ON THE STANLEY DEPTH OF THE PATH IDEAL OF A CYCLE GRAPH

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Abstract. We give tight bounds for the Stanley depth of the quotient ring of the path ideal of a cycle graph. In particular, we prove that it satisfies the Stanley inequality.

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Introduction

Let K be a field and $S = K[x_1,\ldots,x_n]$ the polynomial ring over K. Let M be a $\mathbb{Z}^n$-graded S-module. A *Stanley decomposition* of M is a direct sum $\mathcal{D} : M = \bigoplus_{i=1}^r m_iK[Z_i]$ as a $\mathbb{Z}^n$-graded K-vector space, where $m_i \in M$ is homogeneous with respect to $\mathbb{Z}^n$-grading, $Z_i \subset \{x_1,\ldots,x_n\}$ such that $m_iK[Z_i] = \{um_i : \ u \in K[Z_i]\} \subset M$ is a free $K[Z_i]$-submodule of M. We define $\operatorname{sdepth}(\mathcal{D}) = \min_{i=1,\ldots,r} |Z_i|$ and $\operatorname{sdepth}(M) = \max\{\operatorname{sdepth}(\mathcal{D})|\ \mathcal{D}$ is a Stanley decomposition of $M\}$. The number $\operatorname{sdepth}(M)$ is called the *Stanley depth* of M.

Herzog, Vladoiu and Zheng show in [10] that $\operatorname{sdepth}(M)$ can be computed in a finite number of steps if $M = I/J$, where $J \subset I \subset S$ are monomial ideals. In [13], Rinaldo give a computer implementation for this algorithm, in the computer algebra system `CoCoA` [6]. In [2], J. Apel restated a conjecture firstly given by Stanley in [14], namely that $\operatorname{sdepth}(M) \geq \operatorname{depth}(M)$ for any $\mathbb{Z}^n$-graded S-module M. This conjecture proves to be false, in general, for $M = S/I$ and $M = J/I$, where $0 \neq I \subset J \subset S$ are monomial ideals, see [7]. For a friendly introduction in the thematic of Stanley depth, we refer the reader [11].

Let $\Delta \subset 2^{[n]}$ be a simplicial complex. A face $F \in \Delta$ is called a *facet*, if F is maximal with respect to inclusion. We denote $\mathcal{F}(\Delta)$ the set of facets of Δ. If $F \in \mathcal{F}(\Delta)$, we denote $x_F = \prod_{j\in F} x_j$. Then the *facet ideal* $I(\Delta)$ associated to Δ is the squarefree monomial ideal $I = (x_F \ : \ F \in \mathcal{F}(\Delta))$ of S. The facet ideal was studied by Faridi [8] from the `depth` perspective.

The *line graph* of lenght n, denoted by L_n, is a graph with the vertex set $V = [n]$ and the edge set $E = \{\{1,2\},\{2,3\},\ldots,\{n-1,n\}\}$. Let $\Delta_{n,m}$ be the simplicial complex with the set of facets $\mathcal{F}(\Delta_{n,m}) = \{\{1,2,\ldots,m\},\{2,3,\ldots,m+1\},\ldots,\{n-m+1,n-m+2,\ldots,n\}\}$, where $1 \leq m \leq n$. We denote $I_{n,m} = (x_1x_2\cdots x_m, x_2x_3\cdots x_{m+1},\ldots,x_{n-m+1}x_{n-m+2}\cdots x_n)$, the associated facet ideal. Note that $I_{n,m}$ is the m-path ideal of the graph L_n, provided with the direction given by $1 < 2 < \ldots < n$, see [9] for further details.

According to [9, Theorem 1.2], the projective dimension of $S/I_{n,m}$ is:

$$\operatorname{pd}(S/I_{n,m}) = \begin{cases} \frac{2(n-d)}{m+1},\ n \equiv d(mod\ (m+1))\ with\ 0 \leq d \leq m-1, \\ \frac{2n-m+1}{m+1},\ n \equiv m(mod\ (m+1)). \end{cases}$$

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By Auslander-Buchsbaum formula (see [15]), it follows that $\operatorname{depth}(S/I_{n,m}) = n - \operatorname{pd}(S/I_{n,m})$ and, by a straightforward computation, we can see $\operatorname{depth}(S/I_{n,m}) = n + 1 - \left\lfloor \frac{n+1}{m+1} \right\rfloor - \left\lceil \frac{n+1}{m+1} \right\rceil =: \varphi(n,m)$. We proved in [5] that $\operatorname{sdepth}(S/I_{n,m}) = \varphi(n,m)$.

The *cycle graph* of length n, denoted by C_n, is a graph with the vertex set $V = [n]$ and the edge set $E = \{\{1,2\},\{2,3\},\ldots,\{n-1,n\},\{n,1\}\}$. Let $\bar{\Delta}_{n,m}$ be the simplicial complex with the set of facets $\mathcal{F}(\bar{\Delta}_{n,m}) = \{\{1,2,\ldots,m\},\{2,3,\ldots,m+1\},\cdots,\{n-m+1,n-m+2,\ldots,n\},\{n-m+2,\ldots,n,1\},\ldots,\{n,1,\ldots,m-1\}\}$. We denote $J_{n,m} = (x_1x_2\cdots x_m, x_2x_3\cdots x_{m+1},\ldots,x_{n-m+1}x_{n-m+2}\cdots x_n,\ldots,x_nx_1\cdots x_{m-1})$, the associated facet ideal. Note that $J_{n,m}$ is the m-path ideal of the graph C_n.

Let $p = \left\lfloor \frac{n}{m+1} \right\rfloor$ and $d = n - (m+1)p$. According to [1, Corollary 5.5],

$$\operatorname{pd}(S/J_{n,m}) = \begin{cases} 2p+1, & d \neq 0, \\ 2p, & d = 0. \end{cases}$$

By Auslander-Buchsbaum formula, it follows that $\operatorname{depth}(S/J_{n,m}) = n - \operatorname{pd}(S/J_{n,m}) = n - \left\lfloor \frac{n}{m+1} \right\rfloor - \left\lceil \frac{n}{m+1} \right\rceil = \varphi(n-1,m)$. Our main result is Theorem 1.4, in which we prove that $\varphi(n,m) \geq \operatorname{sdepth}(S/J_{n,m}) \geq \varphi(n-1,m)$. We also prove that, $\operatorname{sdepth}(J_{n,m}/I_{n,m}) = \operatorname{depth}(J_{n,m}/I_{n,m}) = \varphi(n-1,m) + m - 1$, see Proposition 1.6. These results generalize [4, Theorem 1.9] and [4, Proposition 1.10].

1. Main results

First, we recall the well known Depth Lemma, see for instance [15, Lemma 1.3.9].

Lemma 1.1. *(Depth Lemma) If $0 \to U \to M \to N \to 0$ is a short exact sequence of modules over a local ring S, or a Noetherian graded ring with S_0 local, then*
a) $\operatorname{depth} M \geq \min\{\operatorname{depth} N, \operatorname{depth} U\}$.
b) $\operatorname{depth} U \geq \min\{\operatorname{depth} M, \operatorname{depth} N + 1\}$.
c) $\operatorname{depth} N \geq \min\{\operatorname{depth} U - 1, \operatorname{depth} M\}$.

In [12], Asia Rauf proved the analog of Lemma 1.1(a) for sdepth:

Lemma 1.2. *Let $0 \to U \to M \to N \to 0$ be a short exact sequence of $\mathbb{Z}^n$-graded S-modules. Then:* $\operatorname{sdepth}(M) \geq \min\{\operatorname{sdepth}(U), \operatorname{sdepth}(N)\}$.

The following result is well known. However, we present an original proof.

Lemma 1.3. *Let $I \subset S$ be a nonzero proper monomial ideal. Then, I is principal if and only if* $\operatorname{sdepth}(S/I) = n - 1$.

Proof. Assume $\operatorname{sdepth}(S/I) = n-1$ and let $S/I = \bigoplus_{i=1}^r u_iK[Z_i]$ be a Stanley decomposition with $|Z_i| = n-1$ for all i, and $u_i \in S$ monomials. Since $1 \notin I$, we may assume that $u_1 = 1$. Let x_{j_1} be the variable which is not in Z_1. If $x_{j_1} \in I$, since $S/(x_{j_1}) = K[Z_1]$ and $K[Z_1] \subset S/I$, then $I = (x_{j_1})$. Otherwise, we may assume that $u_2 = x_{j_1}$.

Let x_{j_2} be the variable which is not in Z_2. If $x_{j_1}x_{j_2} \in I$, then, one can easily see that $I = (x_{j_1}x_{j_2})$. If $x_{j_1}x_{j_2} \notin I$, then we may assume $u_3 = x_{j_1}x_{j_2}$ and so on. Thus, we have $u_i = x_{j_1}\cdots x_{j_{i-1}}$, for all $1 \leq i \leq r+1$, where x_{j_i} is the variable which is not in Z_i. Moreover, $I = (u_{r+1})$, and therefore I is principal.

In order to prove the other implication, assume that $I = (u)$ and write $u = \prod_{i=1}^r x_{j_i}$. We let $u_i = \prod_{k=1}^{i-1} x_{j_k}$ and $Z_i = \{x_1,\ldots,x_n\} \setminus \{x_{j_i}\}$, for all $1 \leq i \leq r$. Then, $S/I = \bigoplus_{i=1}^r u_iK[Z_i]$ is a Stanley decomposition with $|Z_i| = n-1$ for all i. Therefore $\operatorname{sdepth}(S/I) = n-1$. □

Our main result, is the following theorem.

Theorem 1.4. $\varphi(n,m) \geq \operatorname{sdepth}(S/J_{n,m}) \geq \operatorname{depth}(S/J_{n,m}) = \varphi(n-1,m)$.

Proof. If $n=m$, then $J_{n,n} = (x_1 \ldots x_n)$ is a principal ideal, and, according to Lemma 1.3 we are done. Also, if $m=1$, then $J_{n,1} = (x_1,\ldots,x_n)$ and so there is nothing to prove, since $S/J_{n,1} = K$. The case $m=2$ follows from [4, Proposition 1.8] and [4, Theorem 1.9].

Assume $n > m \geq 3$. If $n = m+1$, then we consider the short exact sequence

$$0 \to S/(J_{n,n-1} : x_n) \to S/J_{n,n-1} \to S/(J_{n,n-1}, x_n) \to 0.$$

Note that $(J_{n,n-1} : x_n) = (x_1 \cdots x_{n-2}, x_2 \cdots x_{n-1}, x_3 \cdots x_{n-1}x_1, \cdots, x_{n-1}x_1 \cdots x_{n-3}) \cong J_{n-1,n-2}S$. Therefore, by induction hypothesis and [10, Lemma 3.6],

$$\operatorname{sdepth}(S/(J_{n,n-1} : x_n)) = \operatorname{depth}(S/(J_{n,n-1} : x_n)) = 1 + \varphi(n-2,n-2) = n-2.$$

Also, $(J_{n,n-1}, x_n) = (x_1 \cdots x_{n-1}, x_n)$ and thus $S/(J_{n,n-1}, x_n) \cong K[x_1,\ldots,x_{n-1}]/(x_1 \cdots x_{n-1})$. Therefore, by Lemma 1.3, we have $\operatorname{sdepth}(S/(J_{n,n-1}, x_n)) = n-2 = \operatorname{depth}(S/(J_{n,n-1}, x_n))$.

Now, assume $n > m+1 > 3$. We consider the ideals $L_0 = J_{n,m}$, $L_{k+1} = (L_k : x_{n-k})$ and $U_k = (L_k, x_{n-k})$, for $0 \leq k \leq m-2$. Note that

$$L_{m-1} = (J_{n,m} : x_{n-m+2} \cdots x_n) = (x_1, x_2 \cdots x_{m+1}, \ldots, x_{n-2m+1} \cdots x_{n-m}, x_{n-m+1}).$$

If $n - 2m \leq 2$, then $L_{m-1} = (x_1, x_{n-m+1})$ and thus $\operatorname{sdepth}(S/L_{m-1}) = \operatorname{depth}(S/L_{m-1}) = n-2 = \varphi(n,m)$, since $\left\lfloor \frac{n+1}{m+1} \right\rfloor = 1$ and $\left\lceil \frac{n+1}{m+1} \right\rceil = 2$.

If $n-2m > 2$, then $S/L_{m-1} \cong K[x_2,\ldots,x_{n-m},x_{n-m+2},\ldots,x_n]/(x_2 \cdots x_{m+1},\ldots,x_{n-2m+1} \cdots x_{n-m})$ and therefore, by [10, Lemma 3.6] and [5, Theorem 1.3], we have $\operatorname{sdepth}(S/L_{m-1}) = \operatorname{depth}(S/L_{m-1}) = n-1-\left\lfloor \frac{n-m}{m+1} \right\rfloor - \left\lceil \frac{n-m}{m+1} \right\rceil = \varphi(n,m)$. On the other hand, for example by [3, Proposition 2.7], $\operatorname{sdepth}(S/L_{m-1}) \geq \operatorname{sdepth}(S/J_{n,m})$. Thus, $\operatorname{sdepth}(S/J_{n,m}) \leq \varphi(n,m)$.

For any $0 < k < m$, we have $L_k = (x_1 \cdots x_{m-k}, x_2 \cdots x_{m+1}, \ldots, x_{n-m-k} \cdots x_{n-k-1}, x_{n-m+1} \cdots x_{n-k}, x_{n-m+2} \cdots x_{n-k}x_1, \ldots, x_{n-k}x_1 \cdots x_{m-k-1})$. Therefore, $U_k = (x_1 \cdots x_{m-k}, x_2 \cdots x_{m+1}, \ldots, x_{n-m-k} \cdots x_{n-k-1}, x_{n-k})$, for $k \leq m-2$. We consider two cases:

(i) If $n-m-k < 2$ and $0 \leq k \leq m-2$, then $U_k = (x_1 \cdots x_{m-k}, x_{n-k})$ and therefore $\operatorname{sdepth}(S/U_k) = \operatorname{depth}(S/U_k) = n-2 = \varphi(n,m)$, since $\left\lfloor \frac{n+1}{m+1} \right\rfloor = 1$ and $\left\lceil \frac{n+1}{m+1} \right\rceil = 2$.

(ii) If $n-m-k \geq 2$, then, for any $0 \leq j \leq k \leq m-2$, we consider the ideals $V_{k,j} := (x_1 \cdots x_{m-j}, x_2 \cdots x_{m+1}, \ldots, x_{n-m-k} \cdots x_{n-k-1})$ in $S_k := K[x_1,\ldots,n_{n-k-1}]$. Note that $S/U_k \cong (S_k/V_{k,k})[x_{n-k+1},\ldots,x_n]$ and thus, by [10, Lemma 3.6], $\operatorname{depth}(S/U_k) = \operatorname{depth}(S_k/V_{k,k}) + k$ and $\operatorname{sdepth}(S/U_k) = \operatorname{sdepth}(S_k/V_{k,k}) + k$.

For any $0 \leq j < k \leq m-2$, we claim that $V_{k,j}/V_{k,j+1}$ is isomorphic to

$$(K[x_{m-j+2},\ldots,x_{n-k-1}]/(x_{m-j+2} \cdots x_{2m-j+1},\ldots,x_{n-m-k} \cdots x_{n-k-1}))[x_1,\ldots,x_{m-j}].$$

Indeed, if $u \in V_{k,j} \setminus V_{k,j+1}$ is a monomial, then $x_1 \cdots x_{m-j} | u$ and $x_{m-j+1} \nmid u$. Also, $x_{m-j+2} \cdots x_{2m-j+1} \nmid u$, $\ldots$, $x_{n-m-k} \cdots x_{n-k-1} \nmid u$. Denoting $v = u/(x_1 \cdots x_{m-j})$, we can write $v = v'v''$, with $v' \in K[x_{m-j+2},\ldots,x_{n-k-1}] \setminus (x_{m-j+2} \cdots x_{2m-j+1},\ldots,x_{n-m-k} \cdots x_{n-k-1})$ and $v'' \in K[x_1,\ldots,x_{m-j}]$.

By [10, Lemma 3.6] and [5, Theorem 1.3], $\operatorname{sdepth}(V_{k,j}/V_{k,j+1}) = \operatorname{depth}(V_{k,j}/V_{k,j+1}) = m-j+\varphi(n-k-m+j-2,m) = n-k-1-\left\lfloor \frac{n-m-1-k+j}{m+1} \right\rfloor - \left\lceil \frac{n-m-1-k+j}{m+1} \right\rceil = n-k+1-\left\lfloor \frac{n-k+j}{m+1} \right\rfloor - \left\lceil \frac{n-k+j}{m+1} \right\rceil \geq \varphi(n,m)-k$. On the other hand, $V_{k,0} = I_{n-k-1,m}$ for any $0 \leq k \leq m-2$ and therefore, by [5, Theorem 1.3], $\operatorname{sdepth}(S/V_{k,0}) = \operatorname{depth}(S/V_{k,0}) = \varphi(n-k-1,m) = n-k-\left\lfloor \frac{n-k}{m+1} \right\rfloor - \left\lceil \frac{n-k}{m+1} \right\rceil \geq \varphi(n,m)-k$, for any $k \geq 1$. From the short exact sequences $0 \to V_{k,j}/V_{k,j+1} \to S/V_{k,j+1} \to S/V_{k,j} \to 0$, $0 \leq j < k$, Lemma 1.1 and Lemma 1.2, it follows that $\operatorname{sdepth}(S/V_{k,j+1}) \geq \operatorname{depth}(S/V_{k,j+1}) = \varphi(n,m)-k$, for all $0 \leq j < k \leq m-2$. Thus $\operatorname{sdepth}(S/U_k) \geq \operatorname{depth}(S/U_k) \geq \varphi(n,m)$, for all $0 < k \leq m-2$.On the other hand, $\operatorname{sdepth}(S/V_{0,0}) = \operatorname{depth}(S/V_{0,0}) = \varphi(n-1,m)$, and thus $\operatorname{sdepth}(S/U_0) = \operatorname{depth}(S/U_0) = \varphi(n-1,m)$.

Now, we consider short exact sequences

$$0 \to S/L_{k+1} \to S/L_k \to S/U_k \to 0, \ \text{ for } 0 \leq k < m.$$

By Lemma 1.1 and Lemma 1.2 we get $\operatorname{sdepth}(S/L_k) \geq \operatorname{depth}(S/L_k) = \varphi(n,m)$, for any $0 < k \leq m-2$, and $\operatorname{sdepth}(S/L_0) \geq \operatorname{depth}(S/L_0) = \varphi(n-1,m)$. $\square$

Corollary 1.5. *If* $\left\lfloor \frac{n+1}{m+1} \right\rfloor = \left\lfloor \frac{n}{m+1} \right\rfloor$ *and* $\left\lceil \frac{n+1}{m+1} \right\rceil = \left\lceil \frac{n}{m+1} \right\rceil$, *then*

$$\operatorname{sdepth}(S/J_{n,m}) = \operatorname{depth}(S/J_{n,m}) = \varphi(n,m).$$

Proposition 1.6. $\operatorname{sdepth}(J_{n,m}/I_{n,m}) \geq \operatorname{depth}(J_{n,m}/I_{n,m}) = \varphi(n-1,m) + m - 1.$

Proof. We claim that $J_{n,m}/I_{n,m}$ is isomorphic to

$$x_{n-m+2}\cdots x_n x_1 \left(\frac{K[x_2,\ldots,x_{n-m}]}{(x_2\cdots x_m, x_3\cdots x_{m+2}\ldots, x_{n-2m+1}\cdots x_{n-m})}\right)[x_{n-m+2},\ldots,x_n,x_1] \oplus$$

$$\oplus x_{n-m+3}\cdots x_n x_1 x_2 \left(\frac{K[x_3,\ldots,x_{n-m+1}]}{(x_3\cdots x_m, x_4\cdots x_{m+3},\ldots, x_{n-2m+2}\cdots x_{n-m+1})}\right)[x_{n-m+3},\ldots,x_n,x_1,x_2] \oplus$$

$$\cdots \oplus x_n x_1 \cdots x_{m-1}\left(\frac{K[x_m,\ldots,x_{n-2}]}{(x_m, x_{m+1}\cdots x_{2m},\ldots,x_{n-m-1}\cdots x_{n-2})}\right)[x_n,x_1\ldots,x_{m-1}].$$

Indeed, let $u \in J_{n,m} \setminus I_{n,m}$ be a monomial. If $x_{n-m+2}\cdots x_n x_1 | u$, then $x_{n-m+1} \nmid u$ and $x_2\cdots x_m \nmid u$. It follows that:

$$u \in x_{n-m+2}\cdots x_n x_1 (K[x_2,\ldots,x_{n-m}]/(x_2\cdots x_m, x_3\cdots x_{m+2}\ldots,x_{n-2m+1}\cdots x_{n-m}))[x_{n-m+2},\ldots,x_n,x_1].$$

If $x_{n-m+2}\cdots x_n x_1 \nmid u$ and $x_{n-m+3}\cdots x_n x_1 x_2 | u$ then $x_{n-m+2} \nmid u$ and $x_3\cdots x_m \nmid u$. Thus:

$$u \in x_{n-m+3}\cdots x_n x_1 x_2 \left(\frac{K[x_3,\ldots,x_{n-m+1}]}{(x_3\cdots x_m, x_4\cdots x_{m+3},\ldots,x_{n-2m+2}\cdots x_{n-m+1})}\right)[x_{n-m+3},\ldots,x_n,x_1,x_2].$$

Finally, if $x_{n-m+2}\cdots x_n x_1 \nmid u$, $\ldots$, $x_{n-1}x_n x_1\cdots x_{m-2} \nmid u$ and $x_n x_1\cdots x_{m-1}|u$, then it follows that $x_{n-1} \nmid u$ and $x_m \nmid u$. Therefore:

$$u \notin x_n x_1\cdots x_{m-1}\left(\frac{K[x_m,\ldots,x_{n-2}]}{(x_m, x_{m+1}\cdots x_{2m},\ldots,x_{n-m-1}\cdots x_{n-2})}\right)[x_n,x_1\ldots,x_{m-1}].$$

As in the proof of Theorem 3.1 (see the computations for $V_{k,j}$'s), by applying Lemma 1.1 and Lemma 1.2, it follows that $\operatorname{sdepth}(J_{n,m}/I_{n,m}) \geq \operatorname{depth}(J_{n,m}/I_{n,m}) = \varphi(n-m-2,m)+m = \varphi(n-1,m)+m-1$, as required. $\square$

Inspired by [4, Conjecture 1.12] and computer experiments [6], we propose the following:

Conjecture 1.7. *For any* $n \geq 3(m+1)+1$, *we have* $\operatorname{sdepth}(S/J_{n,m}) = \varphi(n,m)$.

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