FUNCTIONAL DIFFERENTIAL EQUATIONS WITH STATE-DEPENDENT DELAY AND RANDOM EFFECTS

AMEL BENAISSA AND MOUFFAK BENCHOHRA

ABSTRACT. In this work we study the existence of mild solutions of a functional differential equation with delay and random effects. We use a random fixed point theorem with stochastic domain to show the existence of mild random solutions.

1. INTRODUCTION

Functional evolution equations with state-dependent delay appear frequently in mathematical modeling of several real world problems and for this reason the study of this type of equations has received great attention in the last few years, see for instance [8, 16, 17]. An extensive theory is developed for evolution equations [2, 10]. Uniqueness and existence results have been established recently for different evolution problems in the papers by Baghli and Benchohra for finite and infinite delay in [3, 4, 5]. On the other hand, the nature of a dynamic system in engineering or natural sciences depends on the accuracy of the information we have concerning the parameters that describe that system. If the knowledge about a dynamic system is precise then a deterministic dynamical system arises. Unfortunately in most cases the available data for the description and evaluation of parameters of a dynamic system are inaccurate, imprecise or confusing. In other words, evaluation of parameters of a dynamical system is not without uncertainties. When our knowledge about the parameters of a dynamic system are of statistical nature, that is, the information is probabilistic, the common approach in mathematical modeling of such systems is the use of random differential equations or stochastic differential equations. Random differential equations, as natural extensions of deterministic ones, arise in many applications and have been investigated by many authors; see [19, 20, 21, 25, 26] and references therein. Between them differential equations with random coefficients (see, [7, 25]) offer a natural and rational approach (see [24], Chapter 1), since sometimes we can get the random distributions of some main disturbances by historical experiences and data rather than take all random disturbances into account and assume the noise to be white noises.

In this work we prove the existence of mild solutions of the following functional differential equation with delay and random effects (random parameters) of the form:

$$(1.1) \qquad y'(t,w) = Ay(t,w) + f(t, y_{\rho(t,y_t)}(\cdot, w), w), \quad \text{a.e. } t \in J := [0, T]$$

$$(1.2) \qquad y(t,w) = \phi(t,w), \quad t \in (-\infty, 0],$$

where (Ω, F, P) is a complete probability space, $w \in \Omega$, $f : J \times \mathcal{B} \times \Omega \to E$, $\phi \in \mathcal{B} \times \Omega$ are given random functions which represent random nonlinear of the system, $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $T(t)$, $t \in J$, of bounded linear operators in a Banach space $(E, | \cdot |)$, $\mathcal{B}$ is a phase space to be specified later, $\rho : J \times \mathcal{B} \to (-\infty, +\infty)$, and $(E, |.|)$ is a real Banach space. For any function y defined on $(-\infty, T] \times \Omega$ and any $t \in J$ we denote by $y_t(\cdot, w)$ the element of $\mathcal{B} \times \Omega$ defined by $y_t(\theta, w) = y(t + \theta, w), \theta \in (-\infty, 0]$. Here $y_t(\cdot, w)$ represents the history of the state from time $-\infty$, up to the present time t. We assume that the histories $y_t(\cdot, w)$ belong to the abstract phase $\mathcal{B}$. To our knowledge, the literature on the local existence of random evolution equations with delay is very limited, so the present paper can be considered as a contribution to this question.

2. Preliminaries

We introduce notations, definitions and theorems which are used throughout this paper.

Let $C(J, E)$ be the Banach space of continuous functions from J into E with the norm

$$\|y\|_\infty = \sup \{\ |y(t)|\ :\ t \in J\ \}.$$

Let $B(E)$ denote the Banach space of bounded linear operators from E into E.

A measurable function $y : J \to E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. (For the Bochner integral properties, see the classical monograph of Yosida [27]).

Let $L^1(J, E)$ denote the Banach space of measurable functions $y : J \to E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^T |y(t)|\ dt.$$

Definition 2.1. A map $f : J \times \mathcal{B} \times \Omega \to E$ is said to be Carathéodory if:

   (i) $t \to f(t, y, w)$ is measurable for all $y \in \mathcal{B}$.and for all $w \in \Omega$.
   (ii) $y \to f(t, y, w)$ is continuous for almost each $t \in J$. and for all $w \in \Omega$.
   (iii) $w \to f(t, y, w)$ is measurable for all $y \in \mathcal{B}$, and almost each $t \in J$.

For a given set V of functions $v : (-\infty, T] \to E$, let us denote by

$$V(t) = \{v(t) : v \in V\},\ t \in (-\infty, T]$$

and

$$V(J) = \{v(t) : v \in V,\ t \in (-\infty, T]\}.$$

In this paper, we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [14] and follow the terminology used in [18]. Thus, $(\mathcal{B}, \| \cdot \|_{\mathcal{B}})$ will be a seminormed linear space of functions mapping $(-\infty, 0]$ into E, and satisfying the following axioms :

(A_1) If $y : (-\infty, T) \to E, T > 0$, is continuous on J and $y_0 \in \mathcal{B}$, then for every $t \in J$ the following conditions hold :
   (i) $y_t \in \mathcal{B}$;
   (ii) There exists a positive constant H such that $|y(t)| \leq H\|y_t\|_{\mathcal{B}}$;
   (iii) There exist two functions $K(\cdot), M(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ independent of y with K continuous and M locally bounded such that :

$$\|y_t\|_{\mathcal{B}} \leq K(t) \sup\{\ |y(s)| : 0 \leq s \leq t\} + M(t)\|y_0\|_{\mathcal{B}}.$$

(A_2) For the function y in (A_1), y_t is a $\mathcal{B}-$valued continuous function on J.
(A_3) The space $\mathcal{B}$ is complete.

Denote

$$K_T = \sup\{K(t) : t \in J\},\ M_T = \sup\{M(t) : t \in J\}.$$

Remark 2.2.   1. (ii) is equivalent to $|\phi(0)| \leq H\|\phi\|_{\mathcal{B}}$ for every $\phi \in \mathcal{B}$.

2. Since $\| \cdot \|_{\mathcal{B}}$ is a seminorm, two elements $\phi, \psi \in \mathcal{B}$ can verify $\|\phi - \psi\|_{\mathcal{B}} = 0$ without necessarily $\phi(\theta) = \psi(\theta)$ for all $\theta \leq 0$.

3. From the equivalence of in the first remark, we can see that for all $\phi, \psi \in \mathcal{B}$ such that $\|\phi - \psi\|_{\mathcal{B}} = 0$: We necessarily have that $\phi(0) = \psi(0)$.

We now indicate some examples of phase spaces. For other details we refer, for instance to the book by Hino *et al.* [18].

Example 2.3. Let:

BC the space of bounded continuous functions defined from $(-\infty, 0]$ to E;

BUC the space of bounded uniformly continuous functions defined from $(-\infty, 0]$ to E;

$$C^{\infty} := \left\{ \phi \in BC : \lim_{\theta \to -\infty} \phi(\theta) \text{ exist in } E \right\};$$

$$C^{0} := \left\{ \phi \in BC : \lim_{\theta \to -\infty} \phi(\theta) = 0 \right\}, \text{ endowed with the uniform norm}$$

$$\|\phi\| = \sup\{|\phi(\theta)| : \theta \leq 0\}.$$

We have that the spaces BUC, C^{∞} and C^{0} satisfy conditions $(A_1) - (A_3)$. However, BC satisfies $(A_1), (A_3)$ but (A_2) is not satisfied.

Example 2.4. The spaces C_g, UC_g, C_g^{∞} and C_g^{0}.

Let g be a positive continuous function on $(-\infty, 0]$. We define:

$$C_g := \left\{ \phi \in C((-\infty, 0], E) : \frac{\phi(\theta)}{g(\theta)} \text{ is bounded on } (-\infty, 0] \right\};$$

$$C_g^{0} := \left\{ \phi \in C_g : \lim_{\theta \to -\infty} \frac{\phi(\theta)}{g(\theta)} = 0 \right\}, \text{ endowed with the uniform norm}$$

$$\|\phi\| = \sup\left\{ \frac{|\phi(\theta)|}{g(\theta)} : \theta \leq 0 \right\}.$$

Then we have that the spaces C_g and C_g^{0} satisfy conditions (A_3). We consider the following condition on the function g.

(g_1) For all $a > 0$, $\displaystyle\sup_{0 \leq t \leq a} \sup \left\{ \frac{g(t+\theta)}{g(\theta)} : -\infty < \theta \leq -t \right\} < \infty.$

They satisfy conditions (A_1) and (A_2) if (g_1) holds.

Example 2.5. The space C_{γ}.

For any real positive constant γ, we define the functional space C_{γ} by

$$C_{\gamma} := \left\{ \phi \in C((-\infty, 0], E) : \lim_{\theta \to -\infty} e^{\gamma\theta} \phi(\theta) \text{ exists in } E \right\}$$

endowed with the following norm

$$\|\phi\| = \sup\{e^{\gamma\theta}|\phi(\theta)| : \theta \leq 0\}.$$

Then in the space C_{γ} the axioms $(A_1) - (A_3)$ are satisfied.

Let Y be a separable Banach space with the Borel σ-algebra B_Y. A mapping $y : \Omega \longrightarrow Y$ is said to be a random variable with values in Y if for each $B \in B_Y, y^{-1}(B) \in F$. A mapping $T : \Omega \times Y \longrightarrow Y$ is called a random operator if $T(., y)$ is measurable for each $y \in Y$ and is generally expressed as $T(w, y) = T(w)y$; we will use these two expressions alternatively. Next, we will give a very useful random fixed point theorem with stochastic domain.

Definition 2.6. [9] Let C be a mapping from Ω into 2^Y. A mapping $T : \{(w, y) : w \in \Omega \wedge y \in C(w)\} \longrightarrow Y$ is called random operator with stochastic domain C if C is measurable (i.e., for all closed $A \subseteq Y, \{w \in \Omega : C(w) \cap A \neq \emptyset\} \in F)$ and for all open $D \subseteq Y$ and all $y \in Y, \{w \in \Omega : y \in C(w) \wedge T(w, y) \in D\} \in F$. T will be called continuous if every $T(w)$ is continuous. For a random operator T, a mapping $y : \Omega \longrightarrow Y$ is called 'random (stochastic) fixed point of T' iff for p-almost all $w \in \Omega, y(w) \in C(w)$ and $T(w)y(w) = y(w)$ and for all open $D \subseteq Y, \{w \in \Omega : y(w) \in D\} \in F$('y is measurable').

Remark 2.7. If $C(w) \equiv Y$, then the definition of random operator with stochastic domain coincides with the definition of random operator.

Lemma 2.8. [9] *Let $C : \Omega \longrightarrow 2^Y$ be measurable with $C(w)$ closed, convex and solid (i.e., int $C(w) \neq \emptyset$) for all $w \in \Omega$. We assume that there exists measurable $y_0 : \Omega \longrightarrow Y$ with $y_0 \in$ int $C(w)$ for all $w \in \Omega$. Let T be a continuous random operator with stochastic domain C such that for every $w \in \Omega, \{y \in C(w) : T(w)y = y\} \neq \emptyset$. Then T has a stochastic fixed point.*

Let y be a mapping of $J \times \Omega$ into X. y is said to be a stochastic process if for each $t \in J$, the function $y(t, \cdot)$ is measurable.

Now let us recall some fundamental facts of the notion of Kuratowski measure of noncompactness.

Definition 2.9. [6] Let E be a Banach space and Ω_E the bounded subsets of E. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_E \to [0, \infty)$ defined by

$$\alpha(B) = \inf\{\epsilon > 0 : B \subseteq \cup_{i=1}^n B_i \text{ and } diam(B_i) \leq \epsilon\}; \text{ here } B \in \Omega_E.$$

The Kuratowski measure of noncompactness satisfies the following properties (for more details see [6]).

(a) $\alpha(B) = 0 \Longleftrightarrow \overline{B}$ is compact (B is relatively compact).
(b) $\alpha(B) = \alpha(\overline{B})$.
(c) $A \subset B \Longrightarrow \alpha(A) \leq \alpha(B)$.
(d) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$.
(e) $\alpha(cB) = |c|\alpha(B); c \in \mathbb{R}$
(f) $\alpha(convB) = \alpha(B)$.

Theorem 2.10. *(Mönch)[[1, 22]] Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication*

$$V = \overline{conv}N(V) \quad or \quad V = N(V) \cup 0 \Longrightarrow \alpha(V) = 0$$

holds for every subset V of D, then N has a fixed point.

Lemma 2.11 ([11]). *If $H \subset C(J, E)$ is bounded and equicontinuous, then $\alpha(H(t))$ is continuous on J and*

$$\alpha\left(\left\{\int_J x(s)ds : x \in H\right\}\right) \leq \int_J \alpha(H(s))ds,$$

where $H(s) = \{x(s) : x \in H\}, t \in J$

Lemma 2.12 ([11]). *Let D be a bounded, closed and convex subset of Banach space X. If the operator $N : D \to D$ is a strict set contraction, i.e there is a constant $0 \leq \lambda < 1$ such that $\alpha(N(S)) \leq \lambda\alpha(S)$ for any bounded set $S \subset D$, then N has a fixed point in D.*

3. EXISTENCE OF MILD SOLUTIONS

Now we give our main existence result for problem (1.1)-(1.2). Before starting and proving this result, we give the definition of the mild random solution.

Definition 3.1. A stochastic process $y : J \times \Omega \to E$ is said to be random mild solution of problem (1.1)-(1.2) if $y(t, w) = \phi(t)$, $t \in (-\infty, 0]$ and the restriction of $y(., w)$ to the interval J is continuous and satisfies the following integral equation:

(3.1) $$y(t, w) = T(t)\phi(0, w) + \int_0^t T(t - s)f(s, y_{\rho(s, y_s)}(\cdot, w), w)ds, \ t \in J.$$

Set
$$\mathcal{R}(\rho^-) = \{\rho(s, \varphi) : (s, \varphi) \in J \times \mathcal{B}, \rho(s, \varphi) \leq 0\}.$$

We always assume that $\rho : J \times \mathcal{B} \to (-\infty, T]$ is continuous. Additionally, we introduce following hypothesis:

(H_ϕ) The function $t \to \phi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $L^\phi : \mathcal{R}(\rho^-) \to (0, \infty)$ such that
$$\|\phi_t\|_\mathcal{B} \leq L^\phi(t)\|\phi\|_\mathcal{B} \quad \text{for every } t \in \mathcal{R}(\rho^-).$$

Remark 3.2. The condition (H_ϕ), is frequently verified by functions continuous and bounded. For more details, see for instance [18].

Lemma 3.3. ([15], Lemma 2.4) If $y : (-\infty, T] \to E$ is a function such that $y_0 = \phi$, then
$$\|y_s\|_\mathcal{B} \leq (M_T + L^\phi)\|\phi\|_\mathcal{B} + K_T \sup\{|y(\theta)|; \theta \in [0, max\{0, s\}]\}, \ s \in \mathcal{R}(\rho^-) \cup J,$$
where $L^\phi = \sup_{t \in \mathcal{R}(\rho^-)} L^\phi(t)$.

We will need to introduce the following hypotheses which are assumed there after:

(H_1) The operator solution $T(t)_{t \in J}$ is uniformly continuous for $t > 0$. Let $M = \sup\{\|T\|_{B(E)} : t \geq 0\}$.

(H_2) The function $f : J \times \mathcal{B} \times \Omega \to E$ is Carathéodory.

(H_3) There exist a function $\psi : J \times \Omega \to \mathbb{R}^+$ and $p : J \times \Omega \to \mathbb{R}^+$ such that for each $w \in \Omega$, $\psi(., w)$ is a continuous nondecreasing function and $p(., w)$ integrable with:
$$|f(t, u, w)| \leq p(t, w) \ \psi(\|u\|_\mathcal{B}, w) \text{ for a.e. } t \in J \text{ and each } u \in \mathcal{B}.$$

(H_4) There exists a function $L : J \times \Omega \to \mathbb{R}^+$ with $L(., w) \in L^1(J, \mathbb{R}^+)$ for each $w \in \Omega$ such that for any bounded $B \subseteq E$
$$\alpha(f(t, B, w)) \leq l(t, w)\alpha(B).$$

(H_5) There exist a random function $R : \Omega \to \mathbb{R}^+\backslash\{0\}$ such that:
$$M\|\phi\|_\mathcal{B} + M \ \psi\left((M_T + L^\phi)\|\phi\|_\mathcal{B} + K_T R(w), w\right) \int_0^T p(s, w)ds \leq R(w).$$

(H_6) For each $w \in \Omega, \phi(., w)$ is continuous and for each $t, \phi(t, .)$ is measurable.

Theorem 3.4. Suppose that hypotheses (H_ϕ) and $(H_1) - (H_6)$ are valid, then the problem (1.1)-(1.2) has at least one mild random solution on $(-\infty, T]$.

Proof. Let $Y = \{u \in C(J, E) : u(0, w) = \phi(0, w) = 0\}$ endowed with the uniform convergence topology and $N : \Omega \times Y \to Y$ be the random operator defined by

(3.2) $$(N(w)y)(t) = T(t) \ \phi(0, w) + \int_0^t T(t - s) \ f(s, \overline{y}_{\rho(s, \overline{y}_s)}, w) \ ds, \quad t \in J,$$

where $\overline{y} : (-\infty, T] \times \Omega \to E$ is such that $\overline{y}_0(\cdot, w) = \phi(\cdot, w)$ and $\overline{y}(\cdot, w) = y(\cdot, w)$ on J. Let $\overline{\phi} : (-\infty, T] \times \Omega \to E$ be the extension of ϕ to $(-\infty, T]$ such that $\overline{\phi}(\theta, w) = \phi(0, w) = 0$ on J.

Then we show that the mapping defined by (3.2) is a random operator. To do this, we need to prove that for any $y \in Y$, $N(.)(y) : \Omega \longrightarrow Y$ is a random variable. Then we prove that $N(.)(y) : \Omega \to Y$ is measurable as a mapping $f(t, y, \cdot)$, $t \in J$, $y \in Y$ is measurable by assumptions (H_2) and (H_6).

Let $D : \Omega \to 2^Y$ be defined by:
$$D(w) = \{y \in Y : \|y\| \leq R(w)\}.$$

The set $D(w)$ bounded, closed, convex and solid for all $w \in \Omega$. Then D is measurable by Lemma 17 in [12]. Let $w \in \Omega$ be fixed. If $y \in D(w)$, from Lemma 3.3 it follows that

$$\|\overline{y}_{\rho(t,\overline{y}_t)}\|_{\mathcal{B}} \leq (M_T + L^\phi)\|\phi\|_{\mathcal{B}} + K_T R(w)$$

and for each $y \in D(w)$, by (H_3) and (H_5), we have for each $t \in J$

$$
\begin{aligned}
|(N(w)y)(t)| &\leq M\|\phi\|_{\mathcal{B}} + M \int_0^t |f(s, \overline{y}_{\rho(s,\overline{y}_s)}, w)| ds \\
&\leq M\|\phi\|_{\mathcal{B}} + M \int_0^t p(s,w) \, \psi\left(\|\overline{y}_{\rho(s,\overline{y}_s)}\|_{\mathcal{B}}, w\right) ds \\
&\leq M\|\phi\|_{\mathcal{B}} + M \int_0^t p(s,w) \, \psi\left((M_T + L^\phi)\|\phi\|_{\mathcal{B}} + K_T R(w), w\right) ds \\
&\leq M\|\phi\|_{\mathcal{B}} + M \, \psi\left((M_T + L^\phi)\|\phi\|_{\mathcal{B}} + K_T R(w), w\right) \int_0^T p(s,w) ds \\
&\leq R(w).
\end{aligned}
$$

This implies that N is a random operator with stochastic domain D and $N(w) : D(w) \to D(w)$ for each $w \in \Omega$.

Step 1: N is continuous.
Let y^n be a sequence such that $y^n \to y$ in Y. Then

$$
\begin{aligned}
|(N(w)y^n)(t) - (N(w)y)(t)| &= \Big| T(t)\phi(0,w) \\
&+ \int_0^t T(t-s) \left[f(s, \overline{y^n}_{\rho(s,\overline{y^n}_s)}, w) - f(s, \overline{y}_{\rho(s,\overline{y}_s)}, w) \right] ds \Big| \\
&\leq M \int_0^t |f(s, \overline{y^n}_{\rho(s,\overline{y^n}_s)}, w) - f(s, \overline{y}_{\rho(s,\overline{y}_s)}, w)| ds.
\end{aligned}
$$

Since $f(s, \cdot, w)$ is continuous, we have by the Lebesgue dominated convergence theorem

$$|(N(w)y^n)(t) - (N(w)y)(t)| \to 0 \ \text{ as } \ n \to +\infty.$$

Thus N is continuous.

Step 2: We prove that for every $w \in \Omega, \{y \in D(w) : N(w)y = y\} \neq \emptyset$. For this we apply the Mönch fixed point theorem.

(a) N maps bounded sets into equicontinuous sets in $D(w)$.

Let $\tau_1, \tau_2 \in [0, T]$ with $\tau_2 > \tau_1$, $D(w)$ be a bounded set as in Step 2, and $y \in D(w)$. Then

$$|(N(w)y)(\tau_2) - (N(w)y)(\tau_1)|$$

$$\leq \quad |T(\tau_2) - T(\tau_1)|\|\phi\|_\mathcal{B}$$

$$+ \quad \left| \int_0^{\tau_1} [T(\tau_2 - s) - T(\tau_1 - s)] f(s, \overline{y}_{\rho(s,\overline{y}_s)}, w) ds \right|$$

$$+ \quad \left| \int_{\tau_1}^{\tau_2} T(\tau_2 - s) f(s, \overline{y}_{\rho(s,\overline{y}_s)}, w) \right| ds$$

$$\leq \quad |T(\tau_2) - T(\tau_1)|\|\phi\|_\mathcal{B}$$

$$+ \quad \int_0^{\tau_1} |T(\tau_2 - s) - T(\tau_1 - s)||f(s, \overline{y}_{\rho(s,\overline{y}_s)}, w)| \, ds$$

$$+ \quad \int_{\tau_1}^{\tau_2} |T(\tau_2 - s) f(s, \overline{y}_{\rho(s,\overline{y}_s)}, w)| \, ds$$

$$\leq \quad |T(\tau_2) - T(\tau_1)|\|\phi\|_\mathcal{B} + \psi\left((M_T + L^\phi)\|\phi\|_\mathcal{B} + K_T R(w)\right)$$
$$\int_0^{\tau_1} |T(\tau_2 - s) - T(\tau_1 - s)| p(s, w) ds$$

$$+ \quad M\psi\left((M_T + L^\phi)\|\phi\|_\mathcal{B} + K_T R(w), w\right) \int_{\tau_1}^{\tau_2} p(s, w) ds.$$

The right-hand of the above inequality tends to zero as $\tau_2 - \tau_1 \to 0$, since $T(t)$ is uniformly continuous.

Next, let $w \in \Omega$ be fixed (therefore we do not write 'w' in the sequel) but arbitrary.

(b) Now let V be a subset of $D(w)$ such that $V \subset \overline{conv}\,(N(V) \cup \{0\})$. V is bounded and equicontinuous and therefore the function $v \to v(t) = \alpha(V(t))$ is continuous on $(-\infty, T]$. By (H_4), Lemma 2.11 and the properties of the measure α we have for each $t \in (-\infty, T]$

$$v(t) \quad \leq \quad \alpha\,(N(V))\,(t) \cup \{0\})$$

$$\leq \quad \alpha\,(N(V(t))$$

$$\leq \quad \alpha\left(T(t)\,\phi(0) + \int_0^t T(t - s)\,f(s, \overline{y}_{\rho(s,\overline{y}_s)})\,ds\right)$$

$$\leq \quad \alpha\left(T(t)\,\phi(0)\right) + \alpha\left(\int_0^t T(t - s)\,f(s, \overline{y}_{\rho(s,\overline{y}_s)})\,ds\right)$$

$$\leq \quad M \int_0^t l(s)\alpha\left(\left\{\overline{y}_{\rho(s,y_s)} : \overline{y} \in V\right\}\right) ds$$

$$\leq \quad M \int_0^t l(s) K(s) \sup_{0 \leq \tau \leq s} \alpha(V(\tau)) ds$$

$$\leq \quad \int_0^t l(s) K(s) \alpha(V(s)) ds$$

$$\leq \quad M \int_0^t v(s)\,l(s) K(s) ds$$

$$= \quad M \int_0^t l(s) K(s) v(s) ds.$$

Gronwall's lemma implies that $v(t) = 0$ for each $t \in J$, and then $V(t)$ is relatively compact in E. In view of the Ascoli-Arzelà theorem, V is relatively compact in $D(w)$. Applying now Theorem 2.10 we

conclude that N has a fixed point $y(w) \in D(w)$. Since $\bigcap_{w \in \Omega} D(w) \neq \emptyset$, the hypothesis that a measurable selector of int$D$ exists holds. By Lemma 2.8, the random operator N has a stochastic fixed point $y^*(w)$, which is a mild solution of the random problem (1.1)-(1.2). $\qquad \square$

Proposition 3.5. *Assume that* $(H_\phi), (H_1), (H_2), (H_5), (H_6)$ *are satisfied, then a slight modification of the proof (i.e. use the Darbo's fixed point theorem) guarantees that* (H_4) *could be replaced by*

$(H_4)^*$ *There exists a nonnegative function* $l(., w) \in L^1(J, \mathbb{R}^+)$ *for each* $w \in \Omega$, *such that*

$$\alpha(f(t, B, w)) \leq l(t, w)\alpha(B), \quad t \in J.$$

Proof. Consider the Kuratowski measure of noncompactness α_C defined on the family of bounded subsets of the space $C(J, E)$ by

$$\alpha_C(H) = \sup_{t \in J} e^{-\tau L(t)} \alpha(H(t)),$$

where $L(t) = \int_0^t \widetilde{l}(s)ds$, $\widetilde{l}(t) = Ml(t)K(t)$, $\tau > 1$.

We show that the operator $N : D(w) \to D(w)$ is a strict set contraction for each $w \in \Omega$. We know that $N : D(w) \to D(w)$ is bounded and continuous, we need to prove that there exists a constant $0 \leq \lambda < 1$ such that $\alpha_C(NH) \leq \lambda \alpha_C(H)$ for $H \subset D(w)$. For each $t \in J$ we have

$$\alpha((NH)(t)) \leq M \int_0^t \alpha(f(s, \overline{y}_{\rho(s, \overline{y}_s)}, w)) : \overline{y} \in H)ds.$$

This implies by $(H_4)^*$ and Theorem 2.1 in [13]

$$
\begin{aligned}
\alpha((NH)(t)) &\leq \int_0^t M \, l(s)\alpha(\overline{y}_{\rho(s, \overline{y}_s)} : \overline{y} \in H)ds \\
&\leq \int_0^t Ml(s)K(s) \sup_{0 \leq \tau \leq s} \alpha(H(\tau))ds \\
&\leq \int_0^t Ml(s)K(s)\alpha(H(s))ds \\
&= \int_0^t \widetilde{l}(s)\alpha(H(s))ds \\
&= \int_0^t e^{\tau L(s)} e^{-\tau L(s)} \widetilde{l}(s)\alpha(H(s))ds \\
&\leq \int_0^t \widetilde{l}(s) \, e^{\tau L(s)} \sup_{s \in [0,t]} e^{-\tau L(s)}\alpha(H(s))ds \\
&\leq \sup_{t \in [0,T]} e^{-\tau L(t)}\alpha(H(t)) \int_0^t \widetilde{l}(s)e^{\tau L(s)}ds \\
&= \alpha_C(H) \int_0^t \left(\frac{e^{\tau L(s)}}{\tau}\right)' ds \\
&\leq \alpha_C(H)\frac{1}{\tau}e^{\tau L(t)}.
\end{aligned}
$$

Therefore,

$$\alpha_C(NH) \leq \frac{1}{\tau}\alpha_C(H).$$

So, the operator N is a set contraction. As a consequence of Theorem 2.12, we deduce that N has a fixed point $y(w) \in D(w)$. Since $\bigcap_{w \in \Omega} D(w) \neq \emptyset$, the hypothesis that a measurable selector of int$D$ exists holds. By Lemma 2.8, the random operator N has a stochastic fixed point $y^*(w)$, which is a mild solution of the random problem (1.1)-(1.2). $\qquad \square$

4. An example

Consider the following functional partial differential equation:

$$(4.1) \qquad \frac{\partial}{\partial t}z(t,x,w) = \frac{\partial^2}{\partial x^2}z(t,x,w) + C_0(w)b(t)\int_{-\infty}^{t} F(z(t+\sigma(t,z(t+s,x,w)),x,w))ds,$$

$$x \in [0,\pi], \ t \in [0,T], \ w \in \Omega$$

$$(4.2) \qquad z(t,0,w) = z(t,\pi,w) = 0, \ t \in [0,T], w \in \Omega$$

$$(4.3) \qquad z(s,x,w) = z_0(s,x,w), \ s \in (-\infty,0], \ x \in [0,\pi], \ w \in \Omega,$$

where C_0 are a real-valued random variable, $b \in L^1(J;\mathbb{R}_+)$, $F : \mathbb{R} \to \mathbb{R}$ is continuous, $z_0 :) - \infty, 0] \times [0,\pi] \times \Omega \to \mathbb{R}$ and $\sigma : J \times \mathbb{R} \to \mathbb{R}$ are given functions.

Suppose that $E = L^2[0,\pi]$, (Ω, F, P) is a complete probability space. Define the operator $A : E \to E$ by $Av = v''$ with domain

$$D(A) = \{v \in E, v, v' \text{are absolutely continuous}, \ v'' \in E, \ v(0) = v(\pi) = 0\}.$$

Then

$$Av = \sum_{n=1}^{\infty} n^2(v,v_n)v_n, v \in D(A)$$

where $\omega_n(s) = \sqrt{\frac{2}{\pi}} \sin ns, n = 1, 2, \dots$ is the orthogonal set of eigenvectors in A. It is well know (see [23]) that A is the infinitesimal generator of an analytic semigroup $T(t), t \geq 0$ in E and is given by

$$T(t)v = \sum_{n=1}^{\infty} \exp(-n^2 t)(v,v_n)v_n, v \in E.$$

Since the analytic semigroup $T(t)$ is compact, there exists a positive constant M such that

$$\|T(t)\|_{B(E)} \leq M.$$

Let $\mathcal{B} = BCU(\mathbb{R}^-; E)$ be the space of uniformly bounded continuous functions endowed with the following norm:

$$\|\phi\| = \sup_{s \leq 0} |\phi(s)|, \qquad \text{for } \phi \in \mathcal{B}.$$

If we put $\phi \in BCU(\mathbb{R}^-; E), x \in [0,\pi]$ and $w \in \Omega$

$$y(t,x,w) = z(t,x,w), \ t \in [0,T]$$
$$\phi(s,x,w) = z_0(s,x,w), \ s \in (-\infty,0], \ x \in [0,\pi], \ w \in \Omega.$$

Set

$$f(t,\phi(x),w) = C_0(w)b(t)\int_{-\infty}^{t} F(z(t+\sigma(t,z(t+s,x,w)),x,w))ds,$$

and

$$\rho(t,\phi)(x) = \sigma(t,z(t,x,w)).$$

Let $\phi \in \mathcal{B}$ be such that (H_ϕ) holds, and let $t \to \phi_t$ be continuous on $\mathcal{R}(\rho^-)$, and let f satisfies the conditions $(H_3), (H_4), (H_5)$

Then the problem (1.1)-(1.2) in an abstract formulation of the problem (4.1)-(4.3), and conditions $(H_1) - (H_6)$ are satisfied. Theorem 3.4 implies that the random problem (4.1)-(4.3) has at least one random mild solution.

References

[1] R. P. Agarwal, M. Meechan, and D. O'Regan, *Fixed Point Theory and Applications*, Cambridge University Press, Cambridge, 2001.

[2] N.U. Ahmed, *Semigroup Theory with Applications to Systems and Control*, Harlow John Wiley & Sons, Inc., New York, 1991.

[3] S. Baghli and M. Benchohra, Uniqueness results for partial functional differential equations in Fréchet spaces, *Fixed Point Theory* **9** (2008), 395-406.

[4] S. Baghli and M. Benchohra, Existence results for semilinear neutral functional differential equations involving evolution operators in Fréchet spaces, *Georgian Math. J.* **17** (2010), 423-436.

[5] S. Baghli and M. Benchohra, Global uniqueness results for partial functional and neutral functional evolution equations with infinite delay, *Differential Integral Equations* **23** (2010), 31–50.

[6] J. Banas and K. Goebel, *Measures of Noncompactness in Banach Spaces*, Marcel Dekker, New York, 1980.

[7] J. Charrier, Strong and weak error estimates for elliptic partial differential equations with random coefficients, *SIAM. J. Numer. Anal* **50** (2012), 216–246.

[8] J. P. C. dos Santos, On state-dependent delay partial neutral functional integrodifferential equations, *Appl. Math. Comput.* **100** (2010), 1637-1644.

[9] H. W. Engl, A general stochastic fixed-point theorem for continuos random operators on stochastic domains. *Anal. Appl.* **66** (1978), 220-231.

[10] K.J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Springer-Verlag, New York, 2000.

[11] D. J. Guo, V. Lakshmikantham, X. Liu, *Nonlinear Integral Equations in Abstract Spaces,* Kluwer Academic Publishers, Dordrecht, 1996.

[12] S. Itoh, Random fixed point theorems with an application to random differential equations in Banach space, *Anal. Appl.* **67** (1979), 261-273.

[13] Jong Son Shin, An existence theorem of functional differential equations with infinite delay in a Banach space, *Funkcial. Ekvac.* **30** (1987), 19-29.

[14] J. Hale, and J. Kato, Phase space for retarded equations with infinite delay, *Funkcial. Ekvac* **21** (1978), 11-41.

[15] E. Hernández, R. Sakthivel, and A. Tanaka, Existence results for impulsive evolution differential equations with state-dependent delay, *Electron. J. Differential Equations*, **2008** (2008), 1-11.

[16] E. Hernandez and M. A. McKibben, On state-dependent delay partial neutral functional-differential equations, *Appl. Math. Comput.* **186** (2007), 294-301.

[17] E. Hernandez, A. Prokopczyk, and L. Ladeira, A note on partial functional differential equations with state-dependent delay, *Nonlin. Anal.* **7**, (2006), 510-519.

[18] Y. Hino, S. Murakami, and T. Naito, *Functional Differential Equations with Unbounded Delay*, Springer-Verlag, Berlin, 1991.

[19] J. Li, J. Shi, and J. Sun, Stability of impulsive stochastic differential delay systems and application to impulsive stochastic neutral networks. *Nonlinear Anal, Theory Methods Appl.* **74** (2011), 3099–3111.

[20] C. Li, R. Sun, and J. Sun, Stability analysis of a class of stochastic differential delay equations with nonlinear impulsive sffects, *J. Franklin Inst. Eng. Appl. Math,* **347**, (2010), 1186–1198.

[21] B. Liu, Stability of solutions for stochastic impulsive systems via comparison approach, *IEEE Trans. Aotom. Control*, **53** (2008), 2128–2133.

[22] H. Mönch, Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces, *Nonlinear Anal.* **4** (1980), 985-999.

[23] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[24] T. T. Soong, *Random Differential Equations in Science and Engineering*, Academic Press, New York, 1973.

[25] D. Stanescu, and B.M. Chen-Charpentier, Random coefficient differential equation models for bacterial growth, *Math. Comput. Model.* **50** (2009), 885–895.

[26] X. Yang, and J. Cao, Synchronization of delayed complex dynamical networks with impulsive and stochastic effects, *Nonlinear. Anal, Real World Appl.* **12**, (2011), 2252–2266.

[27] K. Yosida, *Functional Analysis,* 6th ed. Springer-Verlag, Berlin, 1980.

LABORATORY OF MATHEMATICS, UNIVERSITY OF SIDI BEL-ABBÈS, PO BOX 89, 22000 SIDI BEL-ABBÈS, ALGERIA
  E-mail address: amel2222@yahoo.com

LABORATORY OF MATHEMATICS, UNIVERSITY OF SIDI BEL-ABBÈS, PO BOX 89, 22000 SIDI BEL-ABBÈS, ALGERIA
  E-mail address: benchohra@univ-sba.dz

DEPARTMENT OF MATHEMATICS, KING ABDULAZIZ UNIVERSITY, P.O. BOX 80203, JEDDAH 21589, SAUDI ARABIA