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Frame Wavelet Sets and Wavelets in Banach Spaces

Devendra Kumar

Department of Mathematics
M.M.H.College,Model Town,Ghaziabad-201001,U.P.India
e-mail:d_kumar001@rediffmail.com

Shiv Kumar

Department of Mathematics
D.A.V.College,Jalandhar (Punjab),India
(Punjab Technical University)
e-mail:abhi_tulli60@yahoo.com

Abstract. In this paper, we try to study a special class of frame wavelets in Banach spaces whose Fourier transforms are supported by frame wavelet sets. Our results generalize the various results of [2] to Banach spaces other than Hilbert spaces using the Feichtinger and Gröchenig theory.

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1 Introduction

Frame provide stable expansions in Hilbert spaces, but they may be over complete and the coefficients in the frame expansion need not be unique unlike in orthogonal expansions. This redundancy is useful for the application point of view that is to noise reduction or for the reconstruction from lossy data [3,5,12]. The construction of stable orthonormal basis are often difficult in a numerical efficient way than the construction of frames which are more flexible. Sometimes it is reasonable to use the frames to analyze additional properties of functions beyond the Hilbert space. This led to the characterization of an associated family of Banach spaces of functions by the values of the frame coefficients which play an important role in non-linear approximation and in compression algorithms [4]. However, in [6] Gröchenig showed that certain frames for Hilbert spaces extend automatically to Banach frames. Using this theory he derived some results on the construction of non-uniform Gabor frames and solved a problem about non-uniform sampling in shift-invariant spaces. Recently, Kumar [9] studied the convergence of wavelet expansions associated with dilation matrix in the variable L^p spaces using the approximate identity. In an another paper Kumar [10] studied the convergence of non-orthogonal wavelet expansions in $L^p(R), 1 < p < \infty$.

Let H be a Hilbert space. A collection of elements $\{x_i\}$ is called a frame of H if there exist two positive constants $0 < A \leq B$ such that

$$A \parallel f \parallel^2 \leq \sum_i \mid \langle f, x_i \rangle \mid^2 \leq B \parallel f \parallel^2$$

for $f \in H$. The supremum of all such numbers A and infimum of all such numbers B are called the frame bounds of the frame and are denoted by A_0 and B_0 respectively. A frame $\{x_i\}$ is called a tight frame when $A_0 = B_0$, and a normalized tight frame when $A_0 = B_0 = 1$. Frames can be regarded as generalizations of orthogonal bases of Hilbert spaces. The generalization of frames to Banach spaces other than Hilbert spaces generated by a single function through dilations and translations are of special interest due to their close relation to wavelets and will be our focus in this paper.

Let D and T be the standard dilation and translation operators, respectively, on $L^2(R)$, defined by $(Df)(x) = \sqrt{2}f(2x)$ and $(Tf)(x) = f(x-1)$ for any $f \in L^2(R)$. A function $\varphi \in L^2(R)$ is called a frame wavelet for $L^2(R)$ if

$$\{\varphi_{n,\ell}(x)\} = \{2^{n/2}\varphi(2^n x - \ell) : n, \ell \in Z\} = \{D^n T^\ell \varphi : n, \ell \in Z\} \tag{1.1}$$

is a frame of $L^2(R)$, i.e., if there exist two positive constants $0 < A \leq B$ such that

$$A \parallel f \parallel^2 \leq \sum_{n,\ell \in Z} \mid \langle f, D^n T^\ell \varphi \rangle \mid^2 \leq B \parallel f \parallel^2 \tag{1.2}$$

for all $f \in L^2(R)$. φ is called a tight frame wavelet if this frame is tight. Similarly, φ is called a normalized tight frame wavelet if this frame is a normalized tight frame.

Let $\hat{\varphi} = \frac{1}{\sqrt{2\pi}}\chi_E$, where E is a Lebesgue measurable set with finite measure and $\hat{\varphi}$ denote the Fourier transform of φ. We call E a frame wavelet set or just a frame set if the function φ is a frame wavelet for $L^2(R)$. Similarly, E is called a (normalized) tight frame wavelet set if φ is a (normalized) tight frame wavelet. The study of frame sets appears to play a very important role here since for any given positive numbers α,there exist frame sets of measure α. This enables us to construct various frequency domain frame wavelets with support of measure α.

In [1], a complete characterization of tight frame wavelet sets t-sets was obtained, together with some necessary or sufficient conditions for a set E to be an frame wavelet set. Let $\varphi \in L^2(R)$ and let $E_{\hat{\varphi}} = supp(\hat{\varphi})$. It turns out that if $E_{\hat{\varphi}}$ is a t-sets then its Lebesgue measure is at most 2π. Furthermore, the result in [7], regarding frame wavelets can be readily applied to determine whether φ is a frame wavelet. This implies that we are unlikely to obtain new results if we only concentrate on frame wavelets in Banach spaces whose Fourier transforms are supported by f-sets. Our results generalize the various results contained in [2]. Although, V.V.Kisil [8] described a construction of wavelets in Banach spaces generated by admissible group representations. He considered operator-valued Segal-Bargmann-type spaces and the

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 1, p.101-110

Weyl functional calculus. But our approach and results are different from those of V.V.Kisil [8].

Now we discuss the theory of Feichtinger and Gr$\ddot{o}$chenig, which produces coherent state decompositions of a large class of Banach spaces in a way that generalizes the notion of a frame in a Hilbert space. Let H be a Hilbert space, G a topological group with left Haar measure μ and π a representation of G on H. Let $L^2(G)$ denote the Hilbert space of μ-square integrable functions on G, i.e.,

$$L^2(G) = \{F : G \to C :\| F \|_{L^2(G)}= (\int_G | F(x) |^2 d\mu(x))^{1/2} < \infty\}$$

with inner product $\langle F_1, F_2\rangle = \int_G F_1(x)\overline{F_2(x)}d\mu(x)$

Definition 1.1. [13]. A measure μ on a group G is said to be left-invariant provided that for every integrable function f on G and every $y \in G$ we have $\int_G f(y, x)d\mu(x) = \int_G f(x)d\mu(x)$.

A left measure on G, is known as left Haar measure, exists and is unique up to a constant multiple.

Definition 1.2. Let H be a Hilbert space

1. A representation π of G on H is a mapping $\pi : G \to L(H)$ such that π(x.y)=π(x)π(y) for every $x, y \in$G
2. A vector $g \in H$ is admissible if $\int_G | \langle g, \pi(x)g\rangle |^2 d\mu(x) < \infty$, where μ is the left Haar measure on G.
3. A vector $g \in H$ is cyclic if span $\{\pi(x)g\}_{x\in G}$ is dense in H, or equivalently, if the only $f \in H$ such that $\langle f, \pi(x)g\rangle = 0$ for all $x \in G$ is $f = 0$.
4. π is unitary if the map $\pi(x) : H \to H$ is unitary for each $x \in G$.
5. π is irreducible if every $g \in H\backslash\{0\}$ is cyclic.
6. π is square integrable if π is irreducible and there exists an admissible $g \in H\backslash\{0\}$.

The Fourier transform is normalized so that it is unitary operator which implies that (1.2) is equivalent to

$$A \| f \|^2\leq \sum_{n,\ell\in Z} | \langle f, \hat{D}^n\hat{T}^\ell\hat{\varphi}\rangle |^2\leq B \| f \|^2, \forall f \in L^2(R). \tag{1.3}$$

Definition 1.3. Let $\hat{\varphi} \in H\backslash\{0\}$ be admissible. For $f \in H$ we let $V_{\hat{\varphi}}f$ be the complex-valued function on G given by $V_{\hat{\varphi}}f(x) = \langle f, \pi(x)\hat{\varphi}\rangle$, where $\pi(x)\hat{\varphi} = \hat{D}^n\hat{T}^\ell\hat{\varphi}$.

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 1, p.101-110

In view of [7], we call $V_{\hat{\varphi}}f$ the voice transform of f with respect to $\hat{\varphi}$.

Suppose that π is an irreducible, unitary representation of G on H which is integrable, i.e. these is a $\hat{\varphi} \in H \backslash \{0\}$ such that

$$\int_G \mid V_{\hat{\varphi}}\hat{\varphi}(x) \mid d\mu(x) = \int_G \mid \langle \hat{\varphi}, \pi(x)\hat{\varphi} \rangle \mid d\mu(x) < \infty,$$

and which is continuous, i.e., $\pi(x)\hat{\varphi}$ is a continuous map of G into H_o for all $x \in G$ since dilation and translations are continuous.

Let $H_o = \{\hat{\varphi} \in H : V_{\hat{\varphi}}\hat{\varphi} \in L^1(G)\}$ and $H_o^{'} \supset H$ be the dual of H_o. We define the coorbit space $C_o(L^p(G)) = \{f \in H_o^{'} : V_{\hat{\varphi}}f \in L^p(G)\}$ and the norm $\| f \|_{C_o(L^p(G))} = \| V_{\hat{\varphi}}f \|_{L^p(G)}$. Also, we define an appropriate sequence space $\ell^p(Z)$ corresponding to $L^p(G)$.

We see that the following integral operator is a convolution operator on G by

$$\begin{aligned} \int_G V_{\hat{\varphi}}f(x)V_{\hat{\varphi}}\hat{\varphi}(x^{-1}y)d\mu(x) &= \int_G \langle f, \pi(x)\hat{\varphi}\rangle \langle \hat{\varphi}, \pi(x^{-1}y)\hat{\varphi}\rangle d\mu(x) \\ &= \int_G \langle f, \pi(x)\hat{\varphi}\rangle \langle \pi(x)\hat{\varphi}, \pi(y)\hat{\varphi}\rangle d\mu(x) \\ &= \langle f, \pi(y)\hat{\varphi}\rangle \\ &= V_{\hat{\varphi}}f(y). \end{aligned}$$

Here $V_{\hat{\varphi}}\hat{\varphi}(x^{-1}y)$ is an approximate identity. This operator can be approximate by a discrete operator. For this, let $\psi = \{\psi_i\}$ be a collocation of functions on G that satisfy:

1. $\sup_i \| \psi_i \|_\infty < \infty$,
2. there is an open set $O \subset G$ with compact closure and points $x_i \in G$ such that $supp\{\psi_i\} \subset x_i O$ for each i,
3. $\sum_i \psi_i(x) \equiv 2\pi$,
4. $\sup_{x \in G} \sharp\{i \in I : z \in x_i Q\} < \infty$ for each compact set $Q \subset G$.

We say such a ψ bounded uniform partition of 2π. Define the operator T_ψ on $L^p(G)$ associated to a particular bounded uniform partition of ψ, by

$$T_\psi f(y) = \textstyle\sum_i \langle F, \psi_i \rangle V_{\hat{\varphi}}\hat{\varphi}(x_i^{-1}y) \ , \ F(y) = V_\varphi f(y) \in L^p(G) \text{ for some } f \in C_o(L^p(G)).$$

Consider any collection of points $\{x_i\} \subset G$ such that $\cup x_i U = G$ and $x_i V \cap x_j V = \phi$ if $i \neq j$ where U and V are compact neighborhoods of the identity in G. It can be

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 1, p.101-110

shown that for any bounded uniform partition of ψ associated to $\{x_i\}$, there are constants $A, B > 0$ such that

$$A \parallel V_{\hat{\varphi}} f \parallel_{L^p(G)} \leq \parallel \langle V_{\hat{\varphi}} f, \psi_i \rangle \parallel_{\ell^p(Z)} \leq B \parallel V_{\hat{\varphi}} f \parallel_{L^p(G)} . \tag{1.4}$$

The operator T_ψ is continuous and continuously invertible on $S = \{V_{\hat{\varphi}} f \in L^p(G)$ for some $f \in C_o(L^p(G))\}$. Thus for each $f \in C_o(L^p(G))$ we define

$$(H_E f)(\xi) = \sum_{n,\ell \in z} \langle f, \pi(x)\hat{\varphi} \rangle \pi(\xi)\hat{\varphi},$$

where $\hat{\varphi} = \frac{1}{\sqrt{2\pi}} \chi_E$, E be a Lebesgue measurable set with finite measure.

We observe that if $H_E f$ converges to a function in $C_o(L^p(G))$ under the norm $\| \|_{C_o(L^p(G))}$, then the generalization of (1.3) to Banach spaces other than Hilbert spaces is

$$A \parallel f \parallel_{C_o(L^p(G))} \leq \parallel \langle H_E f, f \rangle \parallel_{\ell^p(z)} \leq B \parallel f \parallel_{C_o(L^p(G))} \tag{1.5}$$

for all $f \in C_o(L^p(G))$. Now for each $f \in C_o(L^p(G))$ we can write

$$\begin{aligned}
\langle H_E f, f \rangle &= T_\psi (T_\psi^{-1}) \langle H_E f, f \rangle \\
&= \sum_i \langle T_\psi^{-1} \langle H_E f, f \rangle, \psi_i \rangle V_{\hat{\varphi}} \hat{\varphi}(x_i^{-1} y) \\
&= \sum_i \langle T_\psi^{-1} \langle \sum_{n,\ell \in Z} \langle f, \pi(x)\hat{\varphi} \rangle \pi(y)\hat{\varphi}, f \rangle, \psi_i \rangle \langle \pi(x_i)\hat{\varphi}, \pi(y)\hat{\varphi} \rangle \\
&= \sum_i \langle T_\psi^{-1} V_{\hat{\varphi}} f \langle \pi(y)\hat{\varphi}, f \rangle, \psi_i \rangle \langle \pi(x_i)\hat{\varphi}, \pi(y)\hat{\varphi} \rangle \\
&= \sum_i \langle T_\psi^{-1} V_{\hat{\varphi}} f V_{\hat{\varphi}} f(x_i), \psi_i \rangle \\
&= \sum_i \langle T_\psi^{-1} V_{\hat{\varphi}} f \langle \pi(x_i)\hat{\varphi}, f \rangle, \psi_i \rangle \\
&= \langle \sum_i \langle T_\psi^{-1} V_{\hat{\varphi}} f, \psi_i \rangle \pi(x_i)\hat{\varphi}, f \rangle \\
&= \langle \sum_i \beta_i(f) \pi(x_i)\hat{\varphi}, f \rangle .
\end{aligned}$$

It can be seen that $H_E f = \sum_i \beta_i(f)\pi(x_i)\hat{\varphi}$ and for some constants $A_o, B_o > 0$ we have

$$A_o \parallel f \parallel_{C_o(L^p(G))} \leq \parallel \beta_i(f) \parallel_{\ell^p(Z)} \leq B_o \parallel f \parallel_{C_o(L^p(G))}, \tag{1.6}$$

using (1.6) we get a generalization of frames to Banach spaces other than Hilbert spaces i.e.,(1.5).

If the right inequality in (1.6) holds then it can be easily verified that $\parallel H_E f \parallel_{C_o(L^p(G))} \leq b \parallel f \parallel_{C_o(L^p(G))}$ for some constant $b > 0$, H_E define a bounded linear operator on $C_o(L^p(G))$. In this case we say that E is a Bessel set.

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 1, p.101-110

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 1, p.101-110

2 Some basic concepts on frame wavelet sets

In this section we shall study the basic concepts and terms concerning frame sets. For more detailed information about this topic see [1].

Let E be a measurable set. Two elements $x, y \in E$ are δ equivalent if $x = 2^n y$ for some integer n. The $\delta-$index of a point x in E is the number of elements in its δ equivalent class and is denoted by $\delta_E(x)$. Let $E(\delta, \kappa) = \{x \in E : \delta_E(x) = \kappa\}$.

Then E is the disjoint union of the sets $E(\delta, \kappa)$. two sets E and F with $E = E(\delta, 1)$ and $F = F(\delta, 1)$ are said to be 2-dilation equivalent if every point in E is δ equivalent to a point in F and vice versa.

Lemma 2.1. *If E is a Lebesgue measurable set, then each $E(\delta, \kappa)$ $(\kappa \geq 1)$ is also Lebesgue measurable. Furthermore, each $E(\delta, \kappa)$ is a disjoint union of κ measurable sets $\{E^{(j)}(\delta, \kappa), 1 \leq j \leq \kappa\}$, such that $E^{(j)}(\delta, \kappa)$ δ equivalent$E^{(j')}(\delta, \kappa)$ for any $1 \leq j$, $j' \leq \kappa$.*

If $\Delta(E) = U_{\kappa \in z} E^{(1)}(\delta, \kappa)$, then every point in it has δ-index one. Furthermore, we have $U_{\kappa \in z} 2^\kappa E = U_{\kappa \in z} 2^\kappa \Delta(E)$. A set E is called a 2-dilation generator of R if $E = E(\delta, 1)$ and $U_{\kappa \in z} 2^\kappa E = R$. A 2-dilation generator for a subset of R that is invariant under 2-dilation can be similarly defined.

In the case of translation, we say that $x, y \in E$ are 2π-translation equivalent, denoted by $x\tau y$, if $x = y + 2n\pi$ for some integer n. The $\tau-$index of a point x in E is the number of elements in its τ equivalent class and is denoted by $\tau_E(x)$. Let $E(\tau, \kappa) = \{x \in E : \tau_E(x) = \kappa\}$. Than E is the disjoint union of the sets $E(\tau, \kappa)$. Define $\tau(E) = U_{n \in z}(E \cap ((2n\pi, 2(n+1)\pi) - 2n\pi))$. This is a disjoint union if and only if $E = E(\tau, 1)$. Two sets E and F with $E = E(\tau, 1)$ and $F = F(\tau, 1)$ are said to be 2π-translation equivalent if every point in E is τ equivalent to a point in F and vice versa.

Lemma 2.2. *If E is a Lebesgue measurable set, then each $E(\tau, \kappa)(\kappa \geq 1)$ is also Lebesgue measurable. Furthermore, each $E(\tau, \kappa)$ is a disjoint union of $\kappa-$measurable sets $\{E^{(j)}(\tau, \kappa)\}$, $1 \leq j \leq \kappa$, such that $E^{(j)}(\tau, \kappa)$ τ equivalent $E^{(j')}(\tau, \kappa)$ for any $1 \leq j, j' \leq \kappa$.*

Remark 2.3. The set $E(\tau, 1)$ is uniquely determined by E. Since $E(\tau, 1)$ consists of all points in E that are 2π translation redundancy free and is denoted by $T_{rf}(E)$.

Now we have the following definition

Definition 2.4. A set E is called a basic set if there exists a constant $M > 0$, such that $\mu(E(\delta, m)) = 0$ and $\mu(E(\tau, m)) = 0$ for any $m > M$.

The main aim in this section is a generalization of the following theorem [2] to Banach spaces other than Hilbert spaces.

Theorem 2.5. *Let E be a Lebesgue measurable set with finite measure. Then E is an f-set if E is a basic set and $U_{n\in z}2^n\tau_{rf}(E) = R$.*

But first we shall prove.

Lemma 2.6. *If E is a basic set such that $E(\delta, m) = E(\tau, m) = \phi$ for all $m > M$, where $M > 0$ is a positive integer, then for any $f, g \in C_o(L^p(G))$ we have*

$$\| \langle H_E(f), g\rangle \|_{\ell^p(Z)} \leq M^{5/2} \| f \|_{C_o(L^p(G))} \| g \|_{C_o(L^p(G))} .$$

Proof. The proof follows easily by using the right hand inequality in (1.6) and equation (14) of [1, pp.2051].

Lemma 2.7. *Let E be a basic set with property that $U_{\kappa\in Z}2^\kappa\tau_{rf}(E) = U_{\kappa\in z}2^\kappa E = \Omega$. Then for any f with support in Ω we have*

$$\| \langle H_E(f), f\rangle \|_{\ell^p(Z)} \geq \| f \|_{C_o(L^p(G))} .$$

Proof. We have [1],

$$\sum_{j=1}^{m} f^\kappa_{mj} \cdot \chi_{2^\kappa E(\tau,m)} = \sum_{\ell\in z} \langle f, \hat{D}^\kappa \hat{\tau}^\ell \frac{1}{\sqrt{2\pi}} \chi_{E(\tau,m)}\rangle \hat{D}^\kappa \hat{\tau}^\ell \frac{1}{\sqrt{2\pi}} \chi_{E(\tau,m)}$$

hence

$$\langle \sum_{j=1}^{m} f^\kappa_{mj} \cdot \chi_{2^\kappa E(\tau,m)}, f\rangle = \sum_{\ell\in Z} | \langle f, \hat{D}^\kappa \hat{\tau}^\ell \frac{1}{\sqrt{2\pi}} \chi_{E(\tau,m)}\rangle |^2 \geq 0.$$

By Meyer [11]; the equivalent characterization of $L^p(R)$ i.e., if

$$f \in L^p(R) \Rightarrow [\sum_{j,\kappa} | \langle f, \varphi_{j,\kappa}\rangle |^2 | \varphi_{j,\kappa} |^2]^{1/2} \in L^p(R).$$

Thus

$$\begin{aligned}
\| \langle H_E f, f\rangle \|_{\ell^p(Z)} &= \| \sum_{\kappa\in Z} \sum_{m=1}^{M} \sum_{j=1}^{m} \langle f^\kappa_{mj} \chi_{2^\kappa E(\tau,m)}, f\rangle \|_{\ell^p(z)} \\
&\geq \| \sum_{\kappa\in Z} \langle f \chi_{2^\kappa E(\tau,m)}, f\rangle \|_{\ell^p(Z)} \\
&= \{ \int_G [| f |^2 | \sum_{\kappa\in Z} \chi_{2^\kappa E(\tau,1)} |]^2 d\mu \}^{1/2} \\
&\geq \| f \|_{c_o(L^p(G))},
\end{aligned}$$

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 1, p.101-110

since $\sum_{\kappa\in z}\chi_{2^\kappa(\tau,1)} \geq 1$.

Let E be a basic set and let $\Omega = U_{\kappa\in Z}2^\kappa E$, Set $E_1 = E\cap(U_{\kappa\in Z}2^\kappa T_{rf}(E))$, $\bar{E}_1 = E\backslash E_1$, $E_2 = \bar{E}_1\cap(U_{\kappa\in z}2^\kappa T_{rf}(\bar{E}_1))$, $\bar{E}_2 = \bar{E}_1\backslash E_2$, $E_3 = \bar{E}_2\cap(U_{\kappa\in z}2^\kappa T_{rf}(\bar{E}_2))$; in general set $\bar{E}_n = \bar{E}_{n-1}\backslash E_n$ and define $E_{n+1} = \bar{E}_n\cap(U_{\kappa\in Z}2^\kappa T_{rf}(\bar{E}_n))$. Let $\Omega_j = U_{\kappa\in Z}2^\kappa E_j$. By the definition, $\Omega_i = U_{\kappa\in Z}2^\kappa T_{rf}(\bar{E}_{i-1})$ and $\Omega_i\cap\Omega_j = \phi$ if $i\neq j$.

The following theorem is the generalization of Theorem 2.5.

Theorem 2.8. *Let E be a basic set such that $E(\delta,m) = E(\tau,m) = \phi$ for $m > M$, and let $\Omega = U_{1\leq j\leq n}\Omega_j$ for some $n\geq 1$, Then for any f with support on Ω we have*

$$\| \langle H_E f, f\rangle \|_{\ell^p(z)} \geq a_n \| f \|_{C_o(L^p(G))},$$

where a_n is the n-th term of the sequence of positive numbers defined by $a_\kappa = a_{\kappa-1}^2/(1+4M^{5/2})^2$ and $a_1 = 1/((1+2M^{5/2})^2+1)$, If $\Omega = R$ then E is an frame wavelet set.

Proof. The proof of this theorem follows on the lines of Theorem 3.4 of [2] with Lemma 2.6, 2.7 and [2, Lemma 3.2].

3 Frame wavelets with f-set supported in frequency domain

Now we will discuss the frame wavelets with f-set support in the frequency domain. Let $\varphi\in C_o(L^p(G))$ and let $E_{\hat{\varphi}}$ be the support of $\hat{\varphi}$. We define

$$H_{\hat{\varphi}}(f) = \sum_{n,\ell\in z}\langle f, \hat{D}^n\hat{\tau}^\ell\hat{\varphi}\rangle\hat{D}^n\hat{\tau}^\ell\hat{\varphi}$$

when $H_{\hat{\varphi}}(f)$ defines a bounded linear operator, (1.5) is equivalent to

$$A_0 \| f \|_{C_0(L^p(G))} \leq \| \langle H\hat{\varphi}f\rangle \|_{l^p(Z)} \leq B_0 \| f \|_{C_0(L^p(G))} \tag{3.1}$$

for all $f\in C_0(L^p(G))$. When E is a basic set,we have the following lemma:

Lemma 3.1. *If E is a basic set and $|\hat{\varphi}| \leq b$ for some constant $b > 0$ on $E_{\hat{\varphi}}$, then $H_{\hat{\varphi}}$ define a bounded linear operator. Furthermore, we have*

$$H_{\hat{\varphi}}(f) = \sum_{\kappa\in z}H^\kappa_{\hat{\varphi}}(f)\forall f\in C_o(L^p(G)), \tag{3.2}$$

where

$$H^\kappa_{\hat{\varphi}} = \hat{\varphi}(\xi/2^\kappa)\sum_{j\in Z}f(\xi+2^\kappa 2\pi j)\bar{\hat{\varphi}}(\xi/2^\kappa+2\pi j)$$

and the above sums all converge under the $C_o(L^p(G))$ norm topology.

Proof. The proof follows by simple manipulation in formulas given by [2, Lemma 3.1.].

Let E be an f-set such that $E(\delta, m) = E(\tau, m) = \phi$ for $m > M$, and $R = U_{1\leq j\leq n}\Omega_j$ for some n,$\Omega_j = \Omega_{\kappa\in Z}2^\kappa\tau_{rf}(\bar{E}_{j-1})$. We define the core of E by

$$S(E) = U_{1\leq j\leq n}A(\tau_{rf}(\bar{E}_{j-1}))$$

where $\bar{E}_0 = E_1$.It is known that $S(E)$ is a 2-dilation generator of R and it is not unique in general. Now we will prove the following theorem which is the generalization of Theorem 4.2 of [2] to Banach spaces other than Hilbert spaces.

Theorem 3.2. *Let E be an f-set satisfying the conditions of Theorem 2.8. If the support of $\hat{\varphi}$ is contained in E and there exists a constant $a > 0$ such that $\mid \hat{\varphi}(\xi) \mid \geq a$ a.e. on a core of E, then φ is a frame set.*

Proof. We can generalize the Lemma 4.5 of [2] to Banach space following the Lemma 2.7. Now this theorem can be proved by induction similar to the proof of Theorem 2.8. using [2, Lemma 4.4].

Remark 3.3. Theorem 3.2. cannot be applied to a frame set without a core, the following theorem ensures that we can still construct frame wavelets with a certain level of flexibility on an arbitrary frame set with or without a core.

Theorem 3.4. *Let E be a frame set. Then $\varphi \in C_o(L^p(G))$ is a frame wavelet if $\hat{\varphi}$ is bounded, supp$(\hat{\varphi}) = E$, $\mid \hat{\varphi} \mid \geq a > 0$ on E for some constant $a > 0$, and $\hat{\varphi}(s) = \hat{\varphi}(2^\kappa s)$ whenever s and $2^\kappa s$ are both in E for any integer κ.*

Proof. Since E is a frame set, there exists a positive constant $C > 0$ such that

$$\| \langle H_E f, f\rangle \|_{\ell^p(Z)} \geq C \| f \|_{C_o(L^p(G))}$$

for any $f \in C_o(L^p(G))$. Furthermore, by Lemma 3.1, for any $f \in C_o(L^p(G))$, we have

$$\begin{aligned} H^\kappa_{\hat{\varphi}}(f) &= \hat{\varphi}(\xi/2^\kappa)\sum_{j\in Z} f(\xi + 2^\kappa 2\pi j)\bar{\hat{\varphi}}(\xi/2^\kappa + 2\pi j) \\ &= \hat{\varphi}(\xi)\sum_{j\in Z} f(\xi + 2^\kappa 2\pi j)\bar{\hat{\varphi}}(\xi + 2^\kappa.2\pi j) \\ &= \hat{\varphi}.H^\kappa_E(f.\bar{\hat{\varphi}}), \end{aligned}$$

which implies $H^\kappa_{\hat{\varphi}}(f) = \hat{\varphi}.H^\kappa_E(f.\bar{\hat{\varphi}})$ and we get

$$\| \langle H_{\hat{\varphi}} f, f\rangle \|_{\ell^p(Z)} = \| \langle \hat{\varphi} H_E(f.\bar{\hat{\varphi}}), f\rangle \|_{\ell^p(Z)}$$

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 1, p.101-110

$$
\begin{aligned}
&= \ \| \langle H_E(f.\bar{\hat{\varphi}}), f\bar{\hat{\varphi}} \rangle \|_{\ell^p(Z)} \\
&\geq \ C \| f\bar{\hat{\varphi}} \|_{C_o(L^p(G))} \\
&\geq \ Ca^2 \| f \|_{C_o(L^p(G))} .
\end{aligned}
$$

Thus $\hat{\varphi}$ is a frequency frame wavelet. Therefore, φ is a frame wavelet.

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