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Regularity of quasi-symbolic and bracket powers of Borel type ideals

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Abstract

In this paper, we show that the regularity of the q-th quasi-symbolic power $I^{((q))}$ and the regularity of the q-th bracket power $I^{[q]}$ of a monomial ideal of Borel type I, satisfy the relations $\operatorname{reg}(I^{((q))}) \leq q \operatorname{reg}(I)$, respectively $\operatorname{reg}(I^{[q]}) \geq q \operatorname{reg}(I)$. Also, we give an upper bound for $\operatorname{reg}(I^{[q]})$.

Keywords: Monomial ideals, Borel type ideals, Mumford-Castelnuovo regularity.

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Introduction

Let K be an infinite field, and let $S = K[x_1, \dots, x_n], n \geq 2$ the polynomial ring over K. Bayer and Stillman [1] note that Borel fixed ideals $I \subset S$ satisfy the following property:

$$(*) \quad (I : x_j^{\infty}) = (I : (x_1, \dots, x_j)^{\infty}) \; for \; all \; j = 1, \dots, n.$$

Herzog, Popescu and Vladoiu [8] define a monomial ideal I to be of *Borel type* if it satisfies $(*)$. We mention that this concept appears in [3, Definition 1.3] as the so called *weakly stable ideal*. Also, this concept appears in [2, Definition 3.1], as the so called *monomial ideal of nested type*. We further studied this class of monomial ideals in [4] and [5].

In the first section, we recall some results regarding ideals of Borel type. Also, we discuss the relation between the sequential chain of an ideal of Borel type I, defined in [8], and the primary decomposition of I.

Let $I \subset S$ be a monomial ideal and $I = \bigcap_{i=1}^r Q_i$ the an irreduntant primary decomposition of I, obtained in a canonical way. We define $I^{((q))} := \bigcap_{i=1}^r Q_i^q$, the q-th *quasi-symbolic* power of I, see Definition 2.1. We prove that if I is an ideal of Borel type, then $I^{((q))}$ and $I^{[q]}$ are also ideals of Borel type, where $I^{[q]} = (u^q \; : \; u \in I \text{ monomial})$ is the q-th bracket power of I.

In [5], we proved that $\operatorname{reg}(I^q) \leq q \operatorname{reg}(I)$. We give a similar result for the q-th quasi-symbolic power. More precisely, we prove that $\operatorname{reg}(I^{((q))}) \leq q \operatorname{reg}(I)$, see Theorem 2.4. Also, we prove that $\operatorname{reg}(I^{[q]}) \geq q \operatorname{reg}(I)$, see Theorem 2.6. In Proposition 2.11, we prove that $\operatorname{reg}(I^{[q]}) \leq q \operatorname{reg}(I) + (q-1)(n-1)$.

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1 Some basic facts on Borel type ideals.

Firstly, we recall the following equivalent characterizations of ideals of Borel type given in [8] and in [2].

Proposition 1.1. *Let $I \subset S$ be a monomial ideal. The following conditions are equivalent:*
(a) I is an ideal of Borel type.
(b) For any $1 \leq j < i \leq n$, we have $(I : x_i^\infty) \subset (I : x_j^\infty)$.
(c) Each $P \in Ass(S/I)$ has the form $P = (x_1, \ldots, x_m)$ for some $1 \leq m \leq n$.

Let $I \subset S$ be a monomial ideal of Borel type. Since each prime ideal $P \in Ass(S/I)$ is of the form $P = (x_1, \ldots, x_m)$ for some $1 \leq m \leq n$, we can assume that I has an irredundant primary decomposition:

$$I = \bigcap_{i=1}^{r} Q_i; \;\; such\ that\ P_i := \sqrt{Q_i} = (x_1, \ldots, x_{n_{i-1}}), \;\; n \geq n_0 > n_1 > \cdots > n_{r-1} \geq 1. \quad (1)$$

For each $0 \leq i \leq r-1$, we define $I_i := \bigcap_{j=i+1}^{r} Q_j$. We claim that $I_{i+1} = (I_i : x_{n_i}^\infty)$ for all $0 \leq i \leq r-1$. Indeed, since Q_{i+1} is P_{i+1}-primary, it follows that there exists a positive integer k such that $x_{n_i}^k \in Q_{i+1}$. So $(I_i : x_{n_i}^\infty) \supseteq ((Q_{i+1} \cdot I_{i+1}) : x_{n_i}^\infty) \supseteq (x_{n_i}^k \cdot I_{i+1} : x_{n_i}^\infty) = I_{i+1}$. For the converse inclusion, note that $(I_i : x_{n_i}^\infty) \subseteq (Q_{i+1} : x_{n_i}^\infty) \cap (I_{i+1} : x_{n_i}^\infty) = S \cap I_{i+1} = I_{i+1}$.

Thus, the chain of ideals $I = I_0 \subset I_1 \subset \cdots \subset I_{r-1} \subset I_r := S$ is the *sequential chain* of I, as it was defined in [8]. Note that $n_i = \max\{j : \; x_j | u \;\text{ for some } u \in G(I_i)\}$, where we denoted by $G(I_i)$ the set of minimal monomial generators of I_i.

Let J_i be the monomial ideal generated by $G(I_i)$ in $S_i := K[x_1, \ldots, x_{n_i}]$, $0 \leq i \leq r$. Then, the saturation $J_i^{sat} = (J_i : \mathbf{m}_i^\infty)$ is generated by the elements of $G(I_{i+1})$, where $\mathbf{m}_i = (x_1, \ldots, x_{n_i})S_i$. It follows that $I_{i+1}/I_i \cong (J_i^{sat}/J_i)[x_{n_i+1}, \ldots, x_n]$.

It would be appropriate to recall the definition of the Castelnuovo-Mumford regularity. We refer the reader to [6] for further details on the subject.

Definition 1.2. Let K be an infinite field, and let $S = K[x_1, \ldots, x_n]$, $n \geq 2$ the polynomial ring over K. Let M be a finitely generated graded S-module. The Castelnuovo-Mumford regularity $\operatorname{reg}(M)$ of M is

$$\max_{i,j}\{j - i : \; \beta_{ij}(M) \neq 0\},$$

where $\beta_{ij}(M) = dim_K(Tor_i(K, M))_j$ denotes the ij-th graded Betti number of M.

If $M = \bigoplus_{t \geq 0} M_t$ is an artinian graded S-module, we denote $s(M) = \max\{t : \; M_t \neq 0\}$. Herzog, Popescu and Vlădoiu proved the following formula for the regularity of a monomial ideal of Borel type:

Proposition 1.3. *[8, Corollary 2.7] If I is a Borel type ideal, with the notations above, we have*

$$\operatorname{reg}(I) = \max\{s(J_0^{sat}/J_0), \ldots, s(J_{r-1}^{sat}/J_{r-1})\} + 1.$$

Example 1.4. We consider the ideal $Q = (x_1^{a_1}, \ldots, x_m^{a_m}) \subset S$, where $1 \leq m \leq n$ and $a_1 \geq a_2 \geq \cdots \geq a_m \geq 1$. According to Proposition 1.3, $\operatorname{reg}(Q) = s(\bar{S}/\bar{Q}) + 1$, where $\bar{S} = K[x_1, \ldots, x_m]$ and $\bar{Q} = \bar{S} \cap Q$. Since $u = x_1^{a_1-1} \cdots x_m^{a_m-1} \in \bar{S}$ is the monomial of the highest degree which is not contain in $\bar{Q}$, it follows that

$$\operatorname{reg}(Q) = \sum_{i=1}^{m}(a_i - 1) + 1 = a_1 + \cdots + a_m - m + 1.$$

We consider the ideal $Q^q = (x_1^{qa_1}, \ldots, x_m^{qa_m}, x_1^{(q-1)a_1}x_2^{a_2}, \ldots)$. Note that $Q^q \cap \bar{S} = \bar{Q}^q$ and therefore $\operatorname{reg}(Q^q) = s(\bar{S}/\bar{Q}^q) + 1$. One can easily see that $u = x_1^{qa_1-1}x_2^{a_2-1} \cdots x_m^{a_m-1}$ is the monomial of the highest degree which is not contain in $\bar{Q}^q$. Thus:

$$\operatorname{reg}(Q^q) = qa_1 - 1 + \sum_{i=2}^{m}(a_i - 1) + 1 = qa_1 + a_2 + \cdots + a_m - m + 1.$$

Note that $\operatorname{reg}(Q^q) \leq q \operatorname{reg}(Q)$, as we already know from [5, Corollary 1.8], and the equality holds if and only if $a_2 = \cdots = a_m = 1$.

2 Regularity of quasi-symbolic and bracket powers of Borel type ideals

Now, assume $I \subset S$ is an arbitrary monomial ideal. Then I has a unique irreduntant decomposition $I = \bigcap_{i=1}^{s} C_i$, where C_i are irreducible monomial ideals. One obtains from this presentation a *canonical presentation* of I as an intersection of primary ideals, $I = \bigcap_{i=1}^{r} Q_i$, where each Q_i is P_i-primary and is defined to be the intersection of all C_j'-s with $\sqrt{C_j} = P_i$. See [7] for further details.

Definition 2.1. Let q be a positive integer. We define the *q-th quasi-symbolic power of I* to be the ideal

$$I^{((q))} := \bigcap_{i=1}^{r} Q_i^q.$$

Note that, $I^{(q)} \subset I^{((q))}$, where $I^{(q)} := S \cap \bigcap_{P \in \operatorname{Ass}(S/I)} I^q S_P$ is the *q-th symbolic power* of I. The equality holds if all P_i'-s are pairwise incomparable, but, in general, this is not the case. On the other hand, $I^q \subset I^{(q)}$.

Now, assume $I \subset S$ is of Borel type with the primary decomposition (1). One can easily see that $I^q S_{P_1} \cap S = I^q$, since all the minimal monomial generators of I are from $K[x_1, \ldots, x_{n_0}]$ and $P_1 = (x_1, \ldots, x_{n_0})$. Therefore, $I^{(q)} = I^q$.

In the following, we will assume that the primary decomposition (1), of a Borel type ideal $I \subset S$, is canonical in the above sense. We have the following lemma.

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Lemma 2.2. *If $I \subset S$ is an ideal of Borel type and q is a positive integer, then $Ass(S/I^{((q))}) \subset Ass(S/I)$. In particular, $I^{((q))}$ is an ideal of Borel type.*

Proof. Assume $I = \bigcap_{i=1}^{r} Q_i$ is the primary decomposition of I given in (1). It follows that $I^{((q))} := \bigcap_{i=1}^{r} Q_i^q$. This primary decomposition of $I^{((q))}$ is not necessarily irredundant. However, since $\sqrt{Q_i^q} = \sqrt{Q_i}$, it follows that $Ass(S/I^{((q))}) \subset Ass(S/I)$. Therefore, by Proposition 1.1(c), $I^{((q))}$ is an ideal of Borel type. $\square$

Example 2.3. We consider the following ideals, $Q = (x^8, x^6y^2, x^2y^6, y^8) \subset S := K[x, y, z]$, $Q' = Q + (x^4y^4) \subset S$, and $I := (Q, z^2) \cap Q' = (Q, x^4y^4z^2) \subset S$. Since, $Q \subsetneq Q'$, it follows that $(Q, z^2) \cap Q'$ is a primary decomposition of I and thus $Ass(S/I) = \{(x, y), (x, y, z)\}$.

We have $Q = (x^8, y^2) \cap (x^6, y^6) \cap (x^2, y^8)$ and $Q' = (x^8, y^2) \cap (x^4, y^6) \cap (x^6, y^4) \cap (x^2, y^8)$. Therefore, $I = Q' \cap (x^6, y^6, z^2)$ is the canonical primary decomposition of I, and thus $I^{((2))} = Q^2 \cap (x^6, y^6, z^2)^2$. On the other hand,

$$Q'^2 = Q^2 = (x^{16}, x^{14}y^2, x^{12}y^4, x^{10}y^6, x^8y^8, x^6y^{10}, x^4y^{12}, x^2y^{14}, y^{16}),$$

and thus $I^{((2))} = Q^2$, since $Q^2 \subset (x^6, y^6, z^2)^2$. We have $s(K[x, y]/(Q' \cap K[x, y])) = 8$ and $s(Q'/(Q, z^2x^4y^4)) = 11$, and therefore, by Proposition 1.3, we get $\operatorname{reg}(I) = 12$. Also, $s(K[x, y]/(Q \cap K[x, y])^2) = 16$ and thus $\operatorname{reg}(I^{((2))}) = 17$, according to Proposition 1.3.

Let $I \subset S$ be a Borel type ideal with the primary decomposition $I := \bigcap_{i=1}^{r} Q_i$ from (1). We consider the sequential chain $I = I_0 \subset I_1 \subset \cdots \subset I_r = S$ of I, where $I_i := \bigcap_{j=i+1}^{r} Q_j$. Note that $I_i^{((q))} := \bigcap_{j=i+1}^{r} Q_j^q$, since the previous primary decompositions of I_i-'s are canonical. We consider the following chain of ideals

$$I^{((q))} = I_0^{((q))} \subset I_1^{((q))} \subset \cdots I_r^{((q))} = S.$$

In the chain above, we may have some equalities. Nevertheless, if we denote J_i be the monomial ideal generated by $G(I_i)$ in $S_i := K[x_1, \dots, x_{n_i}]$, we have

$$I_{i+1}^{((q))}/I_i^{((q))} \cong ((J_i^{((q))})^{sat}/J_i^{((q))})[x_{n_i+1}, \dots, x_n].$$

Also, the sequential chain of $I_i^{((q))}$ is obtain from the previous chain of ideal, by removing those ideals I_i with $I_i = I_{i-1}$. Thus, by Proposition 1.3,

$$\operatorname{reg}(I^{((q))}) = \max\{s((J_i^{((q))})^{sat}/J_i^{((q))}),\ 0 \leq i \leq r-1\} + 1. \quad (2)$$

Now, we are able to prove the following Theorem.

Theorem 2.4. *With the above notations, we have* $\operatorname{reg}(I^{((q))}) \leq q \cdot \operatorname{reg}(I)$.

Proof. We fix $0 \leq i \leq r-1$. Since $I_i := \bigcap_{j=i+1}^{r} Q_j$, it follows that $J_i = \bigcap_{j=i+1}^{r} \bar{Q}_j$, where $\bar{Q}_j$ is the ideal generated by $G(Q_j)$ in S_i. On the other hand, since J_i^{sat} is generated by the elements of $G(I_{i+1})$, it follows that $J_i^{sat} = \bigcap_{j=i+2}^{r} \bar{Q}_j$. Note that

$$s(J_i^{sat}/J_i) + 1 = \min\{j :\ \mathbf{m}_i^j J_i^{sat} \subset J_i\}$$

and therefore $s(J_i^{sat}/J_i)+1 = \min\{j : \mathbf{m}_i^j\bar{Q}_k \subset \bar{Q}_{i+1} \quad for \quad all \quad k = i+2,\dots,r\ \}$. Analogously, since $I_i^{((q))} := \bigcap_{j=i+1}^{r} Q_j^q$, it follows that

$$s((J_i^{((q))})^{sat}/J_i^{((q))})+1 = \min\{j : \mathbf{m}_i^j\bar{Q}_k^q \subset \bar{Q}_{i+1}^q \ \ for \ \ all \ \ k = i+2,\dots,r\ \}.$$

Note that if $\mathbf{m}_i^j\bar{Q}_k \subset \bar{Q}_{i+1}$ then $\mathbf{m}_i^{jq}\bar{Q}_k^q = (\mathbf{m}_i^j\bar{Q}_k)^q \subset \bar{Q}_{i+1}^q$. Therefore, we get

$$s((J_i^{((q))})^{sat}/J_i^{((q))})+1 \leq q\cdot(s(J_i^{sat}/J_i)+1). \tag{3}$$

By applying Proposition 1.3 to I and (3) we get the required conclusion. □

Let $I \subset S$ be a monomial ideal of Borel type. An interesting question is to find a relation between $\operatorname{reg}(I^q)$ and $\operatorname{reg}(I^{((q))})$.

Let $I \subset S$ be a monomial ideal and let q be a nonnegative integer. We define the *q-th bracket power* of I, to be the ideal $I^{[q]}$, generated by all monomials u^q, where $u \in I$ is a monomial. In particular, $I^{[0]} = S$ and $I^{[1]} = I$. Note that if $G(I) = \{u_1,\dots,u_m\}$ is the set of minimal monomial generators of I, then $G(I^{[q]}) = \{u_1^q,\dots,u_m^q\}$. Note that $I^{[q]} \subset I^q$ for all q. In fact, when $q \geq 2$, the equality holds if and only if I is principal. Also, one can easily see that $(I\cap J)^{[q]} = I^{[q]} \cap J^{[q]}$ for any monomial ideals $I, J \subset S$.

Now, assume $I = \bigcap_{i=1}^{r} Q_i$ is an irredundant primary decomposition of I. We claim that $I^{[q]} = \bigcap_{i=1}^{r} Q_i^{[q]}$ is an irredundant primary decomposition of $I^{[q]}$, where q is a positive integer. In order to prove this, we fix an integer i with $1 \leq i \leq r$ and we chose a monomial $u \in Q_i \setminus \bigcap_{j\neq i} Q_j$. Obviously, $u^q \in Q_i^{[q]}$. We claim that $u^q \notin \bigcap_{j\neq i} Q_j$. Assume this is not the case. It follows that $u^q = u_j^q w_j$ for some monomials $u_j \in Q_j$ and $w_j \in S$, for all $j \neq i$. Therefore, $u_j | u$ for all $j \neq i$. It follows that $u \in \bigcap_{j\neq i} Q_j$, a contradiction.

As a consequence, we get the following Lemma.

Lemma 2.5. *If $I \subset S$ be a monomial ideal and q a positive integer, then $Ass(S/I) = Ass(S/I^{[q]})$. In particular, if I is of Borel type, then $I^{[q]}$ is of Borel type.*

Now, we are able to prove the following Theorem.

Theorem 2.6. *Let $I \subset S$ be a monomial ideal of Borel type. Then:*

$$\operatorname{reg}(I^{[q]}) \geq q\cdot \operatorname{reg}(I).$$

Proof. We consider the primary irredundant decomposition $\bigcap_{i=1}^{r} Q_i$ of I from (1) and the sequential chain $I = I_0 \subset I_1 \subset \cdots \subset I_r := S$ of I, where $I_i = \bigcap_{j=i+1}^{r} Q_j$, for $0 \leq i \leq r-1$. Note that the sequential chain of $I^{[q]}$, is $I^{[q]} = I_0^{[q]} \subset I_1^{[q]} \subset \cdots \subset I_r^{[q]} = S$. Indeed, all the inclusions are stricts.

We fix an integer $0 \leq i \leq r-1$. Let J_i be the monomial ideal generated by $G(I_i)$ in $S_i := K[x_1,\dots,x_{n_i}]$. We denote $\bar{Q}_j$, the ideal generated by $G(Q_j)$ in S_i, for all $1 \leq j \leq r$. With these notations, we have $J_i = \bigcap_{j=i+1}^{r} \bar{Q}_j$ and $J_i^{[q]} = \bigcap_{j=i+1}^{r} \bar{Q}_j^{[q]}$. On the other hand, since J_i^{sat} is generated by the elements of $G(I_{i+1})$, it follows that $J_i^{sat} = \bigcap_{j=i+2}^{r} \bar{Q}_j$.

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Let $u \in J_i^{sat} \setminus J_i$ be a nonzero monomial. We claim that $x_1^{q-1}u^q \in (J_i^{[q]})^{sat} \setminus J_i^{[q]}$. It is clear that $x_1^{q-1}u^q \in (J_i^{[q]})^{sat}$. If we assume that $x_1^{q-1}u^q \in J_i^{[q]}$, it follows that $x_1^{q-1}u^q = v^q \cdot w$, where $v \in J_i$ is a monomial and $w \in S$ is a monomial. Since $v^q | x_1^{q-1}u^q$, it follows that $v|u$ and therefore $u \in J_i$, a contradiction.

As a consequence, we get $s((J_i^{[q]})^{sat}/J_i^{[q]}) \geq q \cdot s(J_i^{sat}/J_i) + q - 1$. By applying Proposition 1.3, we get the required conclusion. □

Remark 2.7. The conclusions of Theorem 2.4 and Theorem 2.6 hold for monomial ideals $I \subset S$ with $Ass(S/I)$ totally ordered by inclusion. Indeed, if I is such an ideal, we can define a ring isomorphism $\varphi : S \to S$ given by a reordering of variables, such that $\varphi(I)$ is an ideal of Borel type. Since the Castelnuovo-Mumford regularity is an invariant, it follows that $reg(I) = reg(\varphi(I))$.

Bermejo and Giemenez give in [2] a formula for the regularity of a Borel type ideal I, when the irredundant irreducible decomposition is known. More precisely, they proved the following Proposition.

Proposition 2.8. *[2, Corollary 3.17] Let $I \subset S$ be a monomial ideal of Borel type. Assume $I = \bigcap_{i=1}^{m} C_i$ is the irredundant irreducible decomposition of I. Then:*

$$\operatorname{reg}(I) = \max\{reg(C_i) : \ i = 1, \ldots, m\}.$$

Since C_i-'s are irreducible monomial ideals, they are generated by powers of variables. Since $\sqrt{C_i} \in Ass(S/I)$ and I is of Borel type, we may assume that $C_i = (x_1^{a_{i1}}, \ldots, x_{r_i}^{a_{ir_i}})$, where r_i is an integer with $1 \leq r_i \leq n$ and a_{ij} are some positive integers. Denote $S_i := K[x_1, \ldots, x_{r_i}]$. If we denote $\bar{C}_i$ the ideal generated by $G(C_i)$ in S_i, then, by Proposition 1.3, as in Example 1.5, we have $\operatorname{reg}(C_i) = s(S_i/\bar{C}_i) + 1 = a_{i1} + \cdots + a_{ir_i} - r_i + 1$. Therefore, we get the following corollary.

Corollary 2.9. *With the notations above,*

$$\operatorname{reg}(I) = \max\{a_{i1} + \cdots + a_{ir_i} - r_i + 1 : \ i = 1, \ldots, m\}.$$

Let q be a positive integer and consider the ideal $I^{[q]}$. Since $I = \bigcap_{i=1}^{m} C_i$, it follows that $I^{[q]} = \bigcap_{i=1}^{m} C_i^{[q]}$ and $C_i^{[q]} = (x_1^{qa_{i1}}, \ldots, x_{r_i}^{qa_{ir_i}})$. Note that $\bigcap_{i=1}^{m} C_i^{[q]}$ is the irredundant irreducible decomposition of $I^{[q]}$. Indeed, we can argue in the same way as we did for the irreducible primary decomposition of $I^{[q]}$. Therefore, by Corollary 2.9, we get the following.

Corollary 2.10. *$reg(I^{[q]}) = \max\{qa_{i1} + \cdots + qa_{ir_i} - r_i + 1 : \ i = 1, \ldots, m\}$.*

The above formula leads us to the following upper bound for $\operatorname{reg}(I^{[q]})$.

Proposition 2.11. *Let $I \subset S$ be an ideal of Borel type and let q be a positive integer. Then:*

$$\operatorname{reg}(I^{[q]}) \leq q \operatorname{reg}(I) + (q-1)(n-1) = \alpha q \operatorname{reg}(I) - (n-1),$$

where $\alpha = 1 + \frac{n-1}{reg(I)}$.

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Proof. With the above notations, we may assume $\operatorname{reg}(I) = a_{i1} + \cdots + a_{ir_i} - r_i + 1$ for some $1 \leq i \leq m$. According to Corollary 2.9 and Corollary 2.10, $\operatorname{reg}(I^{[q]}) = qa_{i1}+\cdots+qa_{ii_r}-r_i+1$. Therefore, $\operatorname{reg}(I^{[q]}) = q\operatorname{reg}(I) + (q-1)(r_i - 1)$. Since $r_i - 1 \leq n - 1$, we get the required inequality. The remaining equality is trivial. □

We conclude our paper, with the following example.

Example 2.12. Let $I = (x) \cap (x^2, y) = (x^2, xy) \subset S = K[x, y]$. Let q be a positive integer. It follows that $I^q = (x^{2q}, x^{2q-1}y, \ldots, x^q y^q) = (x^q) \cap (x^{2q}, x^{2q-1}y, \ldots, x^{q+1}y^{q-1}, y^q)$.

Also, we obtain $I^{((q))} = (x^q) \cap (x^2, y)^q = (x^q) \cap (x^{2q}, x^{2q-2}y, \ldots, x^2y^{q-1}, y^q) = (x^{2q}, x^{2q-2}y, \ldots, x^{2q-2\lfloor \frac{q}{2} \rfloor}y^{\lfloor \frac{q}{2} \rfloor}, x^q y^{\lfloor \frac{q}{2} \rfloor + 1})$, where we denoted by $\lfloor \alpha \rfloor$ the integer part of α. On the other hand, $I^{[q]} = (x^q) \cap (x^{2q}, y^q) = (x^{2q}, x^q y^q)$.

We consider the sequential chain of I, $I =: I_0 \subset I_1 \subset I_2 := S$, where $I_1 = (x)$. We have $J_0 = I \subset S$ and $J_1 = (x) \subset K[x]$. Therefore, $J_0^q = I^q$, $J_0^{((q))} = I^{((q))}$ and $J_0^{[q]} = I^{[q]}$. Also, $J_1^q = J_1^{((q))} = J_1^{[q]} = (x^q) \subset K[x]$. We get $J_0^{sat} = (x)S$, $(J_0^q)^{sat} = (J_0^{((q))})^{sat} = (J_0^{[q]})^{sat} = (x^q)S$ and $J_1^{sat} = (J_1^q)^{sat} = (J_1^{((q))})^{sat} = (J_1^{[q]})^{sat} = K[x]$.

We have $s(J_1^{sat}/J_1) = 0$ and $s((J_1^q)^{sat}/J_1^q) = s((J_1^{((q))})^{sat}/J_1^{((q))}) = s((J_1^{[q]})^{sat}/J_1^{[q]}) = q - 1$.

Also, one can easily compute $s(J_0^{sat}/J_0) = 1$, $s((J_0^q)^{sat}/J_0^q) = 2q-1$, $s((J_0^{((q))})^{sat}/J_0^{((q))}) = 2q - 1$ and $s((J_0^{[q]})^{sat}/J_0^{[q]}) = 3q - 2$. By Proposition 1.3, it follows that $\operatorname{reg}(I) = 2$, $\operatorname{reg}(I^q) = \operatorname{reg}(I^{((q))}) = 2q$ and $\operatorname{reg}(I^{[q]}) = 3q - 1$.

Since $I = (x) \cap (x^2, y)$ is also the irreducible irredundant decomposition of I, by Corollary 2.8 and Corollary 2.9, we can compute directly $\operatorname{reg}(I) = \max\{1-1+1, 2+1-2+1\} = 2$ and, respectively, $\operatorname{reg}(I^{[q]}) = \max\{q - 1 + 1, 2q + q - 2 + 1\} = 3q - 1$.

Note that $\operatorname{reg}(I^{[q]}) = q\operatorname{reg}(I) + (q - 1)(2 - 1)$ and therefore, the upper bound given in Proposition 2.11 is the best possible.

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