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ABOUT THE EQUIVARIANT *K* – THEORY

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Abstract

The purpose of this paper is to set down the basic results about Atiyah-Hirzebruch spectral sequence in equivariant K-theory (most of them can be found also in [3], [5] and [7]). One application of this spectral sequence is the finiteness theorem of Segal (see [8]) and we present here a complete new proof of this result.

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1. Introduction

We shall have to start with a collection of definitions and simple results concerning the equivariant K – theory.

Let G be a topological group, then a G **– space** is a topological space X together with a continuous action $G \times X \to X$ satisfying the usual conditions. A **G – vector bundle** (or an **equivariant bundle**) on X is a G – space E together with a G – map $p: E \to X$ such that:

a) (E, p, X) is a complex (real) vector bundle on *E*;

b) for any $g \in G$ and any $x \in X$ the group action on fibres $E_x \to E_{gx}$ is a homomorphism of vector spaces.

One can construct a general cohomology theory by using the equivariant vector bundles on G – spaces: the set of isomorphism classes of G – vector bundles on X forms an abelian semigroup under the direct sum. The associated abelian group is noted by $K_G(X)$: its elements are formal differences $E_1 - E_2$ of G – vector bundles on *X*, modulo the familiar equivalence relation (see [8]). The tensor product of G – vector bundles induces a structure of commutative ring in $K_G(X)$.

In the rest of this paper I shall assume that G is a compact Lie group; let U = $(U_\alpha)_{\alpha \in S}$ a finite covering of the compact G – space *X*, by G – stable closed sets and one

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denotes by N_U the nerve of U: the finite simplicial complex whose simplexes are the finite subsets $A \subset S$ such that

$$U_A = \bigcap_{\alpha \in A} U_\alpha \text{ is non-empty.}$$

Then

$$W_U = W(X, U) = \bigcup_A (U_A \times |A|)$$

is a subspace of the product $X \times |N_U|$, where by $|N_U|$ one denotes the geometrical realization of the nerve N_U. Because W_U is closed in $X \times |N_U|$ then it is a compact G – space.

If $(f, \theta) : (X, U) \to (Y, V = (V_\beta)_{\beta \in T})$ is a morphism (i.e. $f : X \to Y$ is a G – map and $\theta : S \to T$ such that $f(U_\alpha) \subset V_{\theta(\alpha)}$ for any $\alpha \in S$) it is obviously to see that the product map $f \times |\theta|$ applies $W(X, U)$ into $W(Y, V)$. We have the following two simple results (see [2]):

Lemma 1. *Let $(f, \theta_0), (f, \theta_1) : (X, U) \to (Y, V)$ be two morphisms ; then the induced maps $W(X, U) \to W(Y, V)$ are G – homotopic.*

Proof. Using the above definition we can define the homotopy

$$h : |N_U| \times [0, 1] \to |N_V|$$

by the formula: $h(\alpha, t) = (1 - t)\,\theta_0 + t\,\theta_1$ for any $\alpha \in |N_U|$ and $t \in [0, 1]$; one finds that the induced applications $f \times |\theta_0|$ and $f \times |\theta_1|$ are G – homotopic by the map

$$(f, h) : W(X, U) \times [0, 1] \to W(Y, V)$$ □

Using the universal property of the direct product and the fact that K_G is a contravariant functor we obtain:

Lemma 2. *Let Y be a compact G – space, $(V_\alpha)_{\alpha \in S}$ a finite covering of Y by G – stable closed sets, and one considers the following notations:*

$$Y_{\alpha\beta} = Y_\alpha \cap Y_\beta,\ Y'_\alpha = \bigcup_{\beta \neq \alpha} Y_{\alpha\beta},\ Y' = \bigcup_\alpha Y'_\alpha .$$

Then there exists the relative homeomorphism

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$$\coprod_{\alpha\in S}(Y_\alpha, Y'_\alpha)\to(Y, Y')$$

which induces the isomorphism in equivariant $K-$ *theory:*

$$K_G^*(Y, Y') \cong \prod_{\alpha\in S} K_G^*(Y_\alpha, Y'_\alpha)$$

□

Using Lemma 1 and fixing the pair (X, U) one can omit the index U for the spaces W_U and N_U because the map K_G^* $(W(Y, V)) \to K_G^*$ $(W(X, U))$ does not depends at the choice of the morphism $(f, \theta) : (X, U) \to (Y, V)$.

Proposition 1. *The projection onto the first factor* $p{:}W \to X$ *induces the isomorphism*

$$K_G^*(X) \to K_G^*(W)$$

Proof. Let a filtration of the space X by $G-$ stable closed sets

$$X = X_0 \supset X_1 \supset \ldots \supset X_r \supset \ldots$$

where X_r is the subset of points of X which are contained in at least $r+1$ of the sets U_α. Define also

$$W_r = p^{-1}(X_r) = \bigcup_{\dim A\ge r}(U_A \times |A|) \subset W$$

and one considers the following diagram:

$$\begin{array}{ccc} \coprod_{\dim A\ge r}(U_A \setminus U'_A)\times|A| & \to & W_r \setminus W_{r+1} \\ \downarrow & & \downarrow \\ \coprod_{\dim A\ge r}(U_A \setminus U'_A) & \to & X_r \setminus X_{r+1} \end{array}$$

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where $U'_A = U_A \cap X_{r+1}$. The horizontal arrows are homeomorphisms (using Lemma 2) and the vertical arrow on the left is a homotopy – equivalence because $|A|$ is contractible. This implies that the induced morphism

$$p^* : K_G^*(X_r \setminus X_{r+1}) \to K_G^*(W_r \setminus W_{r+1})$$

is an isomorphism - i.e. $K_G^*(X_r, X_{r+1}) \cong K_G^*(W_r, W_{r+1})$.

Using now the exact sequences associated to the triples (X_{r-1}, X_r, X_{r+1}), respectively (W_{r-1}, W_r, W_{r+1}) it follows that

$$K_G^*(X, X_r) \to K_G^*(W, W_r)$$

is an isomorphism for all r; after a little manipulation (for a great number r the spaces X_r and W_r are empty) one obtains the desired result

$$p^* : K_G^*(\mathrm{X}) \xrightarrow{\approx} K_G^*(\mathrm{W})$$ □

2. The spectral sequence

We shall associate to the space W_U a filtration by G – subspaces:

$$W_U \supset \ldots\ldots \supset W^1 \supset W^0$$

such that $K_G^*(X) \to K_G^*(W_U)$ is an isomorphism (see Proposition 1) and when V is a refinement of the covering U there exists a G – map $W_V \to W_U$, respecting the filtrations and the projections on to X.

If $q : W \to |N|$ is the projection onto the second factor we define W^p as his inverse image of the p – skeleton of $|N|$, i.e.

$$W^p = \bigcup_{\dim A \le p} (U_A \times |A|) .$$

Using the method of Cartan – Eilenberg (see [4]) to the above filtration of W there corresponds the Atiyah – Hirzebruch spectral sequence:

Theorem 1. *To the finite covering* $U = (U_\alpha)_{\alpha\in S}$ *of the compact* $G-$*space* X *by* $G-$ *stable closed sets there exists a spectral sequence terminating in* $K_G^*(X)\cong K_G^*(W_U)$:

$$E_2^{p,q} = H^p(N, K_G^q(U)) \Rightarrow K_G^*(X)$$

where $K_G^*(U)$ *denotes the coefficient system:* $A \mapsto K_G^q(U_A)$.

Proof. Firstly we define a filtration of $K_G^*(W)$ by:

$$(K_G^*(W))_p = Ker(K_G^*(W) \to K_G^*(W^{p-1})) = \mathrm{Im}(K_G^*(W, W^{p-1}) \to K_G^*(W));$$

thus $K_G^*(W)$ is a filtered ring in the sense that $(K_G^*(W))_p \cdot (K_G^*(W))_q \subset (K_G^*(W))_{p+q}$ (see [8] pag. 145-146) We are going to construct the spectral sequence by setting:

$$\mathrm{Z}_\mathrm{r}^\mathrm{p} = \mathrm{Im}\left(\mathrm{K}_\mathrm{G}^*\left(\mathrm{W}^{\mathrm{p}-1+\mathrm{r}}, \mathrm{W}^{\mathrm{p}-1}\right) \to \mathrm{K}_\mathrm{G}^*\left(\mathrm{W}^\mathrm{p}, \mathrm{W}^{\mathrm{p}-1}\right)\right)$$

$$\mathrm{B}_\mathrm{r}^\mathrm{p} = \mathrm{Im}\left(\mathrm{K}_\mathrm{G}^*\left(\mathrm{W}^{\mathrm{p}-1}, \mathrm{W}^{\mathrm{p}-\mathrm{r}}\right) \xrightarrow{\delta} \mathrm{K}_\mathrm{G}^*\left(\mathrm{W}^\mathrm{p}, \mathrm{W}^{\mathrm{p}-1}\right)\right)$$

and $E_r^p(W) = Z_r^p / B_r^p$, where δ is the differential from the exact sequence associated to a triple.

Because the graduation of $K_G^*(W)$ is compatible with the above filtration one can introduce on the terms E_r a bigraduation by setting:

$$Z_r^{p,q} = Z_r^p \cap K_G^{p+q}, \quad B_r^{p,q} = B_r^p \cap K_G^{p+q} \quad \text{and} \quad E_r^{p,q} = Z_r^{p,q} / B_r^{p,q}.$$

Then the differential is

$$d_r : E_r^{p,q} \to E_r^{p+r, q-r+1}$$

with $E_1^{p,q} = K_G^{p+q}(W^p, W^{p-1})$. But the covering U has finite dimension, so there exists an integer p such that $W^p = W$ and then we obtain the infinite term of the spectral sequence:

$$\mathrm{E}_\infty^{\mathrm{p,q}} = \left(\mathrm{K}_\mathrm{G}^{\mathrm{p+q}}(\mathrm{W})\right)_\mathrm{p} / \left(\mathrm{K}_\mathrm{G}^{\mathrm{p+q}}(\mathrm{W})\right)_{\mathrm{p+1}}$$

(analogous with [4], lemma 1.1, pag. 316)

Using now the Proposition 1 it follows that the above spectral sequence is convergent to $K_G^*(W) \cong K_G^*(X)$ and the terms $E_2^{p,q}$ are just the p – cohomology of the complex

$$K_G^q(W^0) \to K_G^{q+1}(W^1, W^0) \to \ldots \to K_G^{p+q}(W^p, W^{p-1}) \to \ldots$$

On the other hand there exist the isomorphisms:

$$K_G^{p+q}(W^p, W^{p-1}) \cong \prod_{\dim A = p} K_G^{p+q}(U_A \times \overset{\circ}{A}) \cong \prod_{\dim A = p} K_G^q(U_A) = C^p(N; K_G^q(U))$$

where one denotes by $\overset{\circ}{A}$ the interior of the simplex $|A|$ and by $C^p(N, K_G^q(U))$ the complex of p – cochains of the nerve N with coefficients in the system $K_G^q(U)$. The first isomorphism is induced by the relative homeomorphism

$$\coprod_{\dim A = p} (U_A \times \overset{\circ}{A}) \to W^p \setminus W^{p-1}$$

(using Lemma 2) and for the second one we just use the definition of the group $K_G^q(U)$ (see [2]).

It remains to show that the differential

$$d_1 : E_1^{p,q} = K_G^{p+q}(W^p, W^{p-1}) \to E_1^{p+1,q} = K_G^{p+q+1}(W^{p+1}, W^p)$$

associated to the triple (W^{p+1}, W^p, W^{p-1}) is corresponding to the differential of the above complex of cochains of the nerve N.

Using Lemma 2 we obtain the following isomorphisms:

$$K_G^{p+q}(W^p, W^{p-1}) \cong \prod_{\dim A = p} K_G^{p+q}(U_A \times |A|, U_A \times |\overline{A}|)$$

and

$$K_G^{p+q}(U_A \times |A|, U_A \times |\overline{A}|) \cong K_G^{p+q}(U_A \times |\overline{B}|, U_A \times (|\overline{B}| \setminus \overset{\circ}{A}))$$

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where B is a $(p+1)$ – simplex of the nerve N and by $\left|\overline{A}\right|$ one denotes the boundary of the simplex A which is a face of B. We shall also use the differential associated to the triple $(U_B \times |B|, U_B \times |\overline{B}|, U_B \times (|\overline{B}| \setminus \overset{\circ}{A}))$:

$$K_G^{p+q}(U_B \times |\overline{B}|, U_B \times (|\overline{B}| \setminus \overset{\circ}{A})) \to K_G^{p+q+1}(U_B \times |B|, U_B \times |\overline{B}|).$$

Then the desired compatibility of differentials follows from the diagram:

$$\begin{array}{ccccc}
K_G^{p+q}(W^p, W^{p-1}) & = & K_G^{p+q}(W^p, W^{p-1}) & \xrightarrow{d_1} & K_G^{p+q+1}(W^{p+1}, W^p) \\
\downarrow & & \downarrow & & \downarrow \\
D \to & & K_G^{p+q}(U_B \times |A|, U_B \times |\overline{A}|) \cong E & \xrightarrow{d} & K_G^{p+q+1}(U_B \times |B|, U_B \times |\overline{B}|) \\
\uparrow\cong & & \uparrow\cong & & \uparrow\cong \\
C \longrightarrow & & K_G^q(U_B) & = & K_G^q(U_B)
\end{array}$$

where one denotes $C = K_G^q(U_A)$, $D = K_G^{p+q}(U_A \times |A|, U_A \times |\overline{A}|)$, $E = K_G^{p+q}(U_B \times |\overline{B}|, U_B \times (|\overline{B}| \setminus \overset{\circ}{A}))$. All the rectangles of the above diagram commute, except the right bottom rectangle: this one commutes, anticommutes or is degenerated if the orientation of the p – simplex A is the same with the orientation of the $(p+1)$ – simplex B of the nerve N, respectively the orientations are different or A is not a face of the simplex B. Thus we have that the map d_1 is just the differential of the complex of cochains of the nerve N. □

More generally we have:

Theorem 2. *If $f: X \to Y$ is a G - map between two compact G – spaces and the group G acts trivially on Y, then there is a spectral sequence*

$$E_2^{p,q} = H^p(Y; K_G^q f) \Rightarrow K_G^*(X)$$

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where $K_G^q f$ *is a sheaf whose stalk at* $y \in Y$ *is* $K_G^q(f^{-1}(y))$.

For a proof of Theorem 2 I refer again to [5].

3. A finiteness theorem

The Atiyah – Hirzebruch spectral sequence has many interesting applications and one of them is the following finiteness theorem.

Definition. A G – space X is *locally* G – *contractible* if each point $x \in X$ has an arbitrarily G_x - stable neighbourhood which is G_x - contractible to x (where by G_x one denotes the isotropy or stabilizer group of x).

For example, a differential manifold X on which a compact Lie group acts smoothly, is locally G – contractible.

Now one can prove the following useful result:

Theorem 3. (see [8]). *If* X *is a locally* G – *contractible compact* G – *space such that the orbit space* X/G *has finite Lebesgue dimension (see* [1] *), then* $K_G^*(X)$ *is a finite* $R(G)$ – *module (where* $R(G)$ *is the representation ring of the compact Lie group* G *).*

Proof. Let $\pi : X \to X/G$ be the projection on the orbit space; using the Theorem 2 for $Y = X/G$ one obtains a spectral sequence terminating in $K_G^*(X)$:

$$E_2^{p,q} = H^p(X/G; K_G^q \pi) \Rightarrow K_G^*(X)$$

where $K_G^q \pi$ is a sheaf on X/G whose stalk at an orbit xG is $R(G_x)$ if q is even and $K_G^q \pi = 0$ if q is odd (for details see [8] and [5]).

Firstly one shows that $H^p(X/G; K_G^q \pi)$ are finite $R(G)$ – modules and so it suffices to prove the theorem.

Because X is locally G – contractible, for each orbit xG there is a small neighbourhood V such that $K_G^* \pi(V) \cong R(G_x)$, which is finite over $R(G)$ if q is even (see Proposition 3.2 in [9]). Now we choose a finite covering Ω of the orbit space by small open sets V such the above isomorphisms are satisfied; but there exists the isomorphism: $H^*(\Omega; K_G^* \pi) \xrightarrow{\cong} H^*(X/G; K_G^* \pi)$ (see for example [6]) and because the space of orbits has finite Lebesgue dimension that means $H^p(X/G; K_G^* \pi)$ are finite $R(G)$ – modules.

Using the same assumption about the dimension of X/G one can prove that the associated spectral sequence is convergent:

$$H^p(X/G; K_G^q \pi) \Rightarrow K_G^*(X)$$

On the other hand $E_2^{p,q} = 0$ if $p < 0$ or if q is odd and this implies that $E_r^{p,q} = E_\infty^{p,q} = 0$ for all $r \geq 2$ if $p < 0$ or if q is odd. Then we have:

$$\mathrm{Im}\,(E_r^{-r,q+r-1} \to E_r^{0,q}) = \mathrm{Im}\,(E_{r+p}^{1-r,q+r+p-1} \to E_{r+p}^{p+1,q}) = 0 \text{ for all } r \geq 2 \text{ and } p \geq 0;$$

this implies that $B_0^{p,q} = B_1^{p,q} = \ldots = B_\infty^{p,q}$ for all $p, q \in \mathbb{Z}$ and so we obtain the monomorphism $E_\infty^{p,q} \subset E_2^{p,q}$ using the following sequence of inclusions and equalities:

$$E_2^{p,q} = E_3^{p,q} \supset E_4^{p,q} = E_5^{p,q} \supset \ldots \supset E_\infty^{p,q} .$$

But the ring $R(G)$ is noetherian (see [9]) and because $E_2^{p,q} = H^p(X/G; K_G^q \pi)$ are finite $R(G)$ – modules it follows that the infinite terms are also finite modules.

On the other hand, in the filtration of $K_G^q(X)$:

$$K_G^q(X) = (K_G^q(X))_0 \supset (K_G^q(X))_1 \supset \ldots \supset (K_G^q(X))_p \supset \ldots$$

there exists a natural number k such that $(K_G^q(X))_p = 0$ for any $p > k$ and using now the definition of the infinite term $E_\infty^{p,q} \cong (K_G^{p+q}(X))_p / (K_G^{p+q}(X))_{p+1}$ we have the following exact sequences:

$$0 \to (K_G^q(X))_1 \to K_G^q(X) \to E_\infty^{0,q} \to 0$$

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$$0 \to (K_G^q(X))_2 \to (K_G^q(X))_1 \to E_\infty^{1,q-1} \to 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$0 \to (K_G^q(X))_k \to (K_G^q(X))_{k-1} \to E_\infty^{k-1,q-k+1} \to 0$$

$$0 \to (K_G^q(X))_k \to E_\infty^{k,q-k} \to 0$$

Because the finite $R(G)$ – modules are noetherians and they form a Serre class it follows that $K_G^q(X)$ are also finite $R(G)$ – modules.

□

Remark. The hypothesis that the space of orbits X / G has finite Lebesgue dimension is satisfied in the case of a smooth G – manifold because X / G is then a finite union of open manifolds.

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