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ON THE STANLEY DEPTH OF EDGE IDEALS OF LINE AND CYCLIC GRAPHS

MIRCEA CIMPOEAŞ

ABSTRACT. We prove that the edge ideals of line and cyclic graphs and their quotient rings satisfy the Stanley conjecture. We compute the Stanley depth for the quotient ring of the edge ideal associated to a cycle graph of length n, given a precise formula for $n \equiv 0, 2 \pmod 3$ and tight bounds for $n \equiv 1 \pmod 3$. Also, we give bounds for the Stanley depth of a quotient of two monomial ideals, in combinatorial terms.

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INTRODUCTION

Let K be a field and $S = K[x_1, \ldots, x_n]$ the polynomial ring over K. Let M be a $\mathbb{Z}^n$-graded S-module. A *Stanley decomposition* of M is a direct sum $\mathcal{D} : M = \bigoplus_{i=1}^r m_i K[Z_i]$ as a $\mathbb{Z}^n$-graded K-vector space, where $m_i \in M$ is homogeneous with respect to $\mathbb{Z}^n$-grading, $Z_i \subset \{x_1, \ldots, x_n\}$ such that $m_i K[Z_i] = \{um_i : u \in K[Z_i]\} \subset M$ is a free $K[Z_i]$-submodule of M. We define $\operatorname{sdepth}(\mathcal{D}) = \min_{i=1,\ldots,r} |Z_i|$ and $\operatorname{sdepth}_S(M) = \max\{\operatorname{sdepth}(\mathcal{D}) | \ \mathcal{D} \text{ is a Stanley decomposition of } M\}$. The number $\operatorname{sdepth}_S(M)$ is called the *Stanley depth* of M. In [1], J. Apel restated a conjecture firstly given by Stanley in [15], namely that $\operatorname{sdepth}_S(M) \geq \operatorname{depth}_S(M)$ for any $\mathbb{Z}^n$-graded S-module M. This conjecture proves to be false, in general, for $M = S/I$ and $M = J/I$, where $I \subset J \subset S$ are monomial ideals, see [9].

Herzog, Vladoiu and Zheng show in [10] that $\operatorname{sdepth}_S(M)$ can be computed in a finite number of steps if $M = I/J$, where $J \subset I \subset S$ are monomial ideals. However, it is difficult to compute this invariant, even in some very particular cases. In [14], Rinaldo give a computer implementation for this algorithm, in the computer algebra system *CoCoA* [8]. However, it is difficult to compute this invariant, even in some very particular cases. For instance in [2] Biro et al. proved that $\operatorname{sdepth}(m) = \lceil n/2 \rceil$ where $m = (x_1, \ldots, x_n)$.

Let I_n and J_n be the edges ideals associated to the n-line, respectively n-cycle, graph. Firstly, we prove that $\operatorname{depth}(S/J_n) = \left\lceil \frac{n-1}{3} \right\rceil$, see Proposition 1.3. Alin Ştefan [16] proved that $\operatorname{sdepth}(S/I_n) = \left\lceil \frac{n}{3} \right\rceil$. Using similar techniques, we prove that $\operatorname{sdepth}(S/J_n) = \left\lceil \frac{n-1}{3} \right\rceil$, for $n \equiv 0 \pmod 3$ and $n \equiv 2 \pmod 3$. Also, we prove that $\left\lceil \frac{n-1}{3} \right\rceil \leq \operatorname{sdepth}(S/J_n) \leq \left\lceil \frac{n}{3} \right\rceil$, for $n \equiv 1 \pmod 3$. See Theorem 1.9. In particular, S/J_n satisfies the Stanley conjecture. Also, we note that both I_n and J_n satisfy the Stanley conjecture, see Corollary 1.5. In Proposition 1.10, we prove that $\operatorname{sdepth}(J_n/I_n) = \operatorname{depth}(J_n/I_n) = \left\lceil \frac{n+2}{3} \right\rceil$. In the second section, we give an upper bound for the Stanley depth of a quotient of two square free monomial ideals, in combinatorial terms, see Theorem 2.4. Also, we give a lower bound for the Stanley depth of a quotient of two arbitrary monomial ideals, see Proposition 2.9.

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1. Main results

Let $n \geq 3$ be an integer and let $G = (V, E)$ be a graph with the vertex set $V = [n]$ and edge set E. Then the *edge ideal* $I(G)$ associated to G is the squarefree monomial ideal $I = (x_i x_j : \{i,j\} \in E)$ of S.

We consider the l*ine graph* L_n on the vertex set $[n]$ and with the edge set $E(L_n) = \{(i, i+1) : i \in [n-1]\}$. Then $I_n = I(L_n) = (x_1x_2, \ldots, x_{n-1}x_n) \subset S$. Also, we consider the cyclic graph C_n on the vertex set $[n]$ and with the edge set $E(C_n) = \{(i, i+1) : i \in [n-1]\} \cup \{(n, 1)\}$. Then $J_n = I_n + (x_n x_1) \subset S$.

We recall the well known Depth Lemma, see for instance [18, Lemma 1.3.9] or [17, Lemma 3.1.4].

Lemma 1.1. *(Depth Lemma) If* $0 \to U \to M \to N \to 0$ *is a short exact sequence of modules over a local ring* S*, or a Noetherian graded ring with* S_0 *local, then*

a) $\operatorname{depth} M \geq \min\{\operatorname{depth} N, \operatorname{depth} U\}$.

b) $\operatorname{depth} U \geq \min\{\operatorname{depth} M, \operatorname{depth} N + 1\}$.

c) $\operatorname{depth} N \geq \min\{\operatorname{depth} U - 1, \operatorname{depth} M\}$.

Using Depth Lemma, Morey proved in [11] the following result.

Lemma 1.2. [11, Lemma 2.8] $\operatorname{depth}(S/I_n) = \lceil \frac{n}{3} \rceil$.

In the following, we will prove a similar result for S/J_n.

Proposition 1.3. $\operatorname{depth}(S/J_n) = \lceil \frac{n-1}{3} \rceil$.

Proof. We denote $S_k := K[x_1, \ldots, x_k]$, the ring of polynomials in k variables. We use induction on n. If $n \leq 3$ then is an easy exercise to prove the formula. Assume $n \geq 4$ and consider the short exact sequence

$$0 \longrightarrow S/(J_n : x_n) \xrightarrow{\cdot x_n} S/J_n \longrightarrow S/(J_n, x_n) \longrightarrow 0.$$

Note that $(J_n : x_n) = (x_1, x_{n-1}, x_2x_3, \ldots, x_{n-3}x_{n-2})$ and therefore we get $S/(J_n : x_n) \cong K[x_2, \ldots, x_{n-2}, x_n]/(x_2x_3, \ldots, x_{n-3}x_{n-2}) \cong (S_{n-3}/I_{n-3})[x_n]$.

Also, $(J_n, x_n) = (x_1x_2, \ldots, x_{n-2}x_{n-1}, x_n)$ and therefore $S/(J_n, x_n) \cong S_{n-1}/I_{n-1}$. By Lemma 1.2, we get $\operatorname{depth}(S/(J_n : x_n)) = \lceil \frac{n-3}{3} \rceil + 1 = \lceil \frac{n}{3} \rceil$ and $\operatorname{depth}(S/(J_n, x_n)) = \lceil \frac{n-1}{3} \rceil$. If $n \equiv 0 \pmod 3$ or $n \equiv 2 \pmod 3$, then $\lceil \frac{n-1}{3} \rceil = \lceil \frac{n}{3} \rceil$, and, by using Lemma 1.1, we get $\operatorname{depth}(S/J_n) = \lceil \frac{n-1}{3} \rceil$, as required.

Assume $n \equiv 1 \pmod 3$. We claim that we have the S-module isomorphism

$$\frac{(J_n : x_n)}{J_n} \cong x_{n-1} \left(\frac{K[x_1, \ldots, x_{n-3}]}{(x_1x_2, \ldots, x_{n-4}x_{n-3})} \right) [x_{n-1}] \oplus x_1 \left(\frac{K[x_3, \ldots, x_{n-2}]}{(x_3x_4, \ldots, x_{n-3}x_{n-2})} \right) [x_1].$$

Indeed, if $u \in (J_n : x_n)$ is a monomial such that $u \notin J_n$, then $x_1 | u$ or $x_{n-1} | u$. If $x_{n-1} | u$, then $u = x_{n-1}v$ with $v \in S$. Since $u \notin J_n$, it follows that $v = x_{n-1}^{\alpha} w$, with $\alpha \geq 1$, $w \in K[x_1, \ldots, x_{n-3}]$ and $w \notin (x_1x_2, \ldots, x_{n-4}x_{n-3})$. Similarly, if $x_{n-1} \nmid u$, then $x_1 | u$ and $u = x_1^{\alpha} w$ with $\alpha \geq 1$, $w \in K[x_3, \ldots, x_{n-2}]$ and $w \notin (x_3x_4, \ldots, x_{n-3}x_{n-2})$.

Using the above isomorphism and Lemma 1.2, it follows that

$$\operatorname{depth}\left(\frac{(J_n : x_n)}{J_n} \right) = \operatorname{depth}\left(\frac{K[x_3, \ldots, x_{n-2}]}{(x_3x_4, \ldots, x_{n-3}x_{n-2})} \right) + 1 = \left\lceil \frac{n-4}{3} \right\rceil + 1 = \left\lceil \frac{n-1}{3} \right\rceil.$$

Now, using Lemma 1.1 for the short exact sequence $0 \to \frac{(J_n : x_n)}{J_n} \to S/J_n \to S/(J_n : x_n) \to 0$, we are done. □

Note that the previous Proposition can be seen as a consequence of [3, Proposition 5.0.6]. However, we preferred to give a direct proof in order to relate it with the Stanley depth case. Now, we recall the following result of Okazaki.

Theorem 1.4. [12, Theorem 2.1] *Let* $I \subset S$ *be a monomial ideal (minimally) generated by* m *monomials. Then:*

$$\operatorname{sdepth}(I) \geq \max\{1, n - \left\lfloor \frac{m}{2} \right\rfloor\}.$$

As a direct consequence of Lemma 1.2, Proposition 1.3 and Theorem 1.4, we get.

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Corollary 1.5. $\operatorname{sdepth}(I_n) \geq 1 + \frac{n-1}{2}$ *and* $\operatorname{sdepth}(J_n) \geq \frac{n}{2}$. *In particular,* I_n *and* J_n *satisfy the Stanley conjecture.*

In [16], Alin Ştefan computed the Stanley depth for S/I_n.

Lemma 1.6. [16, Lemma 4] $\operatorname{sdepth}(S/I_n) = \lceil \frac{n}{3} \rceil$.

In [13], Asia Rauf proved the analog of Lemma 1.1(a) for sdepth:

Lemma 1.7. *Let* $0 \to U \to M \to N \to 0$ *be a short exact sequence of* $\mathbb{Z}^n$*-graded* S*-modules. Then:*

$$\operatorname{sdepth}(M) \geq \min\{\operatorname{sdepth}(U), \operatorname{sdepth}(N)\}.$$

Using these lemmas, we are able to prove the following Proposition.

Proposition 1.8. $\operatorname{sdepth}(S/J_n) \geq \lceil \frac{n-1}{3} \rceil$. *In particular,* S/J_n *satisfies the Stanley conjecture.*

Proof. As in the proof of Proposition 1.3, we consider the short exact sequence

$$0 \longrightarrow S/(J_n : x_n) \xrightarrow{\cdot x_n} S/J_n \longrightarrow S/(J_n, x_n) \longrightarrow 0.$$

Since $S/(J_n : x_n) \cong (S_{n-2}/I_{n-2})[x_n]$ and $S/(J_n, x_n) \cong S_{n-1}/I_{n-1}$, by Lemma 1.6 and [10, Lemma 3.6], we get $\operatorname{sdepth}(S/(J_n : x_n)) = \lceil \frac{n-3}{3} \rceil + 1 = \lceil \frac{n}{3} \rceil$ and $\operatorname{sdepth}(S/(J_n, x_n)) = \lceil \frac{n-1}{3} \rceil$. Using Lemma 1.7, we get $\operatorname{sdepth}(S/J_n) \geq \lceil \frac{n-1}{3} \rceil$, as required. □

Let $\mathcal{P} \subset 2^{[n]}$ be a poset and $\mathbf{P} : \mathcal{P} = \bigcup_{i=1}^{r} [F_i, G_i]$ be a partition of $\mathbf{P}$. We denote $\operatorname{sdepth}(\mathbf{P}) := \min_{i \in [r]} |D_i|$. Also, we define the Stanley depth of $\mathcal{P}$, to be the number

$$\operatorname{sdepth}(\mathcal{P}) = \max\{\operatorname{sdepth}(\mathbf{P}) : \mathbf{P} \text{ is a partition of } \mathcal{P}\}.$$

We recall the method of Herzog, Vladoiu and Zheng [10] for computing the Stanley depth of S/I and I, where I is a squarefree monomial ideal. Let $G(I) = \{u_1, \ldots, u_s\}$ be the set of minimal monomial generators of I. We define the following two posets:

$$\mathcal{P}_I := \{\sigma \subset [n] : u_i | x_\sigma := \prod_{j \in \sigma} x_j \text{ for some } i \} \text{ and } \mathcal{P}_{S/I} := 2^{[n]} \setminus \mathcal{P}_I.$$

Herzog Vladoiu and Zheng proved in [10] that $\operatorname{sdepth}(I) = \operatorname{sdepth}(\mathcal{P}_I)$ and $\operatorname{sdepth}(S/I) = \operatorname{sdepth}(\mathcal{P}_{S/I})$. Now, for $d \in \mathbb{N}$ and $\sigma \in \mathcal{P}$, we denote

$$\mathcal{P}_d = \{\tau \in \mathcal{P} \ : \ |\tau| = d\} \ , \ \mathcal{P}_{d,\sigma} = \{\tau \in \mathcal{P}_d \ : \ \sigma \subset \tau\}.$$

With these notations, we are able to prove the following result.

Theorem 1.9. *(1)* $\operatorname{sdepth}(S/J_n) = \lceil \frac{n-1}{3} \rceil$, *for* $n \equiv 0 \pmod 3$ *and* $n \equiv 2 \pmod 3$.
(2) $\operatorname{sdepth}(S/J_n) \leq \lceil \frac{n}{3} \rceil$, *for* $n \equiv 1 \pmod 3$.

Proof. Using Proposition 1.8, it is enough to prove the "$\leq$" inequalities. Let $\mathcal{P} = \mathcal{P}_{S/J_n}$. Firstly, note that if $\sigma \in \mathcal{P}$ such that $P_{d,\sigma} = \emptyset$, then $\operatorname{sdepth}(\mathcal{P}) < d$. Indeed, let $\mathbf{P} : \mathcal{P} = \bigcup_{i=1}^{r} [F_i, G_i]$ be a partition of $\mathcal{P}$ with $\operatorname{sdepth}(\mathcal{P}) = \operatorname{sdepth}(\mathbf{P})$. Since $\sigma \in \mathcal{P}$, it follows that $\sigma \in [F_i, G_i]$ for some i. If $|G_i| \geq d$, then it follows that $\mathcal{P}_{\sigma,d} \neq \emptyset$, since there are subsets in the interval $[F_i, G_i]$ of cardinality d which contain σ, a contradiction. Thus, $|G_i| < d$ and therefore $\operatorname{sdepth}(\mathcal{P}) < d$.

We have three cases to study.

1. If $n = 3k \geq 3$ and $\sigma = \{1, 4, \ldots, 3k-2\}$, then $\mathcal{P}_{k+1,\sigma} = \emptyset$. Indeed, if $u = x_1 x_4 \cdots x_{3k-2}$, one can easily see that $u \cdot x_j \in J_n$ for all $j \in [n] \setminus \sigma$. Therefore, be previous remark, $\operatorname{sdepth}(S/J_n) = \operatorname{sdepth}(\mathcal{P}) \leq k = \lceil \frac{n-1}{3} \rceil$, as required.

2. If $n = 3k + 2 \geq 5$ and $\sigma = \{1, 4, \ldots, 3k+1\}$, then $\mathcal{P}_{k+2,\sigma} = \emptyset$. As above, it follows that $\operatorname{sdepth}(S/J_n) \leq k + 1 = \lceil \frac{n-1}{3} \rceil$.

3. If $n = 3k+1 \geq 7$ and $\sigma = \{1, 4, \ldots, 3k-2, 3k\}$, then $\mathcal{P}_{k+2,\sigma} = \emptyset$ and therefore $\operatorname{sdepth}(\mathcal{P}) \leq k+1 = \lceil \frac{n}{3} \rceil$. □

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Proposition 1.10. $\operatorname{sdepth}(J_n/I_n) = \operatorname{depth}(J_n/I_n) = \lceil \frac{n+2}{3} \rceil$, *for all* $n \geq 3$.

Proof. One can easily check that $\frac{J_3}{I_3} \cong x_1x_3K[x_1, x_3]$. Thus $\operatorname{sdepth}(J_3/I_3) = \operatorname{depth}(J_3/I_3) = 2$, as required. Similarly, for $n = 4$, we have $\frac{J_4}{I_4} \cong x_1x_4K[x_1, x_4]$ and for $n = 5$, we have $\frac{J_5}{I_5} \cong x_1x_5K[x_1, x_3, x_5]$.

Now, assume $n \geq 6$, and let $u \in J_n$ a monomial such that $u \notin I_n$. It follows that $u = x_1x_nv$, with $v \in K[x_1, x_3, \ldots, x_{n-2}, x_n]$. We can write $v = x_1^{\alpha}x_n^{\beta}w$, with $w \in K[x_3, \ldots, x_{n-2}]$. Since $u \notin I_n$, it follows that $w \notin (x_3x_4, \ldots, x_{n-3}x_{n-2})$. Therefore, we have the S-module isomorphism:

$$\frac{J_n}{I_n} = x_1x_n \left(\frac{K[x_3, \ldots, x_{n-2}]}{(x_3x_4, \ldots, x_{n-3}x_{n-2})} \right) [x_1, x_n]$$

and therefore, by Lemma 1.2, Lemma 1.6 and [10, Lemma 3.6], we get $\operatorname{sdepth}(J_n/I_n) = \operatorname{depth}(J_n/I_n) = \lceil \frac{n-4}{3} \rceil + 2 = \lceil \frac{n+2}{3} \rceil$. □

Remark 1.11. If $n = 4$, one can easily see that $\operatorname{sdepth}(S/J_4) = 1$. Also, for $n = 7$, we can show that $\operatorname{sdepth}(S/J_7) = 2$, see Example 2.5. On the other hand, using the *SdepthLib.coc* of *CoCoA*, see [14], we get $\operatorname{sdepth}(S/J_{10}) = 4$ and $\operatorname{sdepth}(S/J_{13}) = 5$. This remark, yields the following conjecture.

Conjecture 1.12. $\operatorname{sdepth}(S/J_n) = \lceil \frac{n}{3} \rceil$, *for all* $n \geq 10$ *with* $n \equiv 1 \pmod 3$.

Even if J_n and I_n are closely related, the difficulty of Conjecture 1.12 should not be underestimate. See for instance [2], where the authors, using fine tools of combinatorics were hardly able to compute the Stanley depth of the maximal monomial ideal $(x_1, \ldots, x_n)$. In the second section we will give a possible approach to this problem, see Example 2.5.

2. Bounds for Sdepth of quotient of monomial ideals

Lemma 2.1. *Let* $n \geq 1$ *and* $0 \leq k \leq n$ *be two integers and let* $\mathcal{P} = \{\sigma \in 2^{[n]} \mid |\sigma| \leq k\}$. *Then, there exists a partition* $\mathbf{P} : \mathcal{P} = \bigcup_{i=1}^{r}[C_i, D_i]$ *with* $|D_i| = k$.

Proof. If $k = n$ or $k = 0$ there is nothing to prove. Assume $1 \leq k \leq n - 1$. Note that $\mathcal{P}$ is the partition associated to $S/I_{n,k+1}$, where $I_{n,k+1}$ is the ideal generated by all the square free monomials of degree $k + 1$. According to [7, Theorem 1.1], $\operatorname{sdepth}(S/I_{n,k+1}) = k$ and thus we are done. □

Proposition 2.2. *Let* $\mathcal{P} \subset 2^{[n]}$ *be a poset such that* $\operatorname{sdepth}(\mathcal{P}) \geq k$. *Then there exists a partition of* $\mathcal{P}$*, such that, for each interval* $[C, D]$ *of it, if* $|C| < k$ *then* $|D| = k$.

In particular, the above assertion holds, if $I \subset J$ *are two monomial square-free ideals such that* $\operatorname{sdepth}(J/I) = k$ *and* $\mathcal{P} = \mathcal{P}_{J/I} := \mathcal{P}_{S/I} \cap \mathcal{P}_J$.

Proof. According to Herzog, Vladoiu and Zheng [10], we have $\operatorname{sdepth}(J/I) = \operatorname{sdepth}(\mathcal{P}_{J/I})$. Since $\operatorname{sdepth}(\mathcal{P}) \geq k$, we can find a partition of $\mathcal{P}$, such that each interval $[C, D]$ in this partition has $|D| \geq k$.

Let $[C, D]$ be an interval of the partition of $\mathcal{P}$. If $|C| \geq s$ or $|D| = s$ there is nothing to do. Assume $|C| < k$ and $|D| > k$. We denote $|C| = t$ and $|D| = s$. Without losing the generality, we may assume that $D = [s]$ and $C = [s] \setminus [s - t]$. Using the previous Lemma, we can find a partition of $[\emptyset, [s - t]] = \bigcup_{i=1}^{r}[\overline{C}_i, \overline{D}_i]$ with $|\overline{D}_i| = k - t$ whenever $|\overline{C}_i| < k - t$. Let $C_i = C \cup \overline{C}_i$ and $D_i = C \cup \overline{D}_i$. It follows that $[C, D] = \bigcup_{i=1}^{r}[C_i, D_i]$ is a partition with $|D_i| = k$, whenever $|C_i| < k$. If we apply this method for each interval in the partition of $\mathcal{P}$, finally, we will get a partition of $\mathcal{P}$, as required. □

Corollary 2.3. *Let* $\mathcal{P} \subset 2^{[n]}$ *be a poset such that* $\operatorname{sdepth}(\mathcal{P}) \geq k$. *Denote* $\mathcal{P}_{\leq k} = \{\sigma \in \mathcal{P} \, |\sigma| \leq k\}$. *Then* $\operatorname{sdepth}(\mathcal{P}_{\leq k}) = k$.

Proof. Obviously, $\operatorname{sdepth}(\mathcal{P}_{\leq k}) \leq k$. According to Proposition 2.2, we can find a partition $\mathbf{P} : \mathcal{P} = \bigcup_{i=1}^{r}[F_i, G_i]$ of $\mathcal{P}$ such that $|G_i| = k$, whenever $|F_i| < k$. Note that

$$[F_i, G_i] \cap \mathcal{P}_{\leq k} = \begin{cases} [F_i, G_i], \ |F_i| < k, \\ [F_i, F_i], \ |F_i| = k, \\ \emptyset, \ |F_i| > k \end{cases}$$

Therefore, $\mathcal{P}_{\le k} = \bigcup_{i=1}^{r} [F_i, G_i] \cap \mathcal{P}_{\le k}$ is a partition of $\mathcal{P}_{\le k}$ with its Stanley depth $\ge k$. □

Let $\mathcal{P} \subset 2^{[n]}$ be a poset such that $\operatorname{sdepth}(\mathcal{P}) \ge k$. We denote $\beta_t = |\{\sigma \in \mathcal{P} : |\sigma| = t\}|$, for all $0 \le t \le k$.

We consider the poset $\mathcal{P}_{\le k} := \{\sigma \in \mathcal{P} : |\sigma| \le k\}$. By Corollary 2.3, we can find a partition $\mathbf{P} : \mathcal{P}_{\le k} = \bigcup_{i=1}^{r} [F_i, G_i]$ with $|G_i| = k$ for all i. We may assume that $|F_i| \le |F_{i+1}|$ for all $i \le r-1$. For all $0 \le j \le k$, we denote $\alpha_j = |\{i : |F_i| = j\}|$. Let $[F, G]$ be an arbitrary interval in the partition $\mathbf{P}$ such that $|F| = j$ for some $j \le k$. Note that in the interval $[F, G]$ we have exactly $\binom{k-j}{t-j}$ sets of cardinality t. Therefore, we get $\beta_t = \sum_{j=0}^{t} \binom{k-j}{t-j} \alpha_j$, for all $0 \le t \le k$. Moreover, $\alpha_0 = \beta_0$, $\alpha_1 = \beta_1 - k\beta_0$, $\alpha_2 = \beta_2 - \binom{k}{2}\alpha_0 - (k-1)\alpha_1$ and so on. Thus, we proved the following Theorem.

Theorem 2.4. *If* $\operatorname{sdepth}(\mathcal{P}) \ge k$*, then* $\alpha_t \ge 0$ *for all* $0 \le t \le k$*, where* $\alpha_0 = \beta_0$ *and* $\alpha_t = \beta_t - \sum_{j=0}^{t-1} \binom{k-j}{t-j} \alpha_j$.

Note that the above theorem give an upper bound for $\operatorname{sdepth}(J/I)$, where $I \subset J$ are square free monomial ideals. Indeed, we can consider the poset $\mathcal{P} := \mathcal{P}_{J/I}$.

Example 2.5. We consider the poset $\mathcal{P} := \mathcal{P}_{S/J_n}$, where $J_n = (x_1x_2, \ldots, x_{n-1}x_n, x_nx_1) \subset S$. We claim that $\beta_t = \binom{n-t+1}{t} - \binom{n-t-1}{t-2}$, for all $0 \le t \le n$.

Indeed, if $\sigma = \{i_1, \ldots, i_t\} \in \mathcal{P}$ is a set of cardinality t such that $1 \le i_1 < i_2 < \cdots < i_t \le n$, then $i_{j+1} \ge i_j + 2$ and $\{i_1, i_k\} \ne \{1, n\}$. There are exactly $\binom{n-t+1}{t}$, t-tuples $1 \le i_1 < i_2 < \ldots < i_t \le n$ with $i_{j+1} \ge i_j + 2$ and exactly $\binom{n-t-1}{t-2}$, t-tuples $1 = i_1 < i_2 < \cdots < i_t = n$ with $i_{j+1} \ge i_j + 2$. (To be more clear, if we denote $l_j := i_j - j + 1$, we have $1 \le l_1 \le l_2 \le \cdots \le l_t \le n-t+1$ with $l_{j+1} > l_j$, and there are exactly $\binom{n-t+1}{t}$, t-tuples like this. If we fix $l_1 = 1$ and $l_t = n-t+1$, we have $2 \le l_2 \le \cdots \le l_{t-1} \le n-t$ and there are exactly $\binom{n-t-1}{t-2}$, $t-2$-tuples like this).

Now, for $n = 7$, one can easily check that $\beta_0 = 1$, $\beta_1 = 7$, $\beta_2 = 14$ and $\beta_3 = 7$. For $k = 3$, we have $\alpha_0 = 1$, $\alpha_1 = 4$, $\alpha_2 = 2$ and $\alpha_3 = -1$. This shows, in the light of Theorem 2.4, that we cannot find a decomposition of the poset associated to S/J_7 with its Stanley depth equal to 3. On the other hand, by Proposition 1.8, we have $\operatorname{sdepth}(S/J_7) \ge 2$, and thus $\operatorname{sdepth}(S/J_7) = 2$.

For $n = 3k-2$, where $k \ge 4$, we expect that $\alpha_0, \ldots, \alpha_k$ are nonnegative, which is indeed the case for small values of k, using computer experimentation. However, this is useful only as an heuristic method to estimate the Stanley depth of S/J_n. In order to compute exactly this invariant, one has to produce a concrete partition of the associated poset.

In the second part of this section, we give a lower bound for the Stanley depth of a quotient of monomial ideals in terms of the minimal number of monomial generators. First, we recall several results.

Proposition 2.6. [4, Proposition 1.2] *Let* $I \subset S$ *be a monomial ideal (minimally) generated by* m *monomials. Then* $\operatorname{sdepth}(S/I) \ge n - m$.

Proposition 2.7. [5, Remark 2.3] *Let* $I, J \subset S$ *be two monomial ideals. Then* $\operatorname{sdepth}((I+J)/I) \ge \operatorname{sdepth}(J) + \operatorname{sdepth}(S/I) - n$.

Lemma 2.8. *Let* $I, L \subset S$ *be two monomial ideals such that* L *is minimally generated by some monomials* $w_1, \ldots, w_s$ *which are not in* I*. Then* $\mathcal{B} = \{w_1 + I, \ldots, w_s + I\}$ *is a system of generators of* J/I*, where* $J := L + I$.

Proof. Denoting $G(I) = \{v_1, \ldots, v_p\}$, it follows that $J = (v_1, \ldots, v_p, w_1, \ldots, w_r)$. So, if $w \in J \setminus I$ is a monomial, then $w_j | w$ for some $j \in [r]$ and therefore $\mathcal{B}$ is a system of generators for J/I. On the other hand, since $w_1, \ldots, w_r$ minimally generated L, we get the minimality of $\mathcal{B}$. □

We consider $I \subset J \subset S$ two monomial ideals. Denote $G(I) = \{v_1, \ldots, v_p\}$ and $G(J) = \{u_1, \ldots, u_q\}$ the sets of minimal monomial generators of I and J. If $u_1 \in I$, then we may assume that $v_1 | u_1$. On the other hand, $I \subset J$ and therefore, there exists an index i such that $u_i | v_1$.

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We get $u_i|u_1$ and thus $u_i = u_1 = v_1$. Using the same argument, we can assume that there exists an integer $r \geq 0$ such that $u_1 = v_1, \dots, u_r = v_r$ and $u_{r+1}, \dots, u_q \notin I$. By Lemma 2.8, $\{u_{r+1} + I, \dots, u_q + I\}$ is a set of generators of J/I. With these notations, we have the following result, which is similar to [6, Theorem 2.4].

Proposition 2.9. $\operatorname{sdepth}(J/I) \geq n - p - \left\lfloor \frac{q-r}{2} \right\rfloor$.

Proof. Denote $J' = (u_{r+1}, \dots, u_q)$. Note that $J/I = (I+J')/I$. By Proposition 2.7, we get $\operatorname{sdepth}(J/I) \geq \operatorname{sdepth}(J') + \operatorname{sdepth}(S/I) - n$. By Theorem 1.4 and Proposition 2.6 we are done. □

References

[1] J. Apel, *On a conjecture of R. P. Stanley; Part II - Quotients Modulo Monomial Ideals*, J. of Alg. Comb. **17**, (2003), 57-74.

[2] C. Biro, D. M. Howard, M. T. Keller, W. T. Trotter, S. J. Young, *Interval partitions and Stanley depth*, Journal of Combinatorial Theory, Series A, **117** (2010) 475-482.

[3] R. R. Bouchat, *Free resolutions of some edge ideals of simple graphs*, J. Commutative Algebra **2** (2010), 1-35.

[4] M. Cimpoeas, *Stanley depth of monomial ideals with small number of generators*, Central European Journal of Mathematics, **vol. 7, no. 4**, (2009), 629-634.

[5] M. Cimpoeas, *Several inequalities regarding Stanley depth*, Romanian Journal of Math. and Computer Science **2(1)**, (2012), 28-40.

[6] M. Cimpoeas, *Stanley Depth of Quotient of Monomial Complete Intersection Ideals*, Communications in Algebra **40(8)** (2014), 2720 - 2731.

[7] M. Cimpoeas, *Stanley depth of squarefree Veronese ideals*, An. St. Univ. Ovidius, **Vol. 21(3)**, (2013), 67-71.

[8] CoCoATeam, *CoCoA: a system for doing Computations in Commutative Algebra*, Avaible at http://cocoa.dima.unige.it

[9] A. M. Duval, B. Goeckneker, C. J. Klivans, J. L. Martine, *A non-partitionable Cohen-Macaulay simplicial complex*, http://arxiv.org/pdf/1504.04279, (2015).

[10] J. Herzog, M. Vladoiu, X. Zheng, *How to compute the Stanley depth of a monomial ideal*, Journal of Algebra **322(9)**, (2009), 3151-3169.

[11] S. Morey, *Depths of powers of the edge ideal of a tree*, Comm. Algebra **38 (11)**, (2010), 4042-4055

[12] R. Okazaki, *A lower bound of Stanley depth of monomial ideals*, J. Commut. Algebra **Vol. 3(1)**, (2011), 83-88.

[13] A. Rauf, *Depth and sdepth of multigraded module*, Communications in Algebra, **vol. 38, Issue 2**, (2010), 773-784.

[14] G. Rinaldo, *An algorithm to compute the Stanley depth of monomial ideals*, Le Matematiche, **Vol. LXIII (ii)**, (2008), 243-256.

[15] R. P. Stanley, Linear Diophantine equations and local cohomology, Invent. Math. **68**, 1982, 175-193.

[16] A. Ştefan, *Stanley depth of powers of the path ideal*, http://arxiv.org/pdf/1409.6072.pdf, (2014).

[17] W. V. Vasconcelos, *Arithmetic of Blowup Algebras. London Math. Soc.*, Lecture Note Series **195**. Cambridge: Cambridge University Press, 1994.

[18] R. H. Villarreal, *Monomial algebras. Monographs and Textbooks in Pure and Applied Mathematics*, **238**. Marcel Dekker, Inc., New York, 2001.

Simion Stoilow Institute of Mathematics, Research unit 5, P.O.Box 1-764, Bucharest 014700, Romania

E-mail address: mircea.cimpoeas@imar.ro

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