma_j$ for all $i \neq j$ and the subsequence $T = \{\alpha_k\}_{k>n_1+...+n_s}$ is purely periodic sequence such that each $\alpha_k \neq \gamma_i$ for all $i = 1, 2, ..., s$.

Let S be a sequence such that $|I_S| = n$ is finite. We may write $I_S = \{i_1, ..., i_s, i_{s+1}, ..., i_n\}$ where $|I_S(\alpha_{i_k})| = n_k$ is finite for $k \leq s$, with $n_1 \leq ... \leq n_s$, and infinite for $k > s$. Then we call the numbers s, $m = n - s$, $n_1, ..., n_s$, *the invariants of the sequence* S.

3. Algebraic properties of the ring $K_S[[X]]$

In order to describe the units of the ring $K_S[[X]]$ we need the following lemma which is an easy consequence of (21).

Lemma 2. *If* $\alpha_{n+1} = \alpha_i$ *for* $i < n$ *then* $f_{n,n} = f_{i-1,i-1}$.

Theorem 1. *An element* $f \in K_S[[X]]$ *is a unit if and only if* $f_{i-1,i-1} \neq 0$ *for all* $i \in I_S$.

Proof. If $f = \sum_{i=0}^{\infty} a_i u_i \in K_S[[X]]$ is a unit then by (18) there exists an element $g = \sum_{j=0}^{\infty} b_j u_j \in K_S[[X]]$ such that $\sum_{j=0}^{\infty}\{\sum_{i=0}^{j} b_i f_{j,i}\} u_j = 1$. Hence $b_0 f_{0,0} = 1$ and $\sum_{i=0}^{j} b_i f_{i,j} = 0$ for all $j = 1, 2, ...$ Suppose contrary that there exists $t + 1 \in I_S$ such that $f_{t,t} = 0$. Since $b_0 f_{0,0} = 1$ and $(\sum_{i=0}^{j} b_i f_{j,i})(\alpha_{t+1} - \alpha_1)(\alpha_{t+1} - \alpha_2)...(\alpha_{t+1} - \alpha_j) = 0$ for all $j = 1, 2, ...t$, we obtain $b_0 f_{0,0} + \sum_{j=1}^{t}\{(\sum_{i=0}^{j} b_i f_{j,i})(\alpha_{t+1} - \alpha_1)(\alpha_{t+1} - \alpha_2)...(\alpha_{t+1} - \alpha_j)\} = 1$, which implies

(26)
$$\sum_{j=0}^{t} b_j \prod_{k=1}^{j}(\alpha_{t+1} - \alpha_k)\{f_{j,j} + \sum_{i=j+1}^{t} f_{i,j}(\alpha_{t+1} - \alpha_{j+1})...(\alpha_{t+1} - \alpha_i)\} = 1,$$

where $\prod_{k=1}^{0}(\alpha_{t+1} - \alpha_k) = 1$. Now by replacing j with t and k with $t - j$ in (20) it follows that for $j = 0, 1, ..., t-1$ $f_{t,t} = f_{j,j} + \sum_{i=j+1}^{t} f_{i,j}(\alpha_{t+1} - \alpha_{j+1})(\alpha_{t+1} - \alpha_{j+2})...(\alpha_{t+1} - \alpha_i)$ and by (26) $f_{t,t} \sum_{j=0}^{t} b_j \prod_{k=1}^{j}(\alpha_{t+1} - \alpha_k) = 1$.

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Since $f_{t,t}=0$ we obtain a contradiction which implies that $f_{i-1,i-1}\neq 0$ for all $i\in I_S$.

Conversely, if $f\in K_S[[X]]$ and $f_{i-1,i-1}\neq 0$ for all $i\in I_S$, we can find b_0 such that $b_0f_{0,0}=1$. Since by Lemma 2 $f_{n,n}\neq 0$ for all $n\in\mathbb{N}$. there exists the elements $b_j\in K$ such that $\sum_{i=0}^{j} b_i f_{i,j}=0$. Hence $g=\sum_{i=0}^{\infty} b_i u_i$ verifies $fg=1$. □

We study when two K-algebras $K_S[[X]]$ and $K_{S'}[[X]]$ are isomorphic.

Lemma 3. *Let $S=\{\alpha_n\}_{n\geq 1}$ and $S'=\{\beta_n\}_{n\geq 1}$ be two sequences of elements of K and let u_i and respectively v_i be the associated polynomials defined in (1). If*

a) for every i

$$u_i = v_i + \sum_{j=n_i}^{i-1} \delta_{j,i} v_j = F_i(v_{n_i}, v_{n_i+1}, ..., v_i), \tag{27}$$

$$v_i = u_i + \sum_{j=m_i}^{i-1} \gamma_{j,i} u_j = G_i(u_{m_i}, u_{m_i+1}, ..., u_i), \tag{28}$$

where $\delta_{j,i}, \gamma_{j,i}\in K$ and

$$F_i(G_{n_i}, G_{n_i+1}, ..., G_i) = u_i, \tag{29}$$

$$G_i(F_{m_i}, F_{m_i+1}, ..., F_i) = v_i; \tag{30}$$

b)

$$\lim_{i\to\infty} n_i = \infty \text{ and } \lim_{i\to\infty} m_i = \infty, \tag{31}$$

then the map $\phi : K_S[[X]] \to K_{S'}[[X]]$ defined by

$$\phi(u_i) = F_i(v_{n_i}, v_{n_i+1}, ..., v_i) \tag{32}$$

and for every $f=\sum_{i=0}^{\infty} a_i u_i \in K_S[[X]]$

$$\phi(f) = \sum_{i=0}^{\infty} a_i \phi(u_i) \tag{33}$$

is a $K-$algebra isomorphism.

Proof. Define $\psi : K_{S'}[[X]] \to K_S[[X]]$ such that

$$\psi(v_i) = G_i(u_{m_i}, u_{m_i+1}, ...u_i) \tag{34}$$

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and for every $g = \sum_{i=0}^{\infty} b_i v_i \in K_{S'}[[X]]$

$$\psi(g) = \sum_{i=0}^{\infty} b_i \psi(v_i). \tag{35}$$

Then by (31),(33) and (35) the maps ϕ and ψ are well defined and continuous maps with respect to the corresponding norms defined by (12). The relations (29) and (30) imply that the restricted mappings ϕ and ψ on $K[X]$ are inverses. Since $K[X]$ is dense in $K_S[[X]]$ and $K_{S'}[[X]]$ we obtain that ϕ and ψ are inverses and hence ϕ is bijective map. Because ϕ is the identity map on $K[X]$ it follows that ϕ is also a $K-$algebra morphism. Hence we obtain that $K_S[[X]]$ and $K_{S'}[[X]]$ are isomorphic $K-$algebras. □

Theorem 2. *Suppose $S = \{\alpha_k\}_{k\geq 1}$ is a sequence of elements of K and $\pi : \mathbb{N}^* \to \mathbb{N}^*$ is a bijective map such that $S' = \{\beta_k\}_{k\geq 1}$, where $\beta_k = \alpha_{\pi(k)}$. Then $K_S[[X]]$ and $K_{S'}[[X]]$ are isomorphic $K-$algebras.*

Proof. Consider $u_i = \prod_{k=1}^{i}(X-\alpha_k)$ and $v_i = \prod_{k=1}^{i}(X-\beta_k)$. By Lemma 1 every u_i can be written in the form

$$u_i = v_i + \sum_{j=0}^{i-1} \gamma_{j,i} v_j, \tag{36}$$

where $\gamma_{0,i} = u_i(\beta_1)$. Let t_1 be the smallest index i such that v_1 divides u_i in $K[X]$. Then for every $i \geq t_1$ $u_i(\beta_1) = 0$ and by (36) we obtain that

$$u_i = v_i + \sum_{j=c_1}^{i-1} \gamma_{j,i} v_j, \tag{37}$$

where $c_1 \geq 1$ and $\gamma_{c_1,t_1} \neq 0$. Consider $q_1 \leq c_1$ the greatest index such that v_{q_1} divides u_{t_1} in $K[X]$. Then by (37), for all $i \geq t_1$,

$$\frac{u_i}{v_{q_1}} = \frac{v_i}{v_{q_1}} + \sum_{j=c_1}^{i-1} \gamma_{j,i} \frac{v_j}{v_{q_1}}. \tag{38}$$

Let t_2 be the smallest index i greater than t_1 such that v_{q_1+1} divides u_{t_2}. Then by (38) $\gamma_{c_1,t_2} = \frac{u_{t_2}}{v_{q_1}}(\beta_{q_1+1}) = 0$ and for all $i \geq t_2$

$$u_i = v_i + \sum_{j=c_2}^{i-1} \gamma_{j,i} v_j, \tag{39}$$

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where $\gamma_{c_2,t_2} \neq 0$ and $c_2 > c_1$. Thus by recurrence we obtain that for every i

$$u_i = v_i + \sum_{j=n_i}^{i-1} \gamma_{j,i} v_j = F_i, \tag{40}$$

where $\gamma_{j,i} \in K$ and $\lim\limits_{i\to\infty} n_i = \infty$. Similarly, since $\alpha_k = \beta_{\pi^{-1}(k)}$, it follows that

$$v_i = u_i + \sum_{j=m_i}^{i-1} \delta_{j,i} u_j = G_i, \tag{41}$$

where $\delta_{j,i} \in K$ and $\lim\limits_{i\to\infty} m_i = \infty$. By Lemma 1, (40) and (41) $F_i(G_{n_i}, ..., G_i) = u_i$ and $G_i(F_{m_i}, ..., F_i) = v_i$. Thus the theorem follows from Lemma 3. □

Remark 1. Suppose that $S = \{\alpha_k\}_{k\geq 1}$ is a sequence such that I_S is a finite set. Then there exists a bijective map $\pi : \mathbb{N}^* \to \mathbb{N}^*$ such that $S' = \{\beta_k\}_{k\geq 1}$, where $\beta_k = \alpha_{\pi(k)}$ has canonical form. By Theorem 2 it follows that $K_S[[X]]$ and $K_{S'}[[X]]$ are isomorphic K-algebras. Thus it is enough to study the algebraic properties of $K_S[[X]]$ in the case when S has canonical form.

Lemma 4. *Let α_k be an element of S. Then $g \in K_S[[X]]$ belongs to the ideal generated by $X - \alpha_k$ if and only if $g_{k-1,k-1} = 0$.*

Proof. We denote by $< X - \alpha_k >$ the ideal generated by $X - \alpha_k$. Then for an element $f = \sum\limits_{i=0}^{\infty} a_i u_i$ of $K_S[[X]]$

$$(X - \alpha_k)f = a_0(\alpha_1 - \alpha_k) + \sum_{i=1}^{\infty} (a_{i-1} + a_i(\alpha_{i+1} - \alpha_k))u_i \tag{42}$$

If $g = \sum\limits_{i=0}^{\infty} b_i u_i$ is an element of $< X - \alpha_k >$, then from (42) we can find the elements a_i such that

$$b_0 = a_0(\alpha_1 - \alpha_k), b_i = a_{i-1} + a_i(\alpha_{i+1} - \alpha_k), \text{ for } i = 1, 2, ... \ . \tag{43}$$

By (43) and (21) it follows that

$$g_{k-1,k-1} = \sum_{r=0}^{k-1} b_{k-1-r} \prod_{s=1}^{k-1-r} (\alpha_k - \alpha_s) = 0.$$

Conversely if $g = \sum\limits_{i=0}^{\infty} b_i u_i$ is an element of $K_S[[X]]$ such that $g_{k-1,k-1} = 0$, then by using (21), (43) and Lemma 2 we can find an element $f =$

$\sum_{i=1}^{\infty} a_i u_i \in K_S[[X]]$ such that $g = (X - \alpha_k)f$ and hence $g \in < X - \alpha_k >$.
□

Lemma 5. *The ideal generated by $X - \alpha_i$ for each $i \in I_S$ is a maximal ideal in $K_S[[X]]$.*

Proof. We consider an element $h \in K_S[[X]]$ such that $h \notin < X - \alpha_k >$. Then by Lemma 4 $h_{k-1,k-1} \neq 0$. Since for every $i \in I_S$, $i \neq k$ the elements b_{i-1} in (43) can be chosen arbitrary, we can find an element $g = \sum_{i=0}^{\infty} b_i u_i \in < X - \alpha_k >$ such that $(g + h)_{i-1,i-1} \neq 0$ for all $i \in I_S$, $i \neq k$. If $i = k$, by Lemma 4 $(g + h)_{k-1,k-1} = g_{k-1,k-1} + h_{k-1,k-1} = 0 + h_{k-1,k-1} = h_{k-1,k-1} \neq 0$. Hence by Theorem 1 $g + h$ is unit in $K_S[[X]]$, which shows that $< X - \alpha_k >$ is a maximal ideal in $K_S[[X]]$.
□

Theorem 3. *$K_S[[X]]$ is a semi-local ring if and only if I_S is a finite set. Moreover in this case all maximal ideals are $M_i =< X - \alpha_i >$, where $i \in I_S$.*

Proof. Suppose the contrary that $K_S[[X]]$ is a semi-local ring and I_S is infinite. Then by Lemma 5 the ideal generated by $X - \alpha_i$, for each $i \in I_S$, is a maximal ideal in $K_S[[X]]$. Hence there exist an infinite number of maximal ideals in $K_S[[X]]$ which is a contradiction. This implies that I_S is a finite set.

Conversely, suppose $I_S = \{i_1, ..., i_n\}$. By Lemma 5 each ideal generated by $X - \alpha_{i_k}$, for each $i_k \in I_S$, is a maximal ideal in $K_S[[X]]$. Let M be a maximal ideal of $K_S[[X]]$ different from all $M_{i_k} =< X - \alpha_{i_k} >$, with $i_k \in I_S$. Since $M + M_{i_1} M_{i_2} ... M_{i_n} = K_S[[X]]$, we can find an element $f \in M$ such that $f \notin < X - \alpha_{i_k} >$, for each $i_k \in I_S$. Then by Lemma 4 $f_{i_k-1,i_k-1} \neq 0$ for all $i_k \in I_S$ and by Theorem 1 f is unit in $K_S[[X]]$, a contradiction which shows that all maximal ideals in $K_S[[X]]$ are M_{i_k} with $i_k \in I_S$. Thus $K_S[[X]]$ is a semi-local ring. □

The following example shows that when I_S is infinite $K_S[[X]]$ contains maximal ideals which are different from $< X - \alpha_k >$, for every k.

Example 1. Consider $S = \{\alpha_n\}_{n \geq 1}$ such that $\alpha_i \neq \alpha_j$ for every $i \neq j$. Because $u_n(\alpha_k) = 0$ for $n \geq k$, every $f = \sum_{i=0}^{\infty} a_i u_i \in K_S[[X]]$ defines a map denoted also by f from S to K. Moreover, $u_k(\alpha_{k+1}) \neq 0$ implies that every f is uniquely determined by its values at α_k, $k \in \mathbb{N}^*$ and $K_S[[X]]$ is isomorphic to the K-algebra of all the functions from S to

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K. For every $j \in \mathbb{N}^*$ we consider $h_j \in K_S[[X]]$ such that

$$h_j(\alpha_k) = \begin{cases} 1, \text{ if } k \le j \\ 0, \text{ if } k > j \end{cases} . \tag{44}$$

Then for every nonzero $h \in I =< h_1, ..., h_n, ... >$ there exist $m,\ n \in \mathbb{N}^*$ such that $h(\alpha_m) \neq 0$ and $h(\alpha_k) = 0$ for every $k \geq n$. Hence it follows that $I \cap K[X] = \{0\}$, $I \not\subset < X - \alpha_k >$ for every k and I is not finitely generated. Thus $K_S[[X]]$ contains maximal ideals which are different from $< X - \alpha_k >$, for every k.

4. Structure of the ring $K_S[[X]]$

In this section we prove that the structure of the K-algebra $K_S[[X]]$ is uniquely determined by the invariants of S, when I_S is a finite set. The following lemma describes an essential isomorphism of K-algebras. We note that canonic isomorphisms of K-vector spaces as $f \to \left(\sum\limits_{i=1}^{n-1} a_i u_i, \sum\limits_{i=n}^{\infty} a_i u_i / u_{n-1}\right)$ are not K-algebra isomorphisms.

Lemma 6. *Let $S = \{\alpha_i\}_{i\geq 1}$ be a sequence of elements of K such that, for a fixed $n \in \mathbb{N}$, $\alpha_1 = \alpha_2 = ... = \alpha_n = \gamma$ and $\alpha_i \neq \gamma$, for all $i > n$. Then $K_S[[X]] \cong K[X]/ < X^n > \oplus K_{S'}[[X]]$, where $S' = \{\beta_i\}_{i\geq 1}$, with $\beta_i = \alpha_{n+i}$.*

Proof. By (18) it follows that an element $f = \sum\limits_{i=0}^{\infty} e_i u_i$ is an idempotent if and only if $\sum\limits_{i=0}^{j} e_i f_{j,i} = e_j$ for all $j \geq 0$. In order to find an idempotent we consider f such that $e_0 = 1, e_1 = e_2 = ... = e_{n-1} = 0$. Then by (17), for every $i = 1, 2, ..., n-1$ and $j = 0, 1, ..., i-1$, it follows that

$$f_{i,j} = 0 \text{ and } f_{i,i} = 1. \tag{45}$$

By successive applications of (17) and (19) we obtain that

$$f_{n+k,n} = e_k + \sum_{l=1}^{n} \left(\sum_{i_1,...,i_l=l}^{n} \prod_{j=1}^{l} (\alpha_{k+1+i_j} - \gamma) \right) e_{k+l}.$$

If we put the conditions

$$f_{n+k,n} = 0, \tag{46}$$

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for every $k = 0, 1, ...$ we find
(47)
$$e_{n+k} = \frac{-1}{(\alpha_{n+k+1} - \gamma)^n} \left(e_k + \sum_{l=1}^{n-1} \left(\sum_{i_1,\dots,i_l=l}^{n} \prod_{j=1}^{l} (\alpha_{k+1+i_j} - \gamma) \right) e_{k+l} \right).$$

Thus

$$f = 1 + \sum_{k=0}^{\infty} e_{n+k} u_{n+k}, \tag{48}$$

where e_{n+k} defined in (47). By (16) and (46) we obtain

$$f u_n = \sum_{j=n}^{\infty} f_{j,n} u_j = 0$$

and hence

$$f u_i = 0 \text{ for all } i \geq n. \tag{49}$$

We show that f given by (48) is idempotent. By (22), for every $k \geq 0$, we can write $f_{n+k,m} = \sum_{r=0}^{m-n} f_{n+k-r,n} P_{n+k-r,n}$, for $n \leq m \leq n+k$. Hence by (46)

$$f_{n+k,m} = 0 \text{ for all } n \leq m \leq n+k \text{ and } k \geq 0. \tag{50}$$

Using (45) and (50) it follows easily that $\sum_{i=0}^{j} e_i f_{j,i} = e_j$ for all $j \geq 0$ and hence f is an idempotent. Thus we can write

$$K_S[[X]] \cong K_S[[X]](f) \oplus K_S[[X]](1-f), \tag{51}$$

where by (49) $K_S[[X]](f) = \{(\sum_{i=0}^{n-1} a_i u_i) f \mid a_i \in K\}$ is a ring with f as identity.

Define $\phi : K_S[[X]](f) \to K[X]/ < X - \gamma >^n$ by

$$\phi((\sum_{i=0}^{n-1} a_i u_i) f) = \overline{\sum_{i=0}^{n-1} a_i u_i} \in K[X]/ < X - \gamma >^n .$$

Since $\overline{\sum_{i=0}^{n-1} a_i u_i} = \overline{0}$ if and only if all a_i are equal to zero, by (45) it follows that ϕ is well defined and one to one. Because by Lemma 1 $\sum_{i=0}^{n-1} \gamma_i X^i \in K[X]$ can be uniquely written as $\sum_{i=0}^{n-1} a_i u_i$ we obtain that $\phi((\sum_{i=0}^{n-1} a_i u_i) f) = \overline{\sum_{i=0}^{n-1} a_i u_i} = \overline{\sum_{i=0}^{n-1} \gamma_i X^i}$ and ϕ is bijective. It is obvious

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that ϕ is a K-linear mapping. Because for $i \le n$ $u_i = (X-\gamma)^i$, by (49) we have

$$\phi((\sum_{i=0}^{n-1} a_i u_i)f)((\sum_{i=0}^{n-1} b_i u_i)f) = \phi((\sum_{i=0}^{n-1} c_i u_i + \sum_{i=n}^{2n-2} d_i u_i)f) = \phi((\sum_{i=0}^{n-1} c_i u_i)f) =$$

$$\overline{\sum_{i=0}^{n-1} c_i u_i} = \overline{\sum_{i=0}^{n-1} c_i u_i + \sum_{i=n}^{2n-2} d_i u_i} = \overline{(\sum_{i=0}^{n-1} a_i u_i)(\sum_{i=0}^{n-1} b_i u_i)} = (\overline{\sum_{i=0}^{n-1} a_i u_i})(\overline{\sum_{i=0}^{n-1} b_i u_i})$$

$$= \phi((\sum_{i=0}^{n-1} a_i u_i)f)\phi((\sum_{i=0}^{n-1} a_i u_i)f),$$

where $c_k = \sum_{i=0}^{k} a_i b_{k-i}$ and $d_i \in K$. Hence we obtain that

$$K_S[[X]](f) \cong K[X]/ < X-\gamma >^n \cong K[X]/ < X^n > . \tag{52}$$

Also by (49) $K_S[[X]](1-f) = \{\sum_{i=n}^{\infty} a_i u_i \mid a_i \in K\}$ is a ring with $1-f$ as identity element. Define $\psi : K_S[[X]] \to K_{S'}[[X]]$ by $\psi(\sum_{i=n}^{\infty} a_i u_i) = g(\sum_{i=0}^{\infty} a_{n+i} v_i) \in K_{S'}[[X]]$, where $g = (X-\gamma)^n = u_n$ and $v_i = \prod_{j=1}^{i}(X-\beta_j)$. Since in $K_{S'}[[X]]$ $g_{i,i} \neq 0$, by Theorem 1 g is unit in $K_{S'}[[X]]$ and ψ is well defined and bijective. Let $h' = \sum_{i=n}^{\infty} a_i u_i$, $h'' = \sum_{i=n}^{\infty} b_i u_i$ be two elements of $K_S[[X]](1-f)$. Then obviously $\psi(\sum_{i=n}^{\infty} a_i u_i + \sum_{i=n}^{\infty} b_i u_i) = \psi(\sum_{i=n}^{\infty} a_i u_i) + \psi(\sum_{i=n}^{\infty} b_i u_i)$ and $\psi(u_n \sum_{i=n}^{\infty} a_i u_i) = \psi(\sum_{i=n}^{\infty} c_i u_i) = g(\sum_{i=0}^{\infty} c_{n+i} v_i) = g(g\sum_{i=0}^{\infty} a_{n+i} v_i) = \psi(u_n)\psi(\sum_{i=n}^{\infty} a_i u_i)$, where $c_i \in K$. Hence by induction

$$\psi(u_k \sum_{i=n}^{\infty} a_i u_i) = \psi(u_k)\psi(\sum_{i=n}^{\infty} a_i u_i), \tag{53}$$

for every $k \ge n$. Because ψ is continuous with respect to the corresponding norms defined in (12), by (53) we have $\psi(h'h'') = \psi(\sum_{i=n}^{\infty} a_i h'' u_i) = \sum_{i=n}^{\infty} \psi(a_i u_i)\psi(h'') = \psi(\sum_{i=n}^{\infty} a_i u_i)\psi(\sum_{i=n}^{\infty} b_i u_i) = \psi(h')\psi(h'')$. Hence

$$K_S[[X]](1-f) \cong K_{S'}[[X]] \tag{54}$$

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and by (51),(52) and (54) we have

$$K_S[[X]] \cong K[X]/ < X^n > \oplus K_{S'}[[X]]. \square \tag{55}$$

Theorem 4. *Suppose $S = \{\alpha_n\}_{n\geq 1}$ is a sequence of elements of K such that*

a) I_S is a finite set;

b) S has canonical form.

Then $K_S[[X]] \cong \bigoplus_{i=1}^{s} K[X]/ < X^{n_i} > \oplus K_{S'}[[X]]$, where S' is a purely periodic sequence.

Proof. Since $\alpha_1 = \alpha_2 = ... = \alpha_{n_1} \neq \alpha_i$, for all $i > n_1$, then by Lemma 6 we have $K_S[[X]] \cong K[X]/ < X^{n_1} > \oplus K_{S^{(1)}}[[X]]$ where $S^{(1)} = \{\alpha_{n_1+i}\}_{i\geq 1}$. Since in $S^{(1)}$ $\alpha_{n_1+1} = ... = \alpha_{n_1+n_2} \neq \alpha_i$, for all $i > n_1 + n_2$, then again by Lemma 6 we have $K_S[[X]] \cong K[X]/ < X^{n_1} > \oplus K[X]/ < X^{n_2} > \oplus K_{S^{(2)}}[[X]]$, where $S^{(2)} = \{\alpha_{n_1+n_2+i}\}_{i\geq 1}$. By recurrence we obtain the theorem. □

Corollary 1. *If I_S is a finite set, then $K_S[[X]]$ is a principal ideal ring.*

Proof. If S is a purely periodic sequence by [2] Theorem 2.3 it follows that $K_S[[X]]$ is a noetherian ring. By Theorem 4 and Remark 1 this is true for every S with I_S a finite set. Since by Theorems 2 and 3 all its maximal ideals are principal, $K_S[[X]]$ is a principal ring (see for example Lemma 2 from [5]). □

Corollary 2. *If I_S is a finite set, for each $i \in I_S$, $\bigcap_{j=0}^{\infty} < X - \alpha_i >^j$ is a prime ideal in $K_S[[X]]$.*

Proof. By Corollary 1 $K_S[[X]]$ is a principal ring. Then $\bigcap_{j=0}^{\infty} < X - \alpha_i >^j = < h^{(i)} >$, $h^{(i)} \in K_S[[X]]$ for all $i \in I_S$. Suppose there exists $f, g \in K_S[[X]]$ such that $f \notin < h^{(i)} >$ and $g \notin < h^{(i)} >$ but $fg \in < h^{(i)} >$. Then there exist two non-negative integers n_1 and n_2 such that

$$f = (X - \alpha_i)^{n_1} f' \tag{56}$$

$$g = (X - \alpha_i)^{n_2} g', \tag{57}$$

where $f', g' \in K_S[[X]]$ but $f', g' \notin < X - \alpha_i >$. Hence by Lemma 5

$$f'g' \notin < X - \alpha_i > . \tag{58}$$

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Now by (56)-(58) we obtain $fg = (X-\alpha_i)^{n_1+n_2} f'g'$ where fg belong to $\bigcap_{j=0}^{n_1+n_2} < X-\alpha_i >^j$ but $fg \notin \bigcap_{j=0}^{n_1+n_2+1} < X-\alpha_i >^j$. Hence $fg \notin < h^{(i)} >$, a contradiction which implies that $< h^{(i)} >$ is the prime ideal in $K_S[[X]]$ for all $i \in I_S$.$\square$

In the next lemma we give the structure of $K_S[[X]]$ when S is a purely periodic sequence.

Lemma 7. *If $S = \{\alpha_n\}_{n\geq 1}$ is a purely periodic sequence of elements of K, having the smallest period $m \in \mathbb{N}^*$, then $K_S[[X]] \cong \bigoplus_{i=1}^{m} K[[X]]$.*

Proof. Suppose

$$M = \bigcap_{i=1}^{m} M_i =< u_m >, \tag{59}$$

where by Lemma 5 for each $i \in I_S$, $M_i =< X-\alpha_i >$ is a maximal ideal in $K_S[[X]]$. Since m is the smallest period of a purely periodic sequence we have $u_m^k = u_{km}$ for all $k \in \mathbb{N}$ and (59) implies that $\bigcap_{j=0}^{\infty} M^j =< 0 >$. We consider on $K_S[[X]]$ the M-adic topology (see for example [9], Ch-VIII) defined by means of

$$\omega(f-g) = \sup\{k \mid f-g \in M^k\}, \tag{60}$$

where $f, g \in K_S[[X]]$ and the distance

$$d(f,g) = e^{-\omega(f-g)},\ e \in \mathbb{R},\ e > 1, \tag{61}$$

which satisfies the ultrametric inequality. We show that $K_S[[X]]$ is complete with respect to M-adic topology which in this case coincides with the topology defined by the norm given in (12). Consider $\{f_n\}_{n\geq 1}$, where $f_n = \sum_{i=0}^{\infty} a_i^{(n)} u_i$, a Cauchy sequence of elements of $K_S[[X]]$. Then for each $\epsilon > 0$ there exists $n_0(\epsilon) \in \mathbb{N}$ such that $d(f_n, f_{n+1}) < \epsilon$ for all $n \geq n_0(\epsilon)$. Thus for each $q \in \mathbb{N}$ there exists $n'(q) \in \mathbb{N}$ such that $\omega(f_n - f_{n+1}) > q$ for all $n \geq n'(q)$ implies $a_i^{(n)} = a_i^{(n'(q))}$ for each $i \leq q$ and $n \geq n'(q)$. We take $f = \sum_{i=0}^{\infty} a_i u_i \in K_S[[X]]$, where $a_q = a_q^{(n'(q))}$ for each $q \geq 0$. Then, for all $q \geq 0$, $\omega(f_n - f) > q$ for each $n \geq \max\{n'(0), n'(1), ..., n'(q)\}$. Hence $f \in K_S[[X]]$ is the limit of the sequence $\{f_n\}_{n\geq 1}$, which implies that $K_S[[X]]$ is complete. By Theorem

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3 and Corollary 2 from [9], Ch-VIII, §8, p.283 we can write

$$K_S[[X]] \cong \bigoplus_{i=1}^{m} R_i, \tag{62}$$

where each $R_i = \bigcap_{n=0}^{\infty} (\prod_{j\neq i}(X-\alpha_j))^n \subset K_S[[X]]$ is a complete local ring. Let $R_i = \bigcap_{n=0}^{\infty} (\prod_{j\neq i}(X-\alpha_j))^n =< f_i >$ with f_i a nonzero idempotent of $K_S[[X]]$. Define the linear application $L_{f_i} : K_S[[X]] \to R_i$ by $L_{f_i}(f) = ff_i$. Then $< h^{(i)} >\subset \mathrm{Ker}(L_{f_i})$, where $< h^{(i)} >= \bigcap_{j=0}^{\infty}(X-\alpha_i)^j$ and $f_i \notin< h^{(i)} >$ because $f_i \in< h^{(i)} >$ implies $f_i \in \bigcap_{j=0}^{\infty} M^j$ and hence $f_i = 0$. Since $f_i \notin< h^{(i)} >$ there exists $p \in \mathbb{N}$ such that $f_i = (X-\alpha_i)^p f_i'$ with $f_i' \notin M_i$.

We prove that

$$\mathrm{Ker}(L_{f_i}) =< h^{(i)} > . \tag{63}$$

Suppose the contrary that there exists $g \in \mathrm{Ker}(L_{f_i})$ and $g \notin< h^{(i)} >$. Then there exists $n \in \mathbb{N}$ such that $g = (X-\alpha_i)^n g'$ with $g' \notin M_i$. Because M_i is a prime ideal $g' f_i' \notin M_i$ and hence $(X-\alpha_i)^{p+n} g' f_i' \neq 0$. This implies that $L_{f_i}(g) = f_i g = (X-\alpha_i)^{p+n} g' f_i' \neq 0$, a contradiction which implies (63). Hence we obtain

$$R_i \cong K_S[[X]]/ < h^{(i)} > . \tag{64}$$

Clearly $K \subset R_i$ for each $i = 1, 2, ..., m$ and hence each R_i is an equicharacteristic ring (see [9], Ch-VIII, §12, p.304). Now by (64) and Corollary 2 each $< h^{(i)} >$ is the prime ideal and R_i is an integral domain. By Corollary 1 and (62) each R_i is principal local ring. Then each ideal in R_i is a power of $\tilde{M}_i$, where $\tilde{M}_i$ is the unique maximal ideal in R_i. Since the dimension of R_i is one and $\tilde{M}_i$ is generated by one element each R_i is regular (see [9], Ch-VIII, §11, p.301). Now using Corollary from [9], Ch-VIII, §12, p.307, we obtained that each R_i is isomorphic to $K_i[[Y_i]]$, where in our case $K_i = R_i/\tilde{M}_i \cong K_S[[X]]/M_i$. Since $K_S[[X]]$ is the completion of ring of polynomials $K[X]$ with respect to filtration defined by the ideals $< u_n >, n \in \mathbb{N}$, $K_S[[X]]/M_i \cong K[X]/ < X-\alpha_i >\cong K$ and we obtain

$$R_i \cong K[[X]]. \tag{65}$$

Now by (65) and (62) it follows that $K_S[[X]] \cong \bigoplus_{i=1}^{m} K[[X]]$. □

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Theorem 5. *Suppose* $S = \{\alpha_n\}_{n\geq 1}$ *is a sequence of elements of* K *such that* I_S *is a finite set. If* s, m, n_1, $\ldots, n_s$ *are the invariants of* S, *then* $K_S[[X]] \cong \bigoplus_{i=1}^{s} K[X]/ < X^{n_i} > \oplus \bigoplus_{i=1}^{m} K[[X]]$.

Proof. By Remark 1 we may suppose that S has canonical form. Now the theorem follows by Theorem 4 and Lemma 7. □.

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