ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2015, VOLUME 5, ISSUE 1, p.35-45

DIFFERENTIAL POLYNOMIALS GENERATED BY MEROMORPHIC SOLUTIONS OF [P,Q] ORDER TO COMPLEX LINEAR DIFFERENTIAL EQUATIONS

BENHARRAT BELAÏDI

Abstract. This paper deals with the complex oscillation of differential polynomials generated by meromorphic solutions of the differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0,$$

where $A_i(z)$ $(i = 0, 1, \cdots, k-1)$ are meromorphic functions of finite $[p,q]$-order in the complex plane.

Mathematics Subject Classification (2010): 34M10, 30D35.
Keywords: Linear differential equations, Differential polynomials, Meromorphic solutions, $[p,q]$-order, $[p,q]$-exponent of convergence of the sequence of distinct zeros.

Article history:
Received 4 February 2015
Received in revised form 6 May 2015
Accepted 6 May 2015

1. Introduction and preliminaries

In this paper, a meromorphic function will always mean meromorphic in the complex plane $\mathbb{C}$. Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory (see [9], [25]). A meromorphic function $\varphi(z)$ is called a small function with respect to $f(z)$ if $T(r,\varphi) = o(T(r,f))$ as $r \to +\infty$ except possibly a set of r of finite linear measure, where $T(r, f)$ is the Nevanlinna characteristic function of f.

To express the rate of fast growth of meromorphic functions, we recall the following definitions. For the definition of the iterated order of a meromorphic function, we use the same definition as in [13], ([6], p. 317), ([14], p. 129). For all $r \in \mathbb{R}$, we define $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp\left(\exp_p r\right)$, $p \in \mathbb{N}$. We also define for all r sufficiently large $\log_1 r := \log r$ and $\log_{p+1} r := \log\left(\log_p r\right)$, $p \in \mathbb{N}$. Moreover, we denote by $\exp_0 r := r$, $\log_0 r := r$, $\log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$.

Definition 1.1. ([13], [14]) Let f be a meromorphic function. Then the iterated $p-$order $\rho_p(f)$ of f is defined as

$$\rho_p(f) = \limsup_{r\to+\infty} \frac{\log_p T(r, f)}{\log r} \quad (p \geq 1 \text{ is an integer}).$$

If f is an entire function, then the iterated $p-$order $\rho_p(f)$ of f is defined by

$$\rho_p(f) = \limsup_{r\to+\infty} \frac{\log_p T(r, f)}{\log r} = \limsup_{r\to+\infty} \frac{\log_{p+1} M(r,f)}{\log r},$$

where $M(r,f) = \max_{|z|=r} |f(z)|$. For $p = 1$, this notation is called order and for $p = 2$ hyper-order.

Definition 1.2. ([13]) The finiteness degree of the order of a meromorphic function f is defined as

$$i(f)=\begin{cases}0, & \text{for } f \text{ rational},\\ \min\{j\in\mathbb{N}:\ \rho_j(f)<+\infty\}, & \text{for } f \text{ transcendental for which}\\ & \text{some } j\in\mathbb{N} \text{ with } \rho_j(f)<+\infty \text{ exists},\\ +\infty, & \text{for } f \text{ with } \rho_j(f)=+\infty \text{ for all } j\in\mathbb{N}.\end{cases}$$

Definition 1.3. ([4], [21]) The iterated $p-$type of a meromorphic function f of iterated $p-$order $\rho_p(f)$ $(0<\rho_p(f)<\infty)$ is defined as

$$\tau_p(f)=\limsup_{r\to+\infty}\frac{\log_{p-1}T(r,f)}{r^{\rho_p(f)}}\quad (p\geq 1 \text{ is an integer}).$$

Similarly, the iterated $p-$type of an entire function f of iterated $p-$order $\rho_p(f)$ $(0<\rho_p(f)<\infty)$ is defined as

$$\tau_{M,p}(f)=\limsup_{r\to+\infty}\frac{\log_{p}M(r,f)}{r^{\rho_p(f)}}\quad (p\geq 1 \text{ is an integer}).$$

Definition 1.4. ([13]) Let f be a meromorphic function. Then the iterated exponent of convergence of the sequence of zeros of $f(z)$ is defined as

$$\lambda_p(f)=\limsup_{r\to+\infty}\frac{\log_p N\left(r,\frac{1}{f}\right)}{\log r}\quad (p\geq 1 \text{ is an integer}),$$

where $N\left(r,\frac{1}{f}\right)$ is the integrated counting function of zeros of $f(z)$ in $\{z:|z|\leq r\}$. For $p=1$, this notation is called exponent of convergence of the sequence of zeros and for $p=2$ hyper-exponent of convergence of the sequence of zeros. Similarly, the iterated exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined as

$$\overline{\lambda}_p(f)=\limsup_{r\to+\infty}\frac{\log_p \overline{N}\left(r,\frac{1}{f}\right)}{\log r}\quad (p\geq 1 \text{ is an integer}),$$

where $\overline{N}\left(r,\frac{1}{f}\right)$ is the integrated counting function of distinct zeros of $f(z)$ in $\{z:|z|\leq r\}$. For $p=1$, this notation is called exponent of convergence of the sequence of distinct zeros and for $p=2$ hyper-exponent of convergence of the sequence of distinct zeros.

Definition 1.5. ([13]) The growth index of the convergence exponent of the sequence of the zeros of $f(z)$ is defined by

$$i_\lambda(f)=\begin{cases}0, & \text{if } N\left(r,\frac{1}{f}\right)=O(\log r),\\ \min\{j\in\mathbb{N},\lambda_j(f)<\infty\}, & \text{if some } j\in\mathbb{N} \text{ with } \lambda_j(f)<\infty,\\ +\infty, & \text{if } \lambda_j(f)=\infty \text{ for all } j\in\mathbb{N}.\end{cases}$$

Remark 1.6. Similarly, we can define the growth index of the convergence exponent of the sequence of distinct zeros $i_{\overline{\lambda}}(f)$ of $f(z)$.

Consider the complex differential equation

$$f^{(k)}+A(z)f=0 \tag{1.1}$$

and the differential polynomial

$$g_f=d_kf^{(k)}+d_{k-1}f^{(k-1)}+\cdots+d_0f, \tag{1.2}$$

where $A(z)$ and $d_j(z)$ $(j=0,1,\cdots,k)$ are meromorphic functions in the complex plane.

Recently, many articles focused on the study of the complex oscillation theory of solutions and differential polynomials generated by solutions of differential equations in the unit disc and in the complex plane $\mathbb{C}$, see ([4], [7], [8], [15], [16], [17], [18], [19]). In [4], the author and Z. Latreuch investigated the growth and oscillation of differential polynomials generated by solutions of (1.1), and obtained the following results:

Theorem 1.7. *([4]) Let $A(z)$ be a meromorphic function of finite iterated $p-$order. Let $d_j(z)$ $(j = 0, 1, \cdots, k)$ be finite iterated $p-$order meromorphic functions that are not all vanishing identically such that*

$$h = \begin{vmatrix} \alpha_{0,0} & \alpha_{1,0} & \cdot & \cdot & \alpha_{k-1,0} \\ \alpha_{0,1} & \alpha_{1,1} & \cdot & \cdot & \alpha_{k-1,1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_{0,k-1} & \alpha_{1,k-1} & \cdot & \cdot & \alpha_{k-1,k-1} \end{vmatrix} \not\equiv 0,$$

where the sequence of functions $\alpha_{i,j}$ $(j = 0, \cdots, k-1)$ are defined by

$$\alpha_{i,j} = \begin{cases} \alpha'_{i,j-1} + \alpha_{i-1,j-1}, \text{ for all } i = 1, \cdots, k-1, \\ \alpha'_{0,j-1} - A\alpha_{k-1,j-1}, \text{ for } i = 0 \end{cases}$$

and

$$\alpha_{i,0} = \begin{cases} d_i, \text{ for all } i = 1, \cdots, k-1, \\ d_0 - d_k A, \text{ for } i = 0. \end{cases}$$

If $f(z)$ is an infinite iterated $p-$order meromorphic solution of (1.1) with $\rho_{p+1}(f) = \rho$, then the differential polynomial (1.2) satisfies

$$\rho_p(g_f) = \rho_p(f) = \infty$$

and

$$\rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho.$$

Furthermore, if f is a finite iterated $p-$order meromorphic solution of (1.1) such that

$$\rho_p(f) > \max\{\rho_p(A), \rho_p(d_j) \quad (j = 0, 1, \cdots, k)\},$$

then

$$\rho_p(g_f) = \rho_p(f).$$

Theorem 1.8. *([4]) Under the hypotheses of Theorem 1.7, let $\varphi(z) \not\equiv 0$ be a meromorphic function with finite iterated $p-$order such that*

$$\psi(z) = \frac{\begin{vmatrix} \varphi & \alpha_{1,0} & \cdot & \cdot & \alpha_{k-1,0} \\ \varphi' & \alpha_{1,1} & \cdot & \cdot & \alpha_{k-1,1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \varphi^{(k-1)} & \alpha_{1,k-1} & \cdot & \cdot & \alpha_{k-1,k-1} \end{vmatrix}}{h(z)}$$

is not a solution of (1.1), where $h \not\equiv 0$ and $\alpha_{i,j}$ $(i = 0, .., k-1; j = 0, \cdots, k-1)$ are defined in Theorem 1.7. If $f(z)$ is an infinite iterated $p-$order meromorphic solution of (1.1) with $\rho_{p+1}(f) = \rho$, then the differential polynomial (1.2) satisfies

$$\overline{\lambda}_p(g_f - \varphi) = \lambda_p(g_f - \varphi) = \rho_p(f) = \infty$$

and

$$\overline{\lambda}_{p+1}(g_f - \varphi) = \lambda_{p+1}(g_f - \varphi) = \rho_{p+1}(f) = \rho.$$

Furthermore, if f is a finite iterated $p-$order meromorphic solution of (1.1) such that

$$\rho_p(f) > \max\{\rho_p(A), \rho_p(\varphi), \rho_p(d_j) \quad (j = 0, 1, \cdots, k)\},$$

then

$$\overline{\lambda}_p\left(g_f-\varphi\right)=\lambda_p\left(g_f-\varphi\right)=\rho_p\left(f\right).$$

In ([11], [12]), Juneja, Kapoor and Bajpai investigated some properties of entire functions of $[p,q]$-order and obtained some results concerning their growth. In [22], in order to maintain accordance with general definitions of the entire function f of iterated $p-$order ([13], [14]), Liu-Tu-Shi gave a minor modification of the original definition of the $[p,q]$-order given in ([11], [12]). By this new concept of $[p,q]$-order, the $[p,q]$-order of solutions of complex linear differential equations (1.1) was investigated in the unit disc and in the complex plane (see, e.g. [1], [2], [3], [5], [10], [20], [21], [23], [24]). Now, we shall introduce the definition of meromorphic functions of $[p,q]$-order, where p, q are positive integers satisfying $p\geq q\geq 1$. In order to keep accordance with Definition 1.1, we will give a minor modification to the original definition of $[p,q]$-order (e.g. see, ([11], [12])).

Definition 1.9. ([21]) Let $p\geq q\geq 1$ be integers. If $f\left(z\right)$ is a transcendental meromorphic function, then the $[p,q]$-order of $f\left(z\right)$ is defined by

$$\rho_{[p,q]}\left(f\right)=\limsup_{r\longmapsto+\infty}\frac{\log_p T(r,f)}{\log_q r}.$$

If f is a transcendental entire function, then the $[p,q]$-order of $f\left(z\right)$ is defined by

$$\rho_{[p,q]}\left(f\right)=\limsup_{r\longmapsto+\infty}\frac{\log_p T(r,f)}{\log_q r}=\limsup_{r\rightarrow+\infty}\frac{\log_{p+1} M\left(r,f\right)}{\log_q r}.$$

It is easy to see that $0\leq\rho_{[p,q]}\left(f\right)\leq\infty$. By Definition 1.9, we have that $\rho_{[1,1]}\left(f\right)=\rho_1\left(f\right)=\rho\left(f\right)$, $\rho_{[2,1]}\left(f\right)=\rho_2\left(f\right)$ and $\rho_{[p+1,1]}\left(f\right)=\rho_{p+1}\left(f\right)$.

Remark 1.10. ([21]) If $f\left(z\right)$ is a meromorphic function satisfying $0\leq\rho_{[p,q]}\left(f\right)\leq\infty$, then
(i) $\rho_{[p-n,q]}=\infty$ $(n<p)$, $\rho_{[p,q-n]}=0$ $(n<q)$, $\rho_{[p+n,q+n]}=1$ $(n<p)$ for $n=1,2,3,\cdots$
(ii) If $[p',q']$ is any pair of integers satisfying $q'=p'+q-p$ and $p'<p$, then $\rho_{[p',q']}=0$ if $0<\rho_{[p,q]}<1$ and $\rho_{[p',q']}=\infty$ if $1<\rho_{[p,q]}<\infty$.
(iii) $\rho_{[p',q']}=\infty$ for $q'-p'>q-p$ and $\rho_{[p',q']}=0$ for $q'-p'<q-p$.

Definition 1.11. ([21]) A transcendental meromorphic function $f\left(z\right)$ is said to have index-pair $[p,q]$ if $0<\rho_{[p,q]}\left(f\right)<\infty$ and $\rho_{[p-1,q-1]}\left(f\right)$ is not a nonzero finite number.

Remark 1.12. ([21]) Suppose that f_1 is a meromorphic function of $[p,q]$-order ρ_1 and f_2 is a meromorphic function of $[p_1,q_1]$-order ρ_2, let $\rho_1\leq\rho_2$. We can easily deduce the result about their comparative growth:
(i) If $p_1-p>q_1-q$, then the growth of f_1 is slower than the growth of f_2.
(ii) If $p_1-p<q_1-q$, then f_1 grows faster than f_2.
(iii) If $p_1-p=q_1-q>0$, then the growth of f_1 is slower than the growth of f_2 if $\rho_2\geq 1$, and the growth of f_1 is faster than the growth of f_2 if $\rho_2<1$.
(iv) Especially, when $p_1=p$ and $q_1=q$ then f_1 and f_2 are of the same index-pair $[p,q]$. If $\rho_1>\rho_2$, then f_1 grows faster than f_2; and if $\rho_1<\rho_2$, then f_1 grows slower than f_2. If $\rho_1=\rho_2$, Definition 1.9 does not show any precise estimate about the relative growth of f_1 and f_2.

Definition 1.13. ([21]) Let $p\geq q\geq 1$ be integers. Let f be a meromorphic function satisfying $0<\rho_{[p,q]}\left(f\right)=\rho<\infty$. Then the $[p,q]-$type of $f\left(z\right)$ is defined by

$$\tau_{[p,q]}\left(f\right)=\limsup_{r\rightarrow+\infty}\frac{\log_{p-1} T\left(r,f\right)}{\left[\log_{q-1} r\right]^{\rho}}.$$

Similarly, the $[p,q]-$type of an entire function f of $[p,q]-$order $0<\rho_{[p,q]}\left(f\right)=\rho<\infty$ is defined as

$$\tau_{M,[p,q]}\left(f\right)=\limsup_{r\rightarrow+\infty}\frac{\log_p M\left(r,f\right)}{\left[\log_{q-1} r\right]^{\rho}}.$$

Definition 1.14. ([21]) Let $p \geq q \geq 1$ be integers. The $[p,q]$-exponent of convergence of the zeros sequence of a meromorphic function $f(z)$ is defined by

$$\lambda_{[p,q]}(f) = \limsup_{r\to+\infty} \frac{\log_p N\left(r, \frac{1}{f}\right)}{\log_q r}.$$

Similarly, the $[p,q]$-exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_{[p,q]}(f) = \limsup_{r\to+\infty} \frac{\log_p \overline{N}\left(r, \frac{1}{f}\right)}{\log_q r}.$$

The remainder of the paper is organized as follows. In Section 2, we shall show our main results which improve and extend many results in the above-mentioned papers. Section 3 is for some lemmas and basic theorems. The last section is for the proofs of our main results.

2. Main Results

In this paper, we continue to consider this subject and investigate the complex oscillation theory of differential polynomials generated by meromorphic solutions of differential equations in the complex plane. The main purpose of this paper is to study the controllability of solutions of the differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0. \tag{2.1}$$

In fact, by making use of the concept of meromorphic functions of $[p,q]$ −order, we study the growth and oscillation of higher order differential polynomial (1.2) with meromorphic coefficients of $[p,q]$ −order generated by solutions of equation (2.1). Before we state our results, we define the sequence of functions $\alpha_{i,j}$ $(j = 0, \cdots, k-1)$ by

$$\alpha_{i,j} = \begin{cases} \alpha'_{i,j-1} + \alpha_{i-1,j-1} - A_i \alpha_{k-1,j-1}, & \text{for all } i = 1, \cdots, k-1, \\ \alpha'_{0,j-1} - A_0 \alpha_{k-1,j-1}, & \text{for } i = 0 \end{cases} \tag{2.2}$$

and

$$\alpha_{i,0} = d_i - d_k A_i, \ \text{for } i = 0, \cdots, k-1. \tag{2.3}$$

We define also h by

$$h = \begin{vmatrix} \alpha_{0,0} & \alpha_{1,0} & \cdot & \cdot & \alpha_{k-1,0} \\ \alpha_{0,1} & \alpha_{1,1} & \cdot & \cdot & \alpha_{k-1,1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_{0,k-1} & \alpha_{1,k-1} & \cdot & \cdot & \alpha_{k-1,k-1} \end{vmatrix} \tag{2.4}$$

and $\psi(z)$ by

$$\psi(z) = \frac{\begin{vmatrix} \varphi & \alpha_{1,0} & \cdot & \cdot & \alpha_{k-1,0} \\ \varphi' & \alpha_{1,1} & \cdot & \cdot & \alpha_{k-1,1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \varphi^{(k-1)} & \alpha_{1,k-1} & \cdot & \cdot & \alpha_{k-1,k-1} \end{vmatrix}}{h(z)}, \tag{2.5}$$

where $h \not\equiv 0$ and $\alpha_{i,j}$ $(i = 0, .., k-1; j = 0, \cdots, k-1)$ are defined in (2.2) and (2.3), and $\varphi \not\equiv 0$ is a meromorphic function with $\rho_{[p,q]}(\varphi) < \infty$. The following theorems are the main results of this paper.

Theorem 2.1. *Let $p \geq q \geq 1$ be integers, and let $A_i(z)$ $(i = 0, 1, \cdots, k-1)$ be meromorphic functions of finite $[p, q]$ −order. Let $d_j(z)$ $(j = 0, 1, \cdots, k)$ be finite $[p, q]$ −order meromorphic functions that are not all vanishing identically such that $h \not\equiv 0$. If $f(z)$ is an infinite $[p, q]$ −order meromorphic solution of (2.1) with $\rho_{[p+1,q]}(f) = \rho$, then the differential polynomial (1.2) satisfies*

$$\rho_{[p,q]}(g_f) = \rho_{[p,q]}(f) = \infty$$

and

$$\rho_{[p+1,q]}(g_f) = \rho_{[p+1,q]}(f) = \rho.$$

Furthermore, if f is a finite $[p, q]$ −order meromorphic solution of (2.1) such that

$$\rho_{[p,q]}(f) > \max\left\{\rho_{[p,q]}(A_i)\,(i = 0, 1, \cdots, k-1), \rho_{[p,q]}(d_j)\quad(j = 0, 1, \cdots, k)\right\}, \tag{2.6}$$

then

$$\rho_{[p,q]}(g_f) = \rho_{[p,q]}(f).$$

Remark 2.2. In Theorem 2.1, if we do not have the condition $h \not\equiv 0$, then the conclusions of Theorem 2.1 cannot hold. For example, if we take $d_i = d_k A_i$ $(i = 0, \cdots, k-1)$, then $h \equiv 0$. It follows that $g_f \equiv 0$ and $\rho_{[p,q]}(g_f) = 0$. So, if $f(z)$ is an infinite $[p, q]$ −order meromorphic solution of (2.1), then $\rho_{[p,q]}(g_f) = 0 < \rho_{[p,q]}(f) = \infty$, and if f is a finite $[p, q]$ −order meromorphic solution of (2.1) such that (2.6) holds, then $\rho_{[p,q]}(g_f) = 0 < \rho_{[p,q]}(f)$.

Theorem 2.3. *Under the hypotheses of Theorem 2.1, let $\varphi(z) \not\equiv 0$ be a meromorphic function of finite $[p, q]$ −order such that $\psi(z)$ is not a solution of (2.1). If $f(z)$ is an infinite $[p, q]$ −order meromorphic solution of (2.1) with $\rho_{[p+1,q]}(f) = \rho$, then the differential polynomial (1.2) satisfies*

$$\overline{\lambda}_{[p,q]}(g_f - \varphi) = \lambda_{[p,q]}(g_f - \varphi) = \rho_{[p,q]}(f) = \infty$$

and

$$\overline{\lambda}_{[p+1,q]}(g_f - \varphi) = \lambda_{[p+1,q]}(g_f - \varphi) = \rho_{[p+1,q]}(f) = \rho.$$

Furthermore, if f is a finite $[p, q]$ −order meromorphic solution of (2.1) such that

$$\rho_{[p,q]}(f) > \max\left\{\rho_{[p,q]}(A_i)\ \ (i = 0, 1, \cdots, k-1), \rho_{[p,q]}(\varphi), \rho_{[p,q]}(d_j)\ \ (j = 0, 1, \cdots, k)\right\}, \tag{2.7}$$

then

$$\overline{\lambda}_{[p,q]}(g_f - \varphi) = \lambda_{[p,q]}(g_f - \varphi) = \rho_{[p,q]}(f).$$

Corollary 2.4. *Let $p \geq q \geq 1$ be integers, and let $A_0(z), \cdots, A_{k-1}(z)$ be entire functions satisfying one of the following two conditions:*
(i) $\max\{\rho_{[p,q]}(A_i) : i = 1, 2, \cdots, k-1\} < \rho_{[p,q]}(A_0) = \rho$ $(0 < \rho < +\infty)$ *or that (ii)* $\max\{\rho_{[p,q]}(A_i) : i = 1, 2, \cdots, k-1\} \leq \rho_{[p,q]}(A_0) = \rho$ $(0 < \rho < +\infty)$ *and* $\max\{\tau_{M,[p,q]}(A_i) : \rho_{[p,q]}(A_i) = \rho_{[p,q]}(A_0)\} < \tau_{M,[p,q]}(A_0) = \tau$ $(0 < \tau < +\infty)$. *Let $d_j(z)$ $(j = 0, 1, \cdots, k)$ be finite $[p, q]$ −order entire functions that are not all vanishing identically such that $h \not\equiv 0$. If $f \not\equiv 0$ is a solution of (2.1), then the differential polynomial (1.2) satisfies*

$$\rho_{[p,q]}(g_f) = \rho_{[p,q]}(f) = \infty$$

and

$$\rho_{[p,q]}(g_f) = \rho_{[p+1,q]}(f) = \rho_{[p,q]}(A_0) = \rho.$$

Corollary 2.5. *Under the hypotheses of Corollary 2.4, let $\varphi(z) \not\equiv 0$ be an entire function of finite $[p, q]$ −order such that $\psi(z) \not\equiv 0$. Then the differential polynomial (1.2) satisfies*

$$\overline{\lambda}_{[p,q]}(g_f - \varphi) = \lambda_{[p,q]}(g_f - \varphi) = \rho_{[p,q]}(f) = \infty$$

and

$$\overline{\lambda}_{[p+1,q]}(g_f - \varphi) = \lambda_{[p+1,q]}(g_f - \varphi) = \rho_{[p+1,q]}(f) = \rho_{[p,q]}(A_0) = \rho.$$

In the following we give two applications of the above results without the additional conditions $h \not\equiv 0$ and ψ is not a solution of (2.1).

Corollary 2.6. *Let $p \geq q \geq 1$ be integers, and let $A(z)$, $B(z)$ be entire functions. Assume that $\rho_{[p,q]}(A) < \rho_{[p,q]}(B) = \rho$ $(0 < \rho < +\infty)$ and that $\tau_{[p,q]}(A) < \tau_{[p,q]}(B) = \tau$ $(0 < \tau < +\infty)$ if $\rho_{[p,q]}(B) = \rho_{[p,q]}(A)$. Let $d_j(z)$ $(j = 0, 1, 2)$ be finite $[p,q]-$order entire functions that are not all vanishing identically such that*

$$\max\left\{\rho_{[p,q]}(d_j) : j = 0, 1, 2\right\} < \rho_{[p,q]}(A).$$

If $f \not\equiv 0$ is a solution of the differential equation

$$f'' + A(z) f' + B(z) f = 0, \tag{2.8}$$

then the differential polynomial $g_f = d_2 f'' + d_1 f' + d_0 f$ satisfies $\rho_{[p,q]}(g_f) = \rho_{[p,q]}(f) = \infty$ and $\rho_{[p+1,q]}(g_f) = \rho_{[p+1,q]}(f) = \rho_{[p,q]}(B)$.

Corollary 2.7. *Under the hypotheses of Corollary 2.6, let $\varphi(z) \not\equiv 0$ be an entire function of finite $[p,q]-$order. Then the differential polynomial $g_f = d_2 f'' + d_1 f' + d_0 f$ satisfies*

$$\overline{\lambda}_{[p,q]}(g_f - \varphi) = \lambda_{[p,q]}(g_f - \varphi) = \rho_{[p,q]}(f) = \infty$$

and

$$\overline{\lambda}_{[p+1,q]}(g_f - \varphi) = \lambda_{[p+1,q]}(g_f - \varphi) = \rho_{[p+1,q]}(f) = \rho_{[p,q]}(B).$$

Remark 2.8. The present article may be understood as an extension and improvement of the recent article of the author and Z. Latreuch [4] from equation (1.1) to equation (2.1) and from p-order to $[p,q]-$order.

3. Some lemmas

Lemma 3.1. *([23]) Let $p \geq q \geq 1$ be integers. Let f be a meromorphic function for which $\rho_{[p,q]}(f) = \beta < +\infty$, and let $k \geq 1$ be an integer. Then for any $\varepsilon > 0$,*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\exp_{p-1}\left\{(\beta + \varepsilon)\log_q r\right\}\right),$$

holds outside of a possible exceptional set E of finite linear measure.

Lemma 3.2. *([21]) Let $p \geq q \geq 1$ be integers, and let $A_0, A_1, \cdots, A_{k-1}, F \not\equiv 0$ be meromorphic functions. If f is a meromorphic solution of the differential equation*

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = F, \tag{3.1}$$

such that $\max\left\{\rho_{[p,q]}(F), \rho_{[p,q]}(A_i) \quad (i = 0, \cdots, k-1)\right\} < \rho_{[p,q]}(f) < +\infty$, then $\overline{\lambda}_{[p,q]}(f) = \lambda_{[p,q]}(f) = \rho_{[p,q]}(f)$.

In the following, we extend Lemma 3.2 when $\rho_{[p,q]}(f) = +\infty$ and $\rho_{[p+1,q]}(f) = \rho < +\infty$.

Lemma 3.3. *Let $p \geq q \geq 1$ be integers, and let $A_0, A_1, \cdots, A_{k-1}, F \not\equiv 0$ be finite $[p,q]-$order meromorphic functions. If f is a meromorphic solution of equation (3.1) with $\rho_{[p,q]}(f) = +\infty$ and $\rho_{[p+1,q]}(f) = \rho < +\infty$, then $\overline{\lambda}_{[p,q]}(f) = \lambda_{[p,q]}(f) = \rho_{[p,q]}(f) = \infty$ and $\overline{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \rho_{[p+1,q]}(f) = \rho$.*

Proof. By (3.1), we can write

$$\frac{1}{f} = \frac{1}{F}\left(\frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + \cdots + A_1\frac{f'}{f} + A_0\right). \tag{3.2}$$

If f has a zero at z_0 of order $\alpha\,(> k)$ and if $A_0, A_1, \cdots, A_{k-1}$ are all analytic at z_0, then F must have a zero at z_0 of order $\alpha - k$. Hence,

$$N\left(r, \frac{1}{f}\right) \leq k\,\overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{F}\right) + \sum_{i=0}^{k-1} N(r, A_i). \tag{3.3}$$

By (3.2), we have

$$m\left(r,\frac{1}{f}\right)\leq\sum_{j=1}^{k}m\left(r,\frac{f^{(j)}}{f}\right)+\sum_{i=0}^{k-1}m\left(r,A_i\right)+m\left(r,\frac{1}{F}\right)+O\left(1\right). \tag{3.4}$$

Applying the Lemma 3.1, we have for $\rho_{[p+1,q]}\left(f\right)=\rho<+\infty$

$$m\left(r,\frac{f^{(j)}}{f}\right)=O\left(\exp_p\left\{\left(\rho+\varepsilon\right)\log_q r\right\}\right)\ \ (j=1,\cdots,k) \tag{3.5}$$

holds for all r outside a set $E\subset(0,+\infty)$ with a linear measure $m\left(E\right)=\delta<+\infty$. By (3.3)-(3.5), we get

$$T\left(r,f\right)=T\left(r,\frac{1}{f}\right)+O\left(1\right)$$

$$\leq k\overline{N}\left(r,\frac{1}{f}\right)+\sum_{i=0}^{k-1}T\left(r,A_i\right)+T\left(r,F\right)+O\left(\exp_p\left\{\left(\rho+\varepsilon\right)\log_q r\right\}\right)\ \ (|z|=r\notin E). \tag{3.6}$$

Set $\sigma=\max\left\{\rho_{[p,q]}\left(A_i\right)\ \ (i=0,\cdots,k-1)\,,\rho_{[p,q]}\left(F\right)\right\}$. Then for sufficiently large r, we have

$$T\left(r,A_0\right)+\cdots+T\left(r,A_{k-1}\right)+T\left(r,F\right)\leq\left(k+1\right)\exp_p\left\{\left(\sigma+\varepsilon\right)\log_q r\right\}. \tag{3.7}$$

Thus, by (3.6) and (3.7), we obtain

$$T\left(r,f\right)\leq k\ \overline{N}\left(r,\frac{1}{f}\right)+\left(k+1\right)\exp_p\left\{\left(\sigma+\varepsilon\right)\log_q r\right\}$$

$$+O\left(\exp_p\left\{\left(\rho+\varepsilon\right)\log_q r\right\}\right)\qquad\left(|z|=r\notin E\right). \tag{3.8}$$

Hence, for any f with $\rho_{[p,q]}\left(f\right)=+\infty$ and $\rho_{[p+1,q]}\left(f\right)=\rho$, by (3.8), we have $\overline{\lambda}_{[p,q]}\left(f\right)\geq\rho_{[p,q]}\left(f\right)=+\infty$ and $\overline{\lambda}_{[p+1,q]}\left(f\right)\geq\rho_{[p+1,q]}\left(f\right)$. Since $\overline{\lambda}_{[p+1,q]}\left(f\right)\leq\lambda_{[p+1,q]}\left(f\right)\leq\rho_{[p+1,q]}\left(f\right)$ we obtain

$$\overline{\lambda}_{[p+1,q]}\left(f\right)=\lambda_{[p+1,q]}\left(f\right)=\rho_{[p+1,q]}\left(f\right)=\rho.$$

□

Lemma 3.4. *Let $p\geq q\geq1$ be integers, and let f, g be non-constant meromorphic functions of $[p,q]$-order. Then we have*

$$\rho_{[p,q]}\left(f+g\right)\leq\max\left\{\rho_{[p,q]}\left(f\right),\rho_{[p,q]}\left(g\right)\right\}$$

and

$$\rho_{[p,q]}\left(fg\right)\leq\max\left\{\rho_{[p,q]}\left(f\right),\rho_{[p,q]}\left(g\right)\right\}.$$

Furthermore, if $\rho_{[p,q]}\left(f\right)>\rho_{[p,q]}\left(g\right)$, then we obtain

$$\rho_{[p,q]}\left(f+g\right)=\rho_{[p,q]}\left(fg\right)=\rho_{[p,q]}\left(f\right).$$

Proof. Set $\rho_{[p,q]}\left(f\right)=\rho_1$ and $\rho_{[p,q]}\left(g\right)=\rho_2$. For any given $\varepsilon>0$, we have

$$T\left(r,f+g\right)\leq T\left(r,f\right)+T\left(r,g\right)+O\left(1\right)$$

$$\leq\exp_p\left\{\left(\rho_1+\varepsilon\right)\log_q r\right\}+\exp_p\left\{\left(\rho_2+\varepsilon\right)\log_q r\right\}+O\left(1\right)$$

$$\leq2\exp_p\left\{\left(\max\left\{\rho_1,\rho_2\right\}+\varepsilon\right)\log_q r\right\}+O\left(1\right) \tag{3.9}$$

and

$$T\left(r,fg\right)\leq T\left(r,f\right)+T\left(r,g\right)\leq2\exp_p\left\{\left(\max\left\{\rho_1,\rho_2\right\}+\varepsilon\right)\log_q r\right\} \tag{3.10}$$

for all r sufficiently large. Since $\varepsilon>0$ is arbitrary, from (3.9) and (3.10), we easily obtain

$$\rho_{[p,q]}\left(f+g\right)\leq\max\left\{\rho_{[p,q]}\left(f\right),\rho_{[p,q]}\left(g\right)\right\} \tag{3.11}$$

and

$$\rho_{[p,q]}(fg) \leq \max\left\{\rho_{[p,q]}(f), \rho_{[p,q]}(g)\right\}. \tag{3.12}$$

Suppose now that $\rho_{[p,q]}(f) > \rho_{[p,q]}(g)$. Considering that

$$T(r,f) = T(r, f+g-g) \leq T(r, f+g) + T(r,g) + O(1) \tag{3.13}$$

and

$$T(r,f) = T\left(r, \frac{fg}{g}\right) \leq T(r, fg) + T\left(r, \frac{1}{g}\right)$$

$$= T(r, fg) + T(r,g) + O(1). \tag{3.14}$$

By (3.13) and (3.14), by the same method as above we obtain that

$$\rho_{[p,q]}(f) \leq \max\left\{\rho_{[p,q]}(f+g), \rho_{[p,q]}(g)\right\} = \rho_{[p,q]}(f+g), \tag{3.15}$$

$$\rho_{[p,q]}(f) \leq \max\left\{\rho_{[p,q]}(fg), \rho_{[p,q]}(g)\right\} = \rho_{[p,q]}(fg). \tag{3.16}$$

By using (3.11) and (3.15) we obtain $\rho_{[p,q]}(f+g) = \rho_{[p,q]}(f)$ and by (3.12) and (3.16), we get $\rho_{[p,q]}(fg) = \rho_{[p,q]}(f)$. □

Lemma 3.5. *Let $p \geq q \geq 1$ be integers, and let f, g be meromorphic functions with $[p,q]-$order $0 < \rho_{[p,q]}(f), \rho_{[p,q]}(g) < \infty$ and $[p,q]-$type $0 < \tau_{[p,q]}(f), \tau_{[p,q]}(g) < \infty$. Then the following statements hold:*
(i) If $\rho_{[p,q]}(g) < \rho_{[p,q]}(f)$, then

$$\tau_{[p,q]}(f+g) = \tau_{[p,q]}(fg) = \tau_{[p,q]}(f). \tag{3.17}$$

(ii) If $\rho_{[p,q]}(f) = \rho_{[p,q]}(g)$ and $\tau_{[p,q]}(g) \neq \tau_{[p,q]}(f)$, then

$$\rho_{[p,q]}(f+g) = \rho_{[p,q]}(fg) = \rho_{[p,q]}(f). \tag{3.18}$$

Proof. (i) Suppose that $\rho_{[p,q]}(f) > \rho_{[p,q]}(g)$. By using the definition of the $[p,q]-$type and since $\rho_{[p,q]}(f+g) = \rho_{[p,q]}(f)$, we get

$$\tau_{[p,q]}(f+g) = \limsup_{r\to+\infty} \frac{\log_{p-1} T(r, f+g)}{\left(\log_{q-1} r\right)^{\rho_{[p,q]}(f+g)}}$$

$$\leq \limsup_{r\to+\infty} \frac{\log_{p-1}\left(T(r,f) + T(r,g) + O(1)\right)}{\left(\log_{q-1} r\right)^{\rho_{[p,q]}(f)}}$$

$$\leq \limsup_{r\to+\infty} \frac{\log_{p-1} T(r,f)}{\left(\log_{q-1} r\right)^{\rho_{[p,q]}(f)}} + \limsup_{r\to+\infty} \frac{\log_{p-1} T(r,g) + O(1)}{\left(\log_{q-1} r\right)^{\rho_{[p,q]}(f)}}$$

$$= \limsup_{r\to+\infty} \frac{\log_{p-1} T(r,f)}{\left(\log_{q-1} r\right)^{\rho_{[p,q]}(f)}} + \limsup_{r\to+\infty} \left(\frac{\log_{p-1} T(r,g)}{\left(\log_{q-1} r\right)^{\rho_{[p,q]}(g)}} \frac{\left(\log_{q-1} r\right)^{\rho_{[p,q]}(g)}}{\left(\log_{q-1} r\right)^{\rho_{[p,q]}(f)}}\right)$$

$$\leq \limsup_{r\to+\infty} \frac{\log_{p-1} T(r,f)}{\left(\log_{q-1} r\right)^{\rho_{[p,q]}(f)}}$$

$$+ \limsup_{r\to+\infty} \frac{\left(\log_{q-1} r\right)^{\rho_{[p,q]}(g)}}{\left(\log_{q-1} r\right)^{\rho_{[p,q]}(f)}} \limsup_{r\to+\infty} \frac{\log_{p-1} T(r,g)}{\left(\log_{q-1} r\right)^{\rho_{[p,q]}(g)}} = \tau_{[p,q]}(f). \tag{3.19}$$

Since $\rho_{[p,q]}(f+g) = \rho_{[p,q]}(f) > \rho_{[p,q]}(g)$, then by (3.19), we obtain

$$\tau_{[p,q]}(f) = \tau_{[p,q]}(f+g-g) \leq \tau_{[p,q]}(f+g).$$

Hence $\tau_{[p,q]}(f+g)=\tau_{[p,q]}(f)$. By the same method as before, we have

$$\tau_{[p,q]}(fg)=\limsup_{r\to+\infty}\frac{\log_{p-1}T(r,fg)}{\left(\log_{q-1}r\right)^{\rho_{[p,q]}(fg)}}\leq\limsup_{r\to+\infty}\frac{\log_{p-1}(T(r,f)+T(r,g))}{\left(\log_{q-1}r\right)^{\rho_{[p,q]}(f)}}$$

$$\leq\limsup_{r\to+\infty}\frac{\log_{p-1}T(r,f)}{\left(\log_{q-1}r\right)^{\rho_{[p,q]}(f)}}+\limsup_{r\to+\infty}\frac{\log_{p-1}T(r,g)+O(1)}{\left(\log_{q-1}r\right)^{\rho_{[p,q]}(f)}}\leq\tau_{[p,q]}(f). \tag{3.20}$$

Since $\rho_{[p,q]}(fg)=\rho_{[p,q]}(f)>\rho_{[p,q]}(g)=\rho_{[p,q]}\left(\frac{1}{g}\right)$, then by (3.20), we obtain

$$\tau_{[p,q]}(f)=\tau_{[p,q]}\left(fg\frac{1}{g}\right)\leq\tau_{[p,q]}(fg).$$

Thus, $\tau_{[p,q]}(fg)=\tau_{[p,q]}(f)$.
(ii) Without loss of generality, we suppose that $\tau_{[p,q]}(f)>\tau_{[p,q]}(g)$. It's easy to see that

$$\rho_{[p,q]}(f+g)\leq\rho_{[p,q]}(f)=\rho_{[p,q]}(g).$$

If we suppose that $\rho_{[p,q]}(f+g)<\rho_{[p,q]}(f)=\rho_{[p,q]}(g)$, then by (3.17)

$$\tau_{[p,q]}(g)=\tau_{[p,q]}(f+g-f)=\tau_{[p,q]}(f)$$

which is a contradiction. Hence $\rho_{[p,q]}(f+g)=\rho_{[p,q]}(f)=\rho_{[p,q]}(g)$. Also, we have

$$\rho_{[p,q]}(fg)\leq\rho_{[p,q]}(f)=\rho_{[p,q]}(g).$$

If we suppose $\rho_{[p,q]}(fg)<\rho_{[p,q]}(f)=\rho_{[p,q]}\left(\frac{1}{f}\right)=\rho_{[p,q]}(g)$, then by (3.17), we can write

$$\tau_{[p,q]}(g)=\tau_{[p,q]}\left(fg\frac{1}{f}\right)=\tau_{[p,q]}(f),$$

which is a contradiction. Hence $\rho_{[p,q]}(fg)=\rho_{[p,q]}(f)=\rho_{[p,q]}(g)$. □

Lemma 3.6. *([24]) Let $p\geq q\geq 1$ be integers, and let $A_0(z),\cdots,A_{k-1}(z)$ be entire functions satisfying one of the following two conditions:*
(i) $\max\{\rho_{[p,q]}(A_i):i=1,2,\cdots,k-1\}<\rho_{[p,q]}(A_0)=\rho$ $(0<\rho<+\infty)$ or that (ii) $\max\{\rho_{[p,q]}(A_i):i=1,2,\cdots,k-1\}\leq\rho_{[p,q]}(A_0)=\rho$ $(0<\rho<+\infty)$ and $\max\{\tau_{M,[p,q]}(A_i):\rho_{[p,q]}(A_i)=\rho_{[p,q]}(A_0)\}<\tau_{M,[p,q]}(A_0)=\tau$ $(0<\tau<+\infty)$. Then every solution $f\not\equiv 0$ of (2.1) satisfies $\rho_{[p,q]}(f)=+\infty$ and $\rho_{[p+1,q]}(f)=\rho_{[p,q]}(A_0)=\rho$.

In the following, we give a special case of the result given by L. M. Li and T. B. Cao in [21]. This result is a similar result to Lemma 3.6 for entire solutions f when the $[p,q]-$order and the $[p,q]-$type of the coefficients of (2.1) are defined by the Nevanlinna characteristic function $T(r,f)$.

Lemma 3.7. *Let $p\geq q\geq 1$ be integers, and let $A_0(z),\cdots,A_{k-1}(z)$ be entire functions satisfying one of the following two conditions:*
(i) $\max\{\rho_{[p,q]}(A_i):i=1,2,\cdots,k-1\}<\rho_{[p,q]}(A_0)=\rho$ $(0<\rho<+\infty)$ or that
(ii) $\max\{\rho_{[p,q]}(A_i):i=1,2,\cdots,k-1\}\leq\rho_{[p,q]}(A_0)=\rho$ $(0<\rho<+\infty)$ and $\max\{\tau_{[p,q]}(A_i):\rho_{[p,q]}(A_i)=\rho_{[p,q]}(A_0)\}<\tau_{[p,q]}(A_0)=\tau$ $(0<\tau<+\infty)$. Then every solution $f\not\equiv 0$ of (2.1) satisfies $\rho_{[p,q]}(f)=+\infty$ and $\rho_{[p+1,q]}(f)=\rho_{[p,q]}(A_0)=\rho$.

Lemma 3.8. *Assume that $f \not\equiv 0$ is a solution of equation (2.1). Then the differential polynomial g_f defined in (1.2) satisfies the system of equations*

$$\begin{cases} g_f = \alpha_{0,0} f + \alpha_{1,0} f' + \cdots + \alpha_{k-1,0} f^{(k-1)}, \\ g'_f = \alpha_{0,1} f + \alpha_{1,1} f' + \cdots + \alpha_{k-1,1} f^{(k-1)}, \\ g''_f = \alpha_{0,2} f + \alpha_{1,2} f' + \cdots + \alpha_{k-1,2} f^{(k-1)}, \\ \cdots \\ g_f^{(k-1)} = \alpha_{0,k-1} f + \alpha_{1,k-1} f' + \cdots + \alpha_{k-1,k-1} f^{(k-1)}, \end{cases}$$

where

$$\alpha_{i,j} = \begin{cases} \alpha'_{i,j-1} + \alpha_{i-1,j-1} - A_i \alpha_{k-1,j-1}, \ \text{for all } i = 1, \cdots, k-1, \\ \alpha'_{0,j-1} - A_0 \alpha_{k-1,j-1}, \ \text{for } i = 0 \end{cases}$$

and

$$\alpha_{i,0} = d_i - d_k A_i, \ \text{for } i = 0, \cdots, k-1.$$

Proof. Suppose that f is a solution of (2.1). We can rewrite (2.1) as

$$f^{(k)} = -\sum_{i=0}^{k-1} A_i f^{(i)} \tag{3.21}$$

which implies

$$g_f = d_k f^{(k)} + d_{k-1} f^{(k-1)} + \cdots + d_1 f' + d_0 f = \sum_{i=0}^{k-1} (d_i - d_k A_i) f^{(i)}. \tag{3.22}$$

We can rewrite (3.22) as

$$g_f = \sum_{i=0}^{k-1} \alpha_{i,0} f^{(i)}, \tag{3.23}$$

where $\alpha_{i,0}$ are defined in (2.3). Differentiating both sides of equation (3.23) and replacing $f^{(k)}$ with $f^{(k)} = -\sum\limits_{i=0}^{k-1} A_i f^{(i)}$, we obtain

$$\begin{aligned} g'_f &= \sum_{i=0}^{k-1} \alpha'_{i,0} f^{(i)} + \sum_{i=0}^{k-1} \alpha_{i,0} f^{(i+1)} = \sum_{i=0}^{k-1} \alpha'_{i,0} f^{(i)} + \sum_{i=1}^{k} \alpha_{i-1,0} f^{(i)} \\ &= \alpha'_{0,0} f + \sum_{i=1}^{k-1} \alpha'_{i,0} f^{(i)} + \sum_{i=1}^{k-1} \alpha_{i-1,0} f^{(i)} + \alpha_{k-1,0} f^{(k)} \\ &= \alpha'_{0,0} f + \sum_{i=1}^{k-1} \alpha'_{i,0} f^{(i)} + \sum_{i=1}^{k-1} \alpha_{i-1,0} f^{(i)} - \sum_{i=0}^{k-1} \alpha_{k-1,0} A_i f^{(i)} \\ &= \left(\alpha'_{0,0} - \alpha_{k-1,0} A_0\right) f + \sum_{i=1}^{k-1} \left(\alpha'_{i,0} + \alpha_{i-1,0} - \alpha_{k-1,0} A_i\right) f^{(i)}. \end{aligned} \tag{3.24}$$

We can rewrite (3.24) as

$$g'_f = \sum_{i=0}^{k-1} \alpha_{i,1} f^{(i)}, \tag{3.25}$$

where

$$\alpha_{i,1} = \begin{cases} \alpha'_{i,0} + \alpha_{i-1,0} - \alpha_{k-1,0} A_i, \ \text{for all } i = 1, \cdots, k-1, \\ \alpha'_{0,0} - A_0 \alpha_{k-1,0}, \ \text{for } i = 0. \end{cases} \tag{3.26}$$

Differentiating both sides of equation (3.25) and replacing $f^{(k)}$ with $f^{(k)} = -\sum_{i=0}^{k-1} A_i f^{(i)}$, we obtain

$$g_f'' = \sum_{i=0}^{k-1}\alpha_{i,1}' f^{(i)} + \sum_{i=0}^{k-1}\alpha_{i,1} f^{(i+1)} = \sum_{i=0}^{k-1}\alpha_{i,1}' f^{(i)} + \sum_{i=1}^{k}\alpha_{i-1,1} f^{(i)}$$

$$= \alpha_{0,1}' f + \sum_{i=1}^{k-1}\alpha_{i,1}' f^{(i)} + \sum_{i=1}^{k-1}\alpha_{i-1,1} f^{(i)} + \alpha_{k-1,1} f^{(k)}$$

$$= \alpha_{0,1}' f + \sum_{i=1}^{k-1}\alpha_{i,1}' f^{(i)} + \sum_{i=1}^{k-1}\alpha_{i-1,1} f^{(i)} - \sum_{i=0}^{k-1} A_i \alpha_{k-1,1} f^{(i)}$$

$$= \left(\alpha_{0,1}' - \alpha_{k-1,1} A_0\right) f + \sum_{i=1}^{k-1}\left(\alpha_{i,1}' + \alpha_{i-1,1} - A_i \alpha_{k-1,1}\right) f^{(i)} \tag{3.27}$$

which implies that

$$g_f'' = \sum_{i=0}^{k-1}\alpha_{i,2} f^{(i)}, \tag{3.28}$$

where

$$\alpha_{i,2} = \begin{cases} \alpha_{i,1}' + \alpha_{i-1,1} - A_i \alpha_{k-1,1}, \text{ for all } i = 1, \cdots, k-1, \\ \alpha_{0,1}' - A_0 \alpha_{k-1,1}, \text{ for } i = 0. \end{cases} \tag{3.29}$$

By using the same method as above we can easily deduce that

$$g_f^{(j)} = \sum_{i=0}^{k-1}\alpha_{i,j} f^{(i)}, \ j = 0, 1, \cdots, k-1, \tag{3.30}$$

where

$$\alpha_{i,j} = \begin{cases} \alpha_{i,j-1}' + \alpha_{i-1,j-1} - A_i \alpha_{k-1,j-1}, \text{ for all } i = 1, \cdots, k-1, \\ \alpha_{0,j-1}' - A_0 \alpha_{k-1,j-1}, \text{ for } i = 0 \end{cases} \tag{3.31}$$

and

$$\alpha_{i,0} = d_i - d_k A_i, \text{ for all } i = 0, 1, \cdots, k-1. \tag{3.32}$$

By (3.23)-(3.32), we obtain the system of equations

$$\begin{cases} g_f = \alpha_{0,0} f + \alpha_{1,0} f' + \cdots + \alpha_{k-1,0} f^{(k-1)}, \\ g_f' = \alpha_{0,1} f + \alpha_{1,1} f' + \cdots + \alpha_{k-1,1} f^{(k-1)}, \\ g_f'' = \alpha_{0,2} f + \alpha_{1,2} f' + \cdots + \alpha_{k-1,2} f^{(k-1)}, \\ \cdots \\ g_f^{(k-1)} = \alpha_{0,k-1} f + \alpha_{1,k-1} f' + \cdots + \alpha_{k-1,k-1} f^{(k-1)}. \end{cases} \tag{3.33}$$

This completes the proof of Lemma 3.8. □

4. Proof of the Theorems and the Corollaries

Proof of Theorem 2.1

Proof. Suppose that f is an infinite $[p,q]$ −order meromorphic solution of (2.1) with $\rho_{[p+1,q]}(f) = \rho$. By Lemma 3.8, g_f satisfies the system of equations

(4.1)
$$\begin{cases} g_f = \alpha_{0,0}f + \alpha_{1,0}f' + \cdots + \alpha_{k-1,0}f^{(k-1)}, \\ g'_f = \alpha_{0,1}f + \alpha_{1,1}f' + \cdots + \alpha_{k-1,1}f^{(k-1)}, \\ g''_f = \alpha_{0,2}f + \alpha_{1,2}f' + \cdots + \alpha_{k-1,2}f^{(k-1)}, \\ \cdots \\ g_f^{(k-1)} = \alpha_{0,k-1}f + \alpha_{1,k-1}f' + \cdots + \alpha_{k-1,k-1}f^{(k-1)}, \end{cases}$$

where

(4.2)
$$\alpha_{i,j} = \begin{cases} \alpha'_{i,j-1} + \alpha_{i-1,j-1} - A_i\alpha_{k-1,j-1}, \text{ for all } i = 1, \cdots, k-1, \\ \alpha'_{0,j-1} - A_0\alpha_{k-1,j-1}, \text{ for } i = 0 \end{cases}$$

and

(4.3)
$$\alpha_{i,0} = d_i - d_kA_i, \text{ for all } i = 0, 1, \cdots, k-1.$$

By Cramer's rule, and since $h \not\equiv 0$, then we have

(4.4)
$$f = \frac{\begin{vmatrix} g_f & \alpha_{1,0} & . & . & \alpha_{k-1,0} \\ g'_f & \alpha_{1,1} & . & . & \alpha_{k-1,1} \\ . & . & . & . & . \\ . & . & . & . & . \\ g_f^{(k-1)} & \alpha_{1,k-1} & . & . & \alpha_{k-1,k-1} \end{vmatrix}}{h}.$$

It follows that

(4.5)
$$f = C_0g_f + C_1g'_f + \cdots + C_{k-1}g_f^{(k-1)},$$

where C_j are finite $[p,q]$ −order meromorphic functions depending on $\alpha_{i,j}$, where $\alpha_{i,j}$ are defined in (4.2) and (4.3).

If $\rho_{[p,q]}(g_f) < +\infty$, then by (4.5) we obtain $\rho_{[p,q]}(f) < +\infty$, which is a contradiction. Hence $\rho_{[p,q]}(g_f) = \rho_{[p,q]}(f) = +\infty$.

Now, we prove that $\rho_{[p+1,q]}(g_f) = \rho_{[p+1,q]}(f) = \rho$. By (1.2), we get $\rho_{[p+1,q]}(g_f) \leq \rho_{[p+1,q]}(f)$ and by (4.5) we have $\rho_{[p+1,q]}(f) \leq \rho_{[p+1,q]}(g_f)$. This yield $\rho_{[p+1,q]}(g_f) = \rho_{[p+1,q]}(f) = \rho$.

Furthermore, if f is a finite $[p,q]$ −order meromorphic solution of equation (2.1) such that

(4.6)
$$\rho_{[p,q]}(f) > \max\left\{\rho_{[p,q]}(A_i), \rho_{[p,q]}(d_j) : i = 0, \cdots, k-1, j = 0, 1, \cdots, k\right\},$$

then

(4.7)
$$\rho_{[p,q]}(f) > \max\left\{\rho_{[p,q]}(\alpha_{i,j}) : i = 0, \cdots, k-1, j = 0, \cdots, k-1\right\}.$$

By (1.2) and (4.6) we have $\rho_{[p,q]}(g_f) \leq \rho_{[p,q]}(f)$. Now, we prove $\rho_{[p,q]}(g_f) = \rho_{[p,q]}(f)$. If $\rho_{[p,q]}(g_f) < \rho_{[p,q]}(f)$, then by (4.5) and (4.7) we get

$$\rho_{[p,q]}(f) \leq \max\left\{\rho_{[p,q]}(C_j) \quad (j = 0, \cdots, k-1), \rho_{[p,q]}(g_f)\right\} < \rho_{[p,q]}(f)$$

which is a contradiction. Hence $\rho_{[p,q]}(g_f) = \rho_{[p,q]}(f)$. □

Remark 4.1. By (4.5), we can see that the condition $h \not\equiv 0$ is equivalent to the condition $g_f, g'_f, g''_f, \cdots, g_f^{(k-1)}$ are linearly independent over the field of meromorphic functions of finite $[p,q]$ −order.

Proof of Theorem 2.3

Proof. Suppose that f is an infinite $[p,q]$ $-$order meromorphic solution of equation (2.1) with $\rho_{[p+1,q]}(f)=\rho$. Set $w(z)=g_f-\varphi$. Since $\rho_{[p,q]}(\varphi)<\infty$, then by Lemma 3.4 and Theorem 2.1 we have $\rho_{[p,q]}(w)=\rho_{[p,q]}(g_f)=\infty$ and $\rho_{[p+1,q]}(w)=\rho_{[p+1,q]}(g_f)=\rho$. To prove $\overline{\lambda}_{[p,q]}(g_f-\varphi)=\lambda_{[p,q]}(g_f-\varphi)=\infty$ and $\overline{\lambda}_{[p+1,q]}(g_f-\varphi)=\lambda_{[p+1,q]}(g_f-\varphi)=\rho$ we need to prove $\overline{\lambda}_{[p,q]}(w)=\lambda_{[p,q]}(w)=\infty$ and $\overline{\lambda}_{[p+1,q]}(w)=\lambda_{[p+1,q]}(w)=\rho$. By $g_f=w+\varphi$, and using (4.5), we get

$$f=C_0w+C_1w'+\cdots+C_{k-1}w^{(k-1)}+\psi(z), \tag{4.8}$$

where

$$\psi(z)=C_0\varphi+C_1\varphi'+\cdots+C_{k-1}\varphi^{(k-1)}.$$

Substituting (4.8) into (2.1), we obtain

$$C_{k-1}w^{(2k-1)}+\sum_{j=0}^{2k-2}\phi_jw^{(j)}=-\left(\psi^{(k)}+A_{k-1}(z)\psi^{(k-1)}+\cdots+A_0(z)\psi\right)=H,$$

where ϕ_j $(j=0,\cdots,2k-2)$ are meromorphic functions of finite $[p,q]$-order. Since $\psi(z)$ is not a solution of (2.1), it follows that $H\not\equiv 0$. Then by Lemma 3.3, we obtain $\overline{\lambda}_{[p,q]}(w)=\lambda_{[p,q]}(w)=\infty$ and $\overline{\lambda}_{[p+1,q]}(w)=\lambda_{[p+1,q]}(w)=\rho$, i. e.,

$$\overline{\lambda}_{[p,q]}(g_f-\varphi)=\lambda_{[p,q]}(g_f-\varphi)=\infty$$

and

$$\overline{\lambda}_{[p+1,q]}(g_f-\varphi)=\lambda_{[p+1,q]}(g_f-\varphi)=\rho.$$

Suppose that f is a finite $[p,q]$ $-$order meromorphic solution of equation (2.1) such that (2.7) holds. Set $w(z)=g_f-\varphi$. Since $\rho_{[p,q]}(\varphi)<\rho_{[p,q]}(f)$, then by Lemma 3.4 and Theorem 2.1 we have $\rho_{[p,q]}(w)=\rho_{[p,q]}(g_f)=\rho_{[p,q]}(f)$. To prove $\overline{\lambda}_{[p,q]}(g_f-\varphi)=\lambda_{[p,q]}(g_f-\varphi)=\rho_{[p,q]}(f)$ we need to prove $\overline{\lambda}_{[p,q]}(w)=\lambda_{[p,q]}(w)=\rho_{[p,q]}(f)$. Using the same reasoning as above, we get

$$C_{k-1}w^{(2k-1)}+\sum_{j=0}^{2k-2}\phi_jw^{(j)}=-\left(\psi^{(k)}+A_{k-1}(z)\psi^{(k-1)}+\cdots+A_0(z)\psi\right)=H,$$

where ϕ_j $(j=0,\cdots,2k-2)$ are meromorphic functions with $[p,q]$ $-$order $\rho_{[p,q]}(\phi_j)<\rho_{[p,q]}(f)$ $(j=0,\cdots,2k-2)$ and

$$\psi(z)=C_0\varphi+C_1\varphi'+\cdots+C_{k-1}\varphi^{(k-1)},\ \rho_{[p,q]}(H)<\rho_{[p,q]}(f).$$

Since $\psi(z)$ is not a solution of (2.1), it follows that $H\not\equiv 0$. Then by Lemma 3.2, we obtain $\overline{\lambda}_{[p,q]}(w)=\lambda_{[p,q]}(w)=\rho_{[p,q]}(f)$, i. e., $\overline{\lambda}_{[p,q]}(g_f-\varphi)=\lambda_{[p,q]}(g_f-\varphi)=\rho_{[p,q]}(f)$. □

Proof of Corollary 2.4

Proof. Suppose $f\not\equiv 0$ is a solution of (2.1). Then by Lemma 3.6, we have $\rho_{[p,q]}(f)=\infty$ and $\rho_{[p+1,q]}(f)=\rho_{[p,q]}(A_0)$. Thus, by Theorem 2.1 we obtain $\rho_{[p,q]}(g_f)=\rho_{[p,q]}(f)=\infty$ and $\rho_{[p+1,q]}(g_f)=\rho_{[p+1,q]}(f)=\rho_{[p,q]}(A_0)$. □

Proof of Corollary 2.5

Proof. Suppose $f\not\equiv 0$ is a solution of (2.1). Then by Lemma 3.6, we have $\rho_{[p,q]}(f)=\infty$ and $\rho_{[p+1,q]}(f)=\rho_{[p,q]}(A_0)$. Since $\varphi(z)\not\equiv 0$ is an entire function of finite $[p,q]$ $-$order such that $\psi(z)\not\equiv 0$, then $\rho_{[p,q]}(\psi)<\infty$ and ψ is not a solution of (2.1). Thus, by Theorem 2.3 we obtain $\overline{\lambda}_{[p,q]}(g_f-\varphi)=\lambda_{[p,q]}(g_f-\varphi)=\rho_{[p,q]}(f)=\infty$ and

$$\overline{\lambda}_{[p+1,q]}(g_f-\varphi)=\lambda_{[p+1,q]}(g_f-\varphi)=\rho_{[p+1,q]}(f)=\rho_{[p,q]}(A_0).$$

□

Proof of Corollary 2.6

Proof. Suppose that f is a nontrivial solution of (2.8). Then by Lemma 3.7, we have

$$\rho_{[p,q]}(f)=\infty \text{ and } \rho_{[p+1,q]}(f)=\rho_{[p,q]}(B).$$

On the other hand, we have

$$g_f=d_2f''+d_1f'+d_0f. \tag{4.9}$$

It follows by Lemma 3.8 that

$$\begin{cases} g_f=\alpha_{0,0}f+\alpha_{1,0}f', \\ g_f'=\alpha_{0,1}f+\alpha_{1,1}f'. \end{cases} \tag{4.10}$$

By (2.3) we obtain

$$\alpha_{i,0}=\begin{cases} d_1-d_2A, & \text{for } i=1, \\ d_0-d_2B, & \text{for } i=0. \end{cases} \tag{4.11}$$

Now, by (2.2) we get

$$\alpha_{i,1}=\begin{cases} \alpha_{1,0}'+\alpha_{1,0}-A\alpha_{1,0}, & \text{for } i=1, \\ \alpha_{0,0}'-B\alpha_{1,0}, & \text{for } i=0. \end{cases}$$

Hence

$$\begin{cases} \alpha_{0,1}=d_2BA-(d_2B)'-d_1B+d_0', \\ \alpha_{1,1}=d_2A^2-(d_2A)'-d_1A-d_2B+d_0+d_1' \end{cases} \tag{4.12}$$

and

$$\begin{aligned} h &= \begin{vmatrix} \alpha_{0,0} & \alpha_{1,0} \\ \alpha_{0,1} & \alpha_{1,1} \end{vmatrix} = -d_2^2B^2-d_0d_2A^2+\left(-d_2d_1+d_1'd_2+2d_0d_2-d_1^2\right)B \\ &\quad +\left(d_2'd_0-d_2d_0'+d_0d_1\right)A+d_1d_2AB-d_1d_2B'+d_0d_2A' \\ &\quad +d_2^2B'A-d_2^2BA'+d_0'd_1-d_0d_1'-d_0^2. \end{aligned}$$

First we suppose that $d_2\not\equiv 0$. By $d_2\not\equiv 0$, $B\not\equiv 0$ and Lemma 3.5 we have $\rho_{[p,q]}(h)=\rho_{[p,q]}(B)>0$. Hence $h\not\equiv 0$. Now suppose $d_2\equiv 0$, $d_1\not\equiv 0$ or $d_2\equiv 0$, $d_1\equiv 0$ and $d_0\not\equiv 0$. Then, by using a similar reasoning as above we get $h\not\equiv 0$. By $h\not\equiv 0$ and (4.10), we obtain

$$f=\frac{\alpha_{1,0}g_f'-\alpha_{1,1}g_f}{h}. \tag{4.13}$$

By (4.9) we have $\rho_{[p,q]}(g_f)\le\rho_{[p,q]}(f)$ $(\rho_{[p+1,q]}(g_f)\le\rho_{[p+1,q]}(f))$ and by (4.13) we have $\rho_{[p,q]}(f)\le\rho_{[p,q]}(g_f)$ $(\rho_{[p+1,q]}(f)\le\rho_{[p+1,q]}(g_f))$. Then $\rho_{[p,q]}(g_f)=\rho_{[p,q]}(f)=\infty$ and $\rho_{[p+1,q]}(g_f)=\rho_{[p+1,q]}(f)=\rho_{[p,q]}(B)$. □

Proof of Corollary 2.7

Proof. Set $w(z)=d_2f''+d_1f'+d_0f-\varphi$. Then, by $\rho_{[p,q]}(\varphi)<\infty$, Lemma 3.4 and Corollary 2.6, we have $\rho_{[p,q]}(w)=\rho_{[p,q]}(g_f)=\rho_{[p,q]}(f)=\infty$ and $\rho_{[p+1,q]}(w)=\rho_{[p+1,q]}(g_f)=\rho_{[p+1,q]}(f)=\rho_{[p,q]}(B)$. To prove $\overline{\lambda}_{[p,q]}(g_f-\varphi)=\lambda_{[p,q]}(g_f-\varphi)=\rho_{[p,q]}(f)=\infty$ and

$$\overline{\lambda}_{[p+1,q]}(g_f-\varphi)=\lambda_{[p+1,q]}(g_f-\varphi)=\rho_{[p+1,q]}(f)=\rho_{[p,q]}(B)$$

we need to prove $\overline{\lambda}_{[p,q]}(w)=\lambda_{[p,q]}(w)=\infty$ and $\overline{\lambda}_{[p+1,q]}(w)=\lambda_{[p+1,q]}(w)=\rho_{[p,q]}(B)$. Using $g_f=w+\varphi$, we get from (4.13)

$$f=\frac{\alpha_{1,0}w'-\alpha_{1,1}w}{h}+\psi, \tag{4.14}$$

where

$$\psi(z)=\frac{\alpha_{1,0}\varphi'-\alpha_{1,1}\varphi}{h}. \tag{4.15}$$

Substituting (4.14) into equation (2.8), we obtain

$$\frac{\alpha_{1,0}}{h}w'''+\phi_2 w''+\phi_1 w'+\phi_0 w=\left(\psi''+A(z)\psi'+B(z)\psi\right)=C, \tag{4.16}$$

where ϕ_j $(j=0,1,2)$ are meromorphic functions with $\rho_{[p,q]}(\phi_j)<\infty$ $(j=0,1,2)$. First, we prove that $\psi\not\equiv 0$. Suppose that $\psi\equiv 0$. Then by (4.15), we obtain

$$\alpha_{1,1}=\alpha_{1,0}\frac{\varphi'}{\varphi}. \tag{4.17}$$

Hence, by Lemma 3.1 we have

$$m(r,\alpha_{1,1})\leq m(r,\alpha_{1,0})+O\left(\exp_{p-1}\left\{(\mu+\varepsilon)\log_q r\right\}\right)\quad\left(\rho_{[p,q]}(\varphi)=\mu<\infty\right),$$

outside of a possible exceptional set E of finite linear measure, that is

$$m(r,d_2A^2-(d_2A)'-d_1A-d_2B+d_0+d_1')\leq m(r,d_1-d_2A)$$
$$+O\left(\exp_{p-1}\left\{(\mu+\varepsilon)\log_q r\right\}\right),\ r\notin E. \tag{4.18}$$

(i) If $d_2\not\equiv 0$, then we obtain the contradiction

$$\rho_{[p,q]}(B)\leq\rho_{[p,q]}(A)$$

when $\rho_{[p,q]}(A)<\rho_{[p,q]}(B)$ and we obtain the contradiction

$$\tau_{[p,q]}(B)\leq\tau_{[p,q]}(A)$$

when $\rho_{[p,q]}(A)=\rho_{[p,q]}(B)$.
(ii) If $d_2\equiv 0$ and $d_1\not\equiv 0$, then we obtain the contradiction

$$\rho_{[p,q]}(A)\leq\rho_{[p,q]}(d_1).$$

(iii) If $d_2=d_1\equiv 0$ and $d_0\not\equiv 0$, then we have by (4.17)

$$d_0=0\times\frac{\varphi'}{\varphi}\equiv 0,$$

which is a contradiction. It is clear now that $\psi\not\equiv 0$ cannot be a solution of (2.8) because $\rho_{[p,q]}(\psi)<\infty$. Hence $C\not\equiv 0$. By Lemma 3.3, we obtain $\overline{\lambda}_{[p,q]}(w)=\lambda_{[p,q]}(w)=\infty$ and $\overline{\lambda}_{[p+1,q]}(w)=\lambda_{[p+1,q]}(w)=\rho_{[p,q]}(B)$, i.e., $\overline{\lambda}_{[p,q]}(g_f-\varphi)=\lambda_{[p,q]}(g_f-\varphi)=\rho_{[p,q]}(f)=\infty$ and $\overline{\lambda}_{[p+1,q]}(g_f-\varphi)=\lambda_{[p+1,q]}(g_f-\varphi)=\rho_{[p+1,q]}(f)=\rho_{[p,q]}(B)$. □

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Department of Mathematics, Laboratory of Pure and Applied Mathematics, University of Mostaganem (UMAB), B. P. 227 Mostaganem-Algeria
E-mail address: belaidi@univ-mosta.dz