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SUM AND PRODUCT THEOREMS OF RELATIVE ORDER AND RELATIVE LOWER ORDER OF ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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Abstract. In this paper, we aim at investigating some basic properties in connection with sum and product of relative order and relative lower order of entire functions of several complex variables.

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1. Introduction, Definitions and Notations.

Let f be an entire function of two complex variables holomorphic in the closed polydisc

$$U = \{(z_1, z_2) :\ |z_i| \leq r_i,\ i = 1, 2 \text{ for all } \ r_1 \geq 0, r_2 \geq 0\}$$

and $M_f(r_1, r_2) = \max\{|f(z_1, z_2)| : |z_i| \leq r_i,\ i = 1, 2\}$. Then in view of maximum principal and Hartogs theorem $\{$[7], p. 2, p. 51$\}$, $M_f(r_1, r_2)$ is an increasing functions of r_1, r_2.

The following definition is well known:

Definition 1.1. $\{$[7], *p. 339,* (*see also* [1])$\}$ *The order* ${}_{v_2}\rho_f$ *and the lower order* ${}_{v_2}\lambda_f$ *of an entire function* f *of two complex variables are defined as*

$$\begin{matrix} {}_{v_2}\rho(f) \\ {}_{v_2}\lambda(f) \end{matrix} = \lim_{r_1, r_2 \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log\log M_f(r_1, r_2)}{\log(r_1 r_2)}.$$

If we consider the above definition for single variable, then the definition coincides with the classical definition of order (see [13]) which is as follows:

Definition 1.2. [13] *The order* $\rho(f)$ *and the lower order* $\lambda(f)$ *of an entire function* f *are defined in the following way:*

$$\begin{matrix} \rho(f) \\ \lambda(f) \end{matrix} = \lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log\log M_f(r)}{\log r},$$

where $M_f(r) = \max\{|f(z)| : |z| = r\}$.

If f is non-constant then $M_f(r)$ is strictly increasing and continuous, and its inverse $M_f{}^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists and is such that $\lim_{s\to\infty} M_f{}^{-1}(s) = \infty$. Bernal [2, 3] introduced the definition of relative order of g with respect to f, denoted by $\rho_f(g)$ as follows :

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : M_f(r) < M_g(r^\mu) \text{ for all } r > r_0(\mu) > 0\} \\ &= \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r} .\end{aligned}$$

The definition coincides with the classical one [13] if $g(z) = \exp z$.

During the past decades, several authors (see [5, 8, 9, 10, 11, 12]) made close investigations on the properties of relative order of entire functions of single variable. In the case of relative order, it was then natural for Banerjee and Dutta [4] to define the relative order of entire functions of two complex variables as follows:

Definition 1.3. [4] *The relative order between two entire functions of two complex variables denoted by ${}_{v_2}\rho_g(f)$ is defined as:*

$$\begin{aligned}{}_{v_2}\rho_g(f) &= \inf\{\mu > 0 : M_f(r_1, r_2) < M_g(r_1^\mu, r_2^\mu); r_1 \geq R(\mu), r_2 \geq R(\mu)\} \\ &= \limsup_{r_1, r_2\to\infty} \frac{\log M_g^{-1} M_f(r_1, r_2)}{\log(r_1 r_2)}\end{aligned}$$

where f and g are entire functions holomorphic in the closed polydisc

$$U = \{(z_1, z_2) : \ |z_i| \leq r_i, \ i = 1, 2 \textit{ for all } \ r_1 \geq 0, r_2 \geq 0\}$$

and the definition coincides with Definition 1.1 {see [4]} if $g(z) = \exp(z_1 z_2)$.

Extending this notion, Dutta [6] introduced the idea of relative order of entire functions of several complex variables in the following way:

Definition 1.4. [6] *Let $f(z_1, z_2, ..., z_n)$ and $g(z_1, z_2, ..., z_n)$ be any two entire functions of n complex variables $z_1, z_2, ..., z_n$ with maximum modulus functions $M_f(r_1, r_2, ..., r_n)$ and $M_g(r_1, r_2, ..., r_n)$ respectively then the relative order of f with respect to g, denoted by ${}_{v_n}\rho_g(f)$ is defined by*

$$\begin{aligned}{}_{v_n}\rho_g(f) = \inf\{\mu > 0 : M_f(r_1, r_2, ..., r_n) < M_g(r_1^\mu, r_2^\mu, ..., r_n^\mu); \\ \textit{for } r_i \geq R(\mu), i = 1, 2, ..., n\}.\end{aligned}$$

The above definition can equivalently be written as

$${}_{v_n}\rho_g(f) = \limsup_{r_1, r_2, ..., r_n\to\infty} \frac{\log M_g^{-1} M_f(r_1, r_2, ..., r_n)}{\log(r_1 r_2 ... r_n)}.$$

Similarly, one can define the relative lower order of f with respect to g denoted by ${}_{v_n}\lambda_g(f)$ as follows :

$${}_{v_n}\lambda_g(f) = \liminf_{r_1, r_2, ..., r_n\to\infty} \frac{\log M_g^{-1} M_f(r_1, r_2, ..., r_n)}{\log(r_1 r_2 ... r_n)}.$$

Further an entire function f of several complex variables for which relative order and relative lower order with respect to another entire function g of several complex variables are the same is called a function of regular relative growth with respect to g. Otherwise, f is said to be irregular relative growth.with respect to g.

In this connection just we state the following two definitions which will be needed in the sequel:

Definition 1.5. [6] *The function $f(z_1, z_2, ..., z_n)$ is said to have Property (R) if for any $\sigma > 1$ and for all large $r_1, r_2, ..., r_n$,*

$$[M_f(r_1, r_2, ..., r_n)]^2 < M_f(r_1^\sigma, r_2^\sigma, ..., r_n^\sigma) .$$

For examples of functions with or without the Property (R), one may see [6].

Definition 1.6. *A pair of functions $f(z_1, z_2, ..., z_n)$ and $g(z_1, z_2, ..., z_n)$ of n complex variables are mutually said to have Property (X) if for all sufficiently large values of $r_1, r_2, ..., r_n$, both*

$$M_{f \cdot g}(r_1, r_2, ..., r_n) > M_f(r_1, r_2, ..., r_n)$$

and

$$M_{f \cdot g}(r_1, r_2, ..., r_n) > M_g(r_1, r_2, ..., r_n)$$

hold simultaneously.

One can easily verify that the functions $f(z_1, z_2, ..., z_n) = \exp(z_1 z_2 ... z_n)$ and $g(z_1, z_2, ..., z_n) = \exp(z_1 z_2 ... z_n)^2$ have the Property (X).

Here, in this paper, we aim at investigating some basic properties of relative order and relative lower order of entire functions of several complex variables with respect to another one under somewhat different conditions. We do not explain the standard definitions and notations in the theory of entire function of several complex variables as those are available in [7].

2. Lemma

In this section we present a lemma which will be needed in the sequel.

Lemma 2.1. [6] *Suppose that f be a non constant entire function of several complex variables, $\alpha > 1$ and $0 < \beta < \alpha$. Then*

$$M_f(\alpha r_1, \alpha r_2, ..., \alpha r_n) > \beta M_f(r_1, r_2, ..., r_n)$$

all sufficiently large $r_1, r_2, ..., r_n$.

3. Theorems.

In this section we present the main results of the paper.

Theorem 3.1. *Let us consider f_1, f_2 and g_1 be any three entire functions of several complex variables. Also let at least f_1 or f_2 is of regular relative growth with respect to g_1. Then*

$${}_{v_n}\lambda_{g_1}(f_1 \pm f_2) \leq \max\{ {}_{v_n}\lambda_{g_1}(f_1) , {}_{v_n}\lambda_{g_1}(f_2)\} .$$

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2018, VOLUME 8, ISSUE 2, p.129-139

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2018, VOLUME 8, ISSUE 2, p.129-139

The equality holds when ${}_{v_n}\lambda_{g_1}(f_i) > {}_{v_n}\lambda_{g_1}(f_j)$ *with at least* f_j *is of regular relative growth with respect to* g_1 *where* $i, j = 1, 2$ *and* $i \neq j$.

Proof. If ${}_{v_n}\lambda_{g_1}(f_1 \pm f_2) = 0$ then the result is obvious. So we suppose that ${}_{v_n}\lambda_{g_1}(f_1 \pm f_2) > 0$. We can clearly assume that ${}_{v_n}\lambda_{g_1}(f_k)$ is finite for $k = 1, 2$. Further let max $\{{}_{v_n}\lambda_{g_1}(f_1), {}_{v_n}\lambda_{g_1}(f_2)\} = \Delta$ and f_2 is of regular relative growth with respect to g_1.

Now for any arbitrary $\varepsilon > 0$ from the definition of ${}_{v_n}\lambda_{g_1}(f_1)$, we have for a sequence values of $r_1, r_2, ..., r_n$ tending to infinity that

$$M_{f_1}(r_1, r_2, ..., r_n) < M_{g_1}\left(r_1^{\left({}_{v_n}\lambda_{g_1}(f_1)+\varepsilon\right)}, r_2^{\left({}_{v_n}\lambda_{g_1}(f_1)+\varepsilon\right)}, ..., r_n^{\left({}_{v_n}\lambda_{g_1}(f_1)+\varepsilon\right)}\right)$$

$$\text{(3.1)} \qquad i.e.,\ M_{f_1}(r_1, r_2, ..., r_n) < M_{g_1}\left(r_1^{(\Delta+\varepsilon)}, r_2^{(\Delta+\varepsilon)}, ..., r_n^{(\Delta+\varepsilon)}\right).$$

Also for any arbitrary $\varepsilon > 0$ from the definition of ${}_{v_n}\rho_{g_1}(f_2) \left(= {}_{v_n}\lambda_{g_1}(f_2)\right)$, we obtain for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$M_{f_2}(r_1, r_2, ..., r_n) < M_{g_1}\left(r_1^{\left({}_{v_n}\lambda_{g_1}(f_2)+\varepsilon\right)}, r_2^{\left({}_{v_n}\lambda_{g_1}(f_2)+\varepsilon\right)}, ..., r_n^{\left({}_{v_n}\lambda_{g_1}(f_2)+\varepsilon\right)}\right)$$

$$\text{(3.2)} \qquad i.e.,\ M_{f_2}(r_1, r_2, ..., r_n) < M_{g_1}\left(r_1^{(\Delta+\varepsilon)}, r_2^{(\Delta+\varepsilon)}, ..., r_n^{(\Delta+\varepsilon)}\right).$$

Since $M_{f_1 \pm f_2}(r_1, r_2, ..., r_n) \leq M_{f_1}(r_1, r_2, ..., r_n) + M_{f_2}(r_1, r_2, ..., r_n)$ for sufficiently for large $r_1, r_2, ..., r_n$, we obtain from (3.1) and (3.2) for a sequence values of $r_1, r_2, ..., r_n$ tending to infinity that

$$\text{(3.3)} \qquad M_{f_1 \pm f_2}(r_1, r_2, ..., r_n) < 2M_{g_1}\left(r_1^{(\Delta+\varepsilon)}, r_2^{(\Delta+\varepsilon)}, ..., r_n^{(\Delta+\varepsilon)}\right).$$

Therefore in view of Lemma 2.1, we obtain from (3.3) for a sequence values of $r_1, r_2, ..., r_n$ tending to infinity that

$$\begin{aligned} M_{f_1 \pm f_2}(r_1, r_2, ..., r_n) &< M_{g_1}\left(3r_1^{(\Delta+\varepsilon)}, 3r_2^{(\Delta+\varepsilon)}, ..., 3r_n^{(\Delta+\varepsilon)}\right) \\ i.e.,\ M_{f_1 \pm f_2}(r_1, r_2, ..., r_n) &< M_{g_1}\left(r_1^{(\Delta+3\varepsilon)}, r_2^{(\Delta+3\varepsilon)}, ..., r_n^{(\Delta+3\varepsilon)}\right). \end{aligned}$$

Since $\varepsilon > 0$ are arbitrary, we get from above that

$${}_{v_n}\lambda_{g_1}(f_1 \pm f_2) \leq \Delta = \max\{{}_{v_n}\lambda_{g_1}(f_1), {}_{v_n}\lambda_{g_1}(f_2)\}.$$

Similarly, if we consider that f_1 is of regular relative growth with respect to g_1 or both f_1 and f_2 are of regular relative growth with respect to g_1, then one can easily verify that

$$\text{(3.4)} \qquad {}_{v_n}\lambda_{g_1}(f_1 \pm f_2) \leq \Delta = \max\{{}_{v_n}\lambda_{g_1}(f_1), {}_{v_n}\lambda_{g_1}(f_2)\}.$$

Now let ${}_{v_n}\lambda_{g_1}(f_1) > {}_{v_n}\lambda_{g_1}(f_2)$ and at least f_2 is of regular relative growth with respect to g_1. Also let $f = f_1 \pm f_2$. Then in view of (3.4) we get that ${}_{v_n}\lambda_{g_1}(f) \leq {}_{v_n}\lambda_{g_1}(f_1)$. As, $f_1 = (f \pm f_2)$ and in this case we obtain that ${}_{v_n}\lambda_{g_1}(f_1) \leq \max\{{}_{v_n}\lambda_{g_1}(f), {}_{v_n}\lambda_{g_1}(f_2)\}$.

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2018, VOLUME 8, ISSUE 2, p.129-139

As we assume that ${}_{v_n}\lambda_{g_1}(f_2) < {}_{v_n}\lambda_{g_1}(f_1)$, therefore we have ${}_{v_n}\lambda_{g_1}(f_1) \leq {}_{v_n}\lambda_{g_1}(f)$ and hence

$$ {}_{v_n}\lambda_{g_1}(f_1 \pm f_2) \geq {}_{v_n}\lambda_{g_1}(f_1) = \max\left\{ {}_{v_n}\lambda_{g_1}(f_1),\ {}_{v_n}\lambda_{g_1}(f_2)\right\}. $$

Further if we consider ${}_{v_n}\lambda_{g_1}(f_1) < {}_{v_n}\lambda_{g_1}(f_2)$ and at least f_1 is of regular relative growth with respect to g_1, then one can also verify that

$$ {}_{v_n}\lambda_{g_1}(f_1 \pm f_2) \geq \Delta = \max\left\{ {}_{v_n}\lambda_{g_1}(f_1),\ {}_{v_n}\lambda_{g_1}(f_2)\right\}. \tag{3.5} $$

So the conclusion of the second part of the theorem follows from (3.4) and (3.5). □

Now we state the following theorem due to Dutta [6] :

Theorem 3.2. [6] *Let us consider f_1, f_2 be any two entire functions of several complex variables with relative order ${}_{v_n}\rho_{g_1}(f_1)$ and ${}_{v_n}\rho_{g_1}(f_2)$ with respect to another entire function g_1 of several complex variables. Then*

$$ {}_{v_n}\rho_{g_1}(f_1 \pm f_2) \leq \max\left\{ {}_{v_n}\rho_{g_1}(f_1),\ {}_{v_n}\rho_{g_1}(f_2)\right\}. $$

The equality holds when ${}_{v_n}\rho_{g_1}(f_1) \neq {}_{v_n}\rho_{g_1}(f_2)$.

Theorem 3.3. *Let f_1, g_1 and g_2 be any three entire functions of several complex variables such that ${}_{v_n}\lambda_{g_1}(f_1)$ and ${}_{v_n}\lambda_{g_2}(f_1)$ exits. Then*

$$ {}_{v_n}\lambda_{g_1 \pm g_2}(f_1) \geq \min\left\{ {}_{v_n}\lambda_{g_1}(f_1),\ {}_{v_n}\lambda_{g_2}(f_1)\right\}. $$

The equality holds when ${}_{v_n}\lambda_{g_1}(f_1) \neq {}_{v_n}\lambda_{g_2}(f_1)$.

Proof. If ${}_{v_n}\lambda_{g_1 \pm g_2}(f_1) = \infty$, then the result is obvious. So we suppose that ${}_{v_n}\lambda_{g_1 \pm g_2}(f_1) < \infty$. We can clearly assume that ${}_{v_n}\lambda_{g_k}(f_1)$ is finite for $k = 1, 2$. Further let $\Psi = \min\left\{ {}_{v_n}\lambda_{g_1}(f_1),\ {}_{v_n}\lambda_{g_2}(f_1)\right\}$. Now for any arbitrary $\varepsilon > 0$ from the definition of ${}_{v_n}\lambda_{g_k}(f_1)$, we have for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$ M_{g_k}\left(r_1^{\left({}_{v_n}\lambda_{g_k}(f_1)-\varepsilon\right)}, r_2^{\left({}_{v_n}\lambda_{g_k}(f_1)-\varepsilon\right)}, ..., r_n^{\left({}_{v_n}\lambda_{g_k}(f_1)-\varepsilon\right)}\right) < M_{f_1}(r_1, r_2, ..., r_n) $$

where $k = 1, 2$.

Therefor from above we get for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$ M_{g_k}\left(r_1^{(\Psi-\varepsilon)}, r_2^{(\Psi-\varepsilon)}, ..., r_n^{(\Psi-\varepsilon)}\right) < M_{f_1}(r_1, r_2, ..., r_n) \quad \text{where } k = 1, 2. \tag{3.6} $$

Since $M_{g_1 \pm g_2}(r_1, r_2, ..., r_n) \leq M_{g_1}(r_1, r_2, ..., r_n) + M_{g_2}(r_1, r_2, ..., r_n)$ for sufficiently for large $r_1, r_2, ..., r_n$, we obtain from above and Lemma 2.1 for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$ M_{g_1 \pm g_2}\left(r_1^{(\Psi-\varepsilon)}, r_2^{(\Psi-\varepsilon)}, ..., r_n^{(\Psi-\varepsilon)}\right) < M_{g_1}\left[r_1^{(\Psi-\varepsilon)}, r_2^{(\Psi-\varepsilon)}, ..., r_n^{(\Psi-\varepsilon)}\right] + M_{g_2}\left[r_1^{(\Psi-\varepsilon)}, r_2^{(\Psi-\varepsilon)}, ..., r_n^{(\Psi-\varepsilon)}\right] $$

$$ i.e.,\ M_{g_1 \pm g_2}\left(r_1^{(\Psi-\varepsilon)}, r_2^{(\Psi-\varepsilon)}, ..., r_n^{(\Psi-\varepsilon)}\right) < 2M_{f_1}(r_1, r_2, ..., r_n) $$

$$i.e.,\ M_{g_1 \pm g_2}\left(\left(\frac{1}{3}\right) r_1^{(\Psi-\varepsilon)}, \left(\frac{1}{3}\right) r_2^{(\Psi-\varepsilon)}, ..., \left(\frac{1}{3}\right) r_n^{(\Psi-\varepsilon)}\right) < M_{f_1}(r_1, r_2, ..., r_n)$$

$$i.e.,\ M_{g_1 \pm g_2}\left(r_1^{(\Psi-3\varepsilon)}, r_2^{(\Psi-3\varepsilon)}, ..., r_n^{(\Psi-3\varepsilon)}\right) < M_{f_1}(r_1, r_2, ..., r_n)$$

Since $\varepsilon > 0$ are arbitrary, we get from above that

$$(3.7) \qquad {}_{v_n}\lambda_{g_1 \pm g_2}(f_1) \geq \Psi = \min\left\{ {}_{v_n}\lambda_{g_1}(f_1),\ {}_{v_n}\lambda_{g_2}(f_1)\right\}.$$

Now let ${}_{v_n}\lambda_{g_1}(f_1) < {}_{v_n}\lambda_{g_2}(f_1)$ and $g = g_1 \pm g_2$. Then in view of (3.7) we get that ${}_{v_n}\lambda_g(f_1) \geq {}_{v_n}\lambda_{g_1}(f_1)$. Further, $g_1 = (g \pm g_2)$ and in this case we obtain that ${}_{v_n}\lambda_{g_1}(f_1) \geq \min\{{}_{v_n}\lambda_g(f_1),\ {}_{v_n}\lambda_{g_2}(f_1)\}$. As we assume that ${}_{v_n}\lambda_{g_1}(f_1) < {}_{v_n}\lambda_{g_2}(f_1)$, therefore we have ${}_{v_n}\lambda_{g_1}(f_1) \geq {}_{v_n}\lambda_g(f_1)$ and hence

$${}_{v_n}\lambda_{g_1 \pm g_2}(f_1) \geq {}_{v_n}\lambda_{g_1}(f_1) = \min\left\{ {}_{v_n}\lambda_{g_1}(f_1),\ {}_{v_n}\lambda_{g_2}(f_1)\right\}.$$

Similarly, if we consider ${}_{v_n}\lambda_{g_1}(f_1) > {}_{v_n}\lambda_{g_2}(f_1)$, then one can also derive that

$$(3.8) \qquad {}_{v_n}\lambda_{g_1 \pm g_2}(f_1) \leq \Psi = \min\left\{ {}_{v_n}\lambda_{g_1}(f_1),\ {}_{v_n}\lambda_{g_2}(f_1)\right\}.$$

So the conclusion of the second part of the theorem follows from (3.7) and (3.8). □

Theorem 3.4. *Let f_1, g_1 and g_2 be any three entire functions of several complex variables such that ${}_{v_n}\rho_{g_1}(f_1)$ and ${}_{v_n}\rho_{g_2}(f_1)$ exits. Also let f_1 is of regular relative growth with respect to at least any one of g_1 or g_2. Then*

$${}_{v_n}\rho_{g_1 \pm g_2}(f_1) \geq \min\{{}_{v_n}\rho_{g_1}(f_1),\ {}_{v_n}\rho_{g_2}(f_1)\}.$$

The equality holds when ${}_{v_n}\rho_{g_i}(f_1) < {}_{v_n}\rho_{g_j}(f_1)$ with at least f_1 is of regular relative growth with respect to g_j where $i, j = 1, 2$ and $i \neq j$.

We omit the proof of Theorem 3.4 as it can easily be carried out in the line of Theorem 3.3.

Theorem 3.5. *Let f_1, f_2, g_1 and g_2 be any four entire functions of several complex variables. Then*

$${}_{v_n}\rho_{g_1 \pm g_2}(f_1 \pm f_2) \leq$$
$$\max\left[\min\{{}_{v_n}\rho_{g_1}(f_1),\ {}_{v_n}\rho_{g_2}(f_1)\}, \min\{{}_{v_n}\rho_{g_1}(f_2),\ {}_{v_n}\rho_{g_2}(f_2)\}\right]$$

when the following two conditions holds:
(i) ${}_{v_n}\rho_{g_i}(f_1) < {}_{v_n}\rho_{g_j}(f_1)$ with at least f_1 is of regular relative growth with respect to g_j for $i = 1, 2$, $j = 1, 2$ and $i \neq j$; and
(ii) ${}_{v_n}\rho_{g_i}(f_2) < {}_{v_n}\rho_{g_j}(f_2)$ with at least f_2 is of regular relative growth with respect to g_j for $i = 1, 2$, $j = 1, 2$ and $i \neq j$.
The equality holds when ${}_{v_n}\rho_{g_1}(f_i) < {}_{v_n}\rho_{g_1}(f_j)$ and ${}_{v_n}\rho_{g_2}(f_i) < {}_{v_n}\rho_{g_2}(f_j)$ holds simultaneously for $i = 1, 2$; $j = 1, 2$ and $i \neq j$.

Proof. Let the conditions (i) and (ii) of the theorem hold. Therefore in view of Theorem 3.2 and Theorem 3.4 we get that

$$\begin{aligned} & \max\left[\min\left\{ {}_{v_n}\rho_{g_1}(f_1), {}_{v_n}\rho_{g_2}(f_1)\right\}, \min\left\{ {}_{v_n}\rho_{g_1}(f_2), {}_{v_n}\rho_{g_2}(f_2)\right\}\right] \\ & = \max\left[{}_{v_n}\rho_{g_1\pm g_2}(f_1), {}_{v_n}\rho_{g_1\pm g_2}(f_2)\right] \\ & \geq {}_{v_n}\rho_{g_1\pm g_2}(f_1 \pm f_2). \end{aligned} \tag{3.9}$$

Since ${}_{v_n}\rho_{g_1}(f_i) < {}_{v_n}\rho_{g_1}(f_j)$ and ${}_{v_n}\rho_{g_2}(f_i) < {}_{v_n}\rho_{g_2}(f_j)$ hold simultaneously for $i = 1, 2;\ j = 1, 2$ and $i \neq j$, we obtain that

$$\text{either } \min\left\{ {}_{v_n}\rho_{g_1}(f_1), {}_{v_n}\rho_{g_2}(f_1)\right\} > \min\left\{ {}_{v_n}\rho_{g_1}(f_2), {}_{v_n}\rho_{g_2}(f_2)\right\} \text{ or}$$
$$\min\left\{ {}_{v_n}\rho_{g_1}(f_2), {}_{v_n}\rho_{g_2}(f_2)\right\} > \min\left\{ {}_{v_n}\rho_{g_1}(f_1), {}_{v_n}\rho_{g_2}(f_1)\right\} \text{ holds.}$$

Now in view of the conditions (i) and (ii) of the theorem, it follows from above that

$$\text{either } {}_{v_n}\rho_{g_1\pm g_2}(f_1) > {}_{v_n}\rho_{g_1\pm g_2}(f_2) \text{ or } {}_{v_n}\rho_{g_1\pm g_2}(f_2) > {}_{v_n}\rho_{g_1\pm g_2}(f_1)$$

which is the condition for holding equality in (3.9).

Hence the theorem follows. □

Theorem 3.6. *Let f_1, f_2, g_1 and g_2 be any four entire functions of several complex variables. Then*

$$\begin{aligned} & {}_{v_n}\lambda_{g_1\pm g_2}(f_1 \pm f_2) \geq \\ & \min\left[\max\left\{ {}_{v_n}\lambda_{g_1}(f_1), {}_{v_n}\lambda_{g_1}(f_2)\right\}, \max\left\{ {}_{v_n}\lambda_{g_2}(f_1), {}_{v_n}\lambda_{g_2}(f_2)\right\}\right] \end{aligned}$$

when the following two conditions holds:
(i) ${}_{v_n}\lambda_{g_1}(f_i) > {}_{v_n}\lambda_{g_1}(f_j)$ with at least f_j is of regular relative growth with respect to g_1 for $i = 1, 2,\ j = 1, 2$ and $i \neq j$; and
(ii) ${}_{v_n}\lambda_{g_2}(f_i) > {}_{v_n}\lambda_{g_2}(f_j)$ with at least f_j is of regular relative growth with respect to g_2 for $i = 1, 2,\ j = 1, 2$ and $i \neq j$.
The equality holds when ${}_{v_n}\lambda_{g_i}(f_1) < {}_{v_n}\lambda_{g_j}(f_1)$ and ${}_{v_n}\lambda_{g_i}(f_2) < {}_{v_n}\lambda_{g_j}(f_2)$ hold simultaneously for $i = 1, 2;\ j = 1, 2$ and $i \neq j$.

Proof. Suppose that the conditions (i) and (ii) of the theorem holds. Therefore in view of Theorem 3.1 and Theorem 3.3, we obtain that

$$\begin{aligned} & \min\left[\max\left\{ {}_{v_n}\lambda_{g_1}(f_1), {}_{v_n}\lambda_{g_1}(f_2)\right\}, \max\left\{ {}_{v_n}\lambda_{g_2}(f_1), {}_{v_n}\lambda_{g_2}(f_2)\right\}\right] \\ & = \min\left[{}_{v_n}\lambda_{g_1}(f_1 \pm f_2), {}_{v_n}\lambda_{g_2}(f_1 \pm f_2)\right] \\ & \geq {}_{v_n}\lambda_{g_1\pm g_2}(f_1 \pm f_2). \end{aligned} \tag{3.10}$$

Since ${}_{v_n}\lambda_{g_i}(f_1) < {}_{v_n}\lambda_{g_j}(f_1)$ and ${}_{v_n}\lambda_{g_i}(f_2) < {}_{v_n}\lambda_{g_j}(f_2)$ holds simultaneously for $i = 1, 2;\ j = 1, 2$ and $i \neq j$, we get that

$$\text{either } \max\left\{ {}_{v_n}\lambda_{g_1}(f_1), {}_{v_n}\lambda_{g_1}(f_2)\right\} < \max\left\{ {}_{v_n}\lambda_{g_2}(f_1), {}_{v_n}\lambda_{g_2}(f_2)\right\} \text{ or}$$
$$\max\left\{ {}_{v_n}\lambda_{g_2}(f_1), {}_{v_n}\lambda_{g_2}(f_2)\right\} < \max\left\{ {}_{v_n}\lambda_{g_1}(f_1), {}_{v_n}\lambda_{g_1}(f_2)\right\} \text{ holds.}$$

Since condition (i) and (ii) of the theorem holds, it follows from above that

$$\text{either } {}_{v_n}\lambda_{g_1}(f_1 \pm f_2) < {}_{v_n}\lambda_{g_2}(f_1 \pm f_2) \text{ or } {}_{v_n}\lambda_{g_2}(f_1 \pm f_2) < {}_{v_n}\lambda_{g_1}(f_1 \pm f_2)$$

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2018, VOLUME 8, ISSUE 2, p.129-139

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2018, VOLUME 8, ISSUE 2, p.129-139

which is the condition for holding equality in (3.10).

Hence the theorem follows. □

Now we state the following two remarks which are immediately follows from our previous discussion:

Remark 3.7. *Let f_1, f_2 and g_1 be any three entire functions of several complex variables. Also let both f_1 and f_2 are of regular relative growth with respect to g_1 with ${}_{v_n}\rho_{g_1}(f_1) \neq {}_{v_n}\rho_{g_1}(f_2)$. Then*

$$ {}_{v_n}\lambda_{g_1}(f_1 \pm f_2) = {}_{v_n}\rho_{g_1}(f_1 \pm f_2) = \max\left\{{}_{v_n}\rho_{g_1}(f_1),\ {}_{v_n}\rho_{g_1}(f_2)\right\}. $$

Remark 3.8. *Let f_1, g_1 and g_2 be any three entire functions of several complex variables. Also let f_1 is of regular relative growth with respect to both of g_1 and g_2 with ${}_{v_n}\rho_{g_1}(f_1) \neq {}_{v_n}\rho_{g_2}(f_1)$. Then*

$$ {}_{v_n}\lambda_{g_1 \pm g_2}(f_1) = {}_{v_n}\rho_{g_1 \pm g_2}(f_1) = \min\left\{{}_{v_n}\rho_{g_1}(f_1),\ {}_{v_n}\rho_{g_2}(f_1)\right\}. $$

Theorem 3.9. *Let f_1, f_2 and g_1 be any three entire functions of several complex variables. Also let at least f_1 or f_2 is of regular relative growth with respect to g_1. Then*

$$ {}_{v_n}\lambda_{g_1}(f_1 \cdot f_2) \leq \max\left\{ {}_{v_n}\lambda_{g_1}(f_1),\ {}_{v_n}\lambda_{g_1}(f_2)\right\} $$

provided g_1 has the Property (R). The equality holds when f_1 and f_2 satisfy Property (X).

Proof. Suppose that ${}_{v_n}\lambda_{g_1}(f_1 \cdot f_2) > 0$. Otherwise if ${}_{v_n}\lambda_{g_1}(f_1 \cdot f_2) = 0$ then the result is obvious. Let us consider that f_2 is of regular relative growth with respect to g_1. Also suppose that $\max\left\{{}_{v_n}\lambda_{g_1}(f_1), {}_{v_n}\lambda_{g_1}(f_2)\right\} = \Delta$. We can clearly assume that ${}_{v_n}\lambda_{g_1}(f_k)$ is finite for $k = 1, 2$. Since $M_{f_1 \cdot f_2}(r_1, r_2, ..., r_n) < M_{f_1}(r_1, r_2, ..., r_n) \cdot M_{f_2}(r_1, r_2, ..., r_n)$ for all large $r_1, r_2, ..., r_n$, we have from (3.1), (3.2) for a sequence values of $r_1, r_2, ..., r_n$ tending to infinity that

$$ M_{f_1 \cdot f_2}(r_1, r_2, ..., r_n) < \left[M_{g_1}\left(r_1^{(\Delta+\varepsilon)}, r_2^{(\Delta+\varepsilon)}, ..., r_n^{(\Delta+\varepsilon)}\right)\right]^2. $$

Also in view of Definition 1.5, we obtain from above for any $\delta > 1$ and for a sequence values of $r_1, r_2, ..., r_n$ tending to infinity that

$$ M_{f_1 \cdot f_2}(r_1, r_2, ..., r_n) < M_{g_1}\left(r_1^{\delta(\Delta+\varepsilon)}, r_2^{\delta(\Delta+\varepsilon)}, ..., r_n^{\delta(\Delta+\varepsilon)}\right), $$

since g_1 has the Property (R). Since $\varepsilon > 0$ is arbitrary, now letting $\delta \to 1^+$, we get from above that

$$ {}_{v_n}\lambda_{g_1}(f_1 \cdot f_2) \leq \Delta = \max\left\{{}_{v_n}\lambda_{g_1}(f_1),\ {}_{v_n}\lambda_{g_1}(f_2)\right\}. $$

Similarly, if we consider that f_1 is of regular relative growth with respect to g_1 or both f_1 and f_2 are of regular relative growth with respect to g_1, then also one can easily verify that

$$ {}_{v_n}\lambda_{g_1}(f_1 \cdot f_2) \leq \Delta = \max\left\{ {}_{v_n}\lambda_{g_1}(f_1),\ {}_{v_n}\lambda_{g_1}(f_2)\right\}. \tag{3.11} $$

Now let f_1 and f_2 are satisfy Property (X), then of course we have $M_{f_1 \cdot f_2}(r_1, r_2, ..., r_n) > M_{f_1}(r_1, r_2, ..., r_n)$ and $M_{f_1 \cdot f_2}(r_1, r_2, ..., r_n) > M_{f_2}(r_1, r_2, ..., r_n)$ for all sufficiently large

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2018, VOLUME 8, ISSUE 2, p.129-139

values of $r_1, r_2, ..., r_n$. Therefore from the definition of relative lower order, we get for a sequence values of $r_1, r_2, ..., r_n$ tending to infinity that

$$M_{f_1}(r_1, r_2, ..., r_n) < M_{f_1 \cdot f_2}(r_1, r_2, ..., r_n)$$
$$\leq M_{g_1}\left(r_1^{\left({}_{v_n}\lambda_{g_1}(f_1 \cdot f_2)+\varepsilon\right)}, r_2^{\left({}_{v_n}\lambda_{g_1}(f_1 \cdot f_2)+\varepsilon\right)}, ..., r_n^{\left({}_{v_n}\lambda_{g_1}(f_1 \cdot f_2)+\varepsilon\right)}\right).$$

Since $\varepsilon > 0$ are arbitrary, we get from above that ${}_{v_n}\lambda_{g_1}(f_1 \cdot f_2) \geq {}_{v_n}\lambda_{g_1}(f_1)$. Similarly ${}_{v_n}\lambda_{g_1}(f_1 \cdot f_2) \geq {}_{v_n}\lambda_{g_1}(f_2)$ and therefore

$$(3.12) \qquad {}_{v_n}\lambda_{g_1}(f_1 \cdot f_2) \geq \Delta = \max\{{}_{v_n}\lambda_{g_1}(f_1), {}_{v_n}\lambda_{g_1}(f_2)\}.$$

Hence the theorem follows from (3.11) and (3.12). □

Remark 3.10. *In Theorem 4.1 (ii) of* [6], *Dutta* [6] *said nothing about the condition of equality but the equality of Theorem 4.1 (ii) of* [6] *holds when f_1 and f_2 are satisfying the Property (X) which can easily be derived in the line of Theorem 3.9.*

Theorem 3.11. *Let f_1, g_1 and g_2 be any three entire functions of several complex variables. Also let $\lambda_{g_1}(f_1)$ and $\lambda_{g_2}(f_1)$ exists. Then*

$${}_{v_n}\lambda_{g_1 \cdot g_2}(f_1) \geq \min\{{}_{v_n}\lambda_{g_1}(f_1), {}_{v_n}\lambda_{g_2}(f_1)\}$$

provided $g_1 \cdot g_2$ has the Property (R). The equality holds when g_1 and g_2 satisfy Property (X).

Proof. Suppose that ${}_{v_n}\lambda_{g_1 \cdot g_2}(f_1) < \infty$. Otherwise if ${}_{v_n}\lambda_{g_1 \cdot g_2}(f_1) = \infty$ then the result is obvious. Also suppose that $\min\{{}_{v_n}\lambda_{g_1}(f_1), {}_{v_n}\lambda_{g_2}(f_1)\} = \Psi$. We can clearly assume that ${}_{v_n}\lambda_{g_k}(f_1)$ is finite for $k = 1, 2$.

As $M_{g_1 \cdot g_2}(r_1, r_2, ..., r_n) < M_{g_1}(r_1, r_2, ..., r_n) \cdot M_{g_2}(r_1, r_2, ..., r_n)$ for all large $r_1, r_2, ..., r_n$, we get in view of (3.6) for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$M_{g_1 \cdot g_2}\left(r_1^{(\Psi-\varepsilon)}, r_2^{(\Psi-\varepsilon)}, ..., r_n^{(\Psi-\varepsilon)}\right) < [M_{f_1}(r_1, r_2, ..., r_n)]^2$$

$$\left[M_{g_1 \cdot g_2}\left(r_1^{(\Psi-\varepsilon)}, r_2^{(\Psi-\varepsilon)}, ..., r_n^{(\Psi-\varepsilon)}\right)\right]^{\frac{1}{2}} < M_{f_1}(r_1, r_2, ..., r_n).$$

Now in view of Definition 1.5 we obtain from above for any $\delta > 1$ and for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$M_{g_1 \cdot g_2}\left(r_1^{\frac{(\Psi-\varepsilon)}{\delta}}, r_2^{\frac{(\Psi-\varepsilon)}{\delta}}, ..., r_n^{\frac{(\Psi-\varepsilon)}{\delta}}\right) < M_{f_1}(r_1, r_2, ..., r_n)$$

since $g_1 \cdot g_2$ has the Property (R). Since $\varepsilon > 0$ is arbitrary, now letting $\delta \to 1^+$, we obtain from above that

$$(3.13) \qquad {}_{v_n}\lambda_{g_1 \cdot g_2}(f_1) \geq \Psi = \min\{{}_{v_n}\lambda_{g_1}(f_1), {}_{v_n}\lambda_{g_2}(f_1)\}.$$

Now let g_1 and g_2 are satisfy Property (X), then of course we have $M_{g_1 \cdot g_2}(r_1, r_2, ..., r_n) > M_{g_1}(r_1, r_2, ..., r_n)$ and $M_{g_1 \cdot g_2}(r_1, r_2, ..., r_n) > M_{g_2}(r_1, r_2, ..., r_n)$ for all sufficiently large values of $r_1, r_2, ..., r_n$. Therefore from the definition of relative lower order, we get for all

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2018, VOLUME 8, ISSUE 2, p.129-139

sufficiently large values of $r_1, r_2, ..., r_n$ that

$$M_{g_1}\left(r_1^{\left({}_{v_n}\lambda_{g_1\cdot g_2}(f_1)-\varepsilon\right)}, r_2^{\left({}_{v_n}\lambda_{g_1\cdot g_2}(f_1)-\varepsilon\right)}, ..., r_n^{\left({}_{v_n}\lambda_{g_1\cdot g_2}(f_1)-\varepsilon\right)}\right) <$$

$$M_{g_1\cdot g_2}\left(r_1^{\left({}_{v_n}\lambda_{g_1\cdot g_2}(f_1)-\varepsilon\right)}, r_2^{\left({}_{v_n}\lambda_{g_1\cdot g_2}(f_1)-\varepsilon\right)}, ..., r_n^{\left({}_{v_n}\lambda_{g_1\cdot g_2}(f_1)-\varepsilon\right)}\right) \leq M_{f_1}(r_1, r_2, ..., r_n).$$

Since $\varepsilon > 0$ are arbitrary, we get from above that ${}_{v_n}\lambda_{g_1}(f_1) \geq {}_{v_n}\lambda_{g_1\cdot g_2}(f_1)$. Similarly ${}_{v_n}\lambda_{g_2}(f_1) \geq {}_{v_n}\lambda_{g_1\cdot g_2}(f_1)$ and therefore

$$(3.14) \qquad {}_{v_n}\lambda_{g_1\cdot g_2}(f_1) \leq \Psi = \min\left\{{}_{v_n}\lambda_{g_1}(f_1), {}_{v_n}\lambda_{g_2}(f_1)\right\}.$$

Hence the theorem follows from (3.13) and (3.14). □

Theorem 3.12. *Let f_1, g_1 and g_2 be any three entire functions of several complex variables. Also let f_1 is of regular relative growth with respect to at least any one of g_1 or g_2. Then*

$$ {}_{v_n}\rho_{g_1\cdot g_2}(f_1) \geq \min\left\{{}_{v_n}\rho_{g_1}(f_1), {}_{v_n}\rho_{g_2}(f_1)\right\}$$

provided $g_1 \cdot g_2$ has the Property (R). The equality holds when g_1 and g_2 satisfy Property (X).

We omit the proof of Theorem 3.12 as it can easily be carried out in the line of Theorem 3.11.

Now we state the following two theorems without their proofs as those can easily be carried out with the help of Remark 3.10, Theorem 3.9, Theorem 3.11 and Theorem 3.12 and
in the line of Theorem 3.5 and Theorem 3.6 respectively.

Theorem 3.13. *Let f_1, f_2, g_1 and g_2 be any four entire functions of several complex variables. Also let $g_1 \cdot g_2$ be satisfy the Property (R). Then,*

$$ {}_{v_n}\rho_{g_1\cdot g_2}(f_1\cdot f_2) = \max\left[\min\left\{{}_{v_n}\rho_{g_1}(f_1), {}_{v_n}\rho_{g_2}(f_1)\right\}, \min\left\{{}_{v_n}\rho_{g_1}(f_2), {}_{v_n}\rho_{g_2}(f_2)\right\}\right],$$

when the following four conditions holds:
(i) f_1 is of regular relative growth with respect to at least any one of g_1 or g_2;
(ii) f_2 is of regular relative growth with respect to at least any one of g_1 or g_2;
(iii) f_1 and f_2 satisfy Property (X); and
(iv) g_1 and g_2 satisfy Property (X).

Theorem 3.14. *Let f_1, f_2, g_1 and g_2 be any four entire functions of several complex variables. Also let $g_1 \cdot g_2$, g_1 and g_2 be satisfy the Property (R). Then,*

$$ {}_{v_n}\lambda_{g_1\cdot g_2}(f_1\cdot f_2) = \min\left[\max\left\{{}_{v_n}\lambda_{g_1}(f_1), {}_{v_n}\lambda_{g_1}(f_2)\right\}, \max\left\{{}_{v_n}\lambda_{g_2}(f_1), {}_{v_n}\lambda_{g_2}(f_2)\right\}\right],$$

when the following four conditions holds:
(i) At least f_1 or f_2 is of regular relative growth with respect to g_1;
(ii) At least f_1 or f_2 is of regular relative growth with respect to g_2;

(iii) f_1 *and* f_2 *satisfy Property (X); and*
(iv) g_1 *and* g_2 *satisfy Property (X).*

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