ON EXPONENTIAL EDGE DOMINATION NUMBER OF A GRAPH

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ABSTRACT. The domination number is an important vulnerability parameter that it has become one of the most widely studied topics in graph theory, and also most often studied property of vulnerability of communication networks. Recently, Dankelmann et al. defined the exponential domination number in [11]. We investigate a refinement that involves the edge exponential domination number of this parameter. Let $G = (V(G), E(G))$ be a simple graph. An exponential edge dominating set of graph G is a kind of distance edge domination subset $D \subseteq E(G)$ such that $\sum_{e \in S}(1/2)^{\overline{d}(e,f)} \geq 1, \forall e \in E(G)$, where $\overline{d}(e, f)$ is the length of a shortest path in $\langle E(G) - (D - \{e\})\rangle$ if such a path exist, and ∞ otherwise. The minimum exponential edge domination number, $\gamma'_e(G)$ is the smallest cardinality of an exponential edge dominating set. In this paper, the above mentioned new parameter is defined and examined. Then upper bounds, lower bounds and exact formulas are obtained for any graph G. Finally, the exact values have been determined for some well-known graph families.

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1. Introduction

Vulnerability is an important concept in network analysis. The vulnerability of a communication network is defined as the measurement of the global strength of its underlying graph. The design of a good communication network must take into account notions such as reliability and vulnerability. When the network requirements are expressed in terms of graph theoretical parameters, the problem of analysis and design of networks becomes finding a graph G satisfying some specified requirement [6, 7].

The communication systems are often exposed to failures and attacks. The vulnerability value of a communication network shows the resistance of the network after the disruption of some centers or connection lines until a communication breakdown [6, 7]. In the literature, various measures have been defined to measure the robustness of network and a variety of graph theoretic parameters have been used to derive formulas to calculate network vulnerability. Graph vulnerability relates to the study of graph when some of its elements (vertices or edges) are removed. The measures of graph vulnerability are usually invariants that measure how a deletion of one or more network elements changes properties of the network. The best known measure of reliability of a graph is its connectivity. The vertex (edge) connectivity is defined to be the minimum number of vertices (edges) whose deletion results in a disconnected or trivial graph [5]. The vertex and edge connectivity are denoted by $k(G)$ and $k'(G)$, respectively. Then toughness [19], integrity [3], domination number [4, 13], and edge domination number [12], etc. have been proposed for measuring the vulnerability of networks. Recently, some average vulnerability parameters such as average lower independence number [2, 8], average lower domination number [8, 18], average

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lower 2-domination number [17], average connectivity number [9], average lower bondage number [15] and average lower reinforcement number [16] have been defined.

Let $G = (V(G), E(G))$ be a simple undirected graph of order n. We begin by recalling some standard definitions that we need throughout this paper. For any vertex $v \in V(G)$, the *open neighborhood* of v is $N_G(v) = \{u \in V(G)|uv \in E(G)\}$ and *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$. The degree of v in G denoted by $deg(v)$, is the size of its open neighborhood [4, 13]. The *distance* $d(u, v)$ between two vertices u and v in G is the length of a shortest path between them. The *diameter* of G, denoted by $diam(G)$ is the largest distance between two vertices in $V(G)$ [4, 13]. Let $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ be two edges of G. The distance between e_1 and e_2 is defined as $d(e_1, e_2) = min\{(u_1, u_2), (u_1, v_2), (v_1, u_2), (v_1, v_2)\}$. If $d(e_1, e_2) = 0$, then these edges are called neighbour edges [13].

A set $S \subseteq V(G)$ is a *dominating set* if every vertex in $V(G) - S$ is adjacent to at least one vertex in S. The minimum cardinality taken over all dominating sets of G is called the *domination number* of G is denoted by $\gamma(G)$ [4, 13]. The concept of edge domination was introduced by Mitchell and Hedetniemi [12]. A subset D of E is called an edge dominating set of G if every edge not in D is adjacent to some edge in D. The edge domination number $\gamma'(G)$ of G is the minimum cardinality taken over all edge dominating sets of G. The literature on domination has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [13, 14].

There are different application of domination problems. For instance, dominating sets in graphs are natural models for facility location problems in operations research [13] or domination number is the one of the most important vulnerability parameter for networks [13, 18].

In 2009, Dankelmann introduced the concept of *exponential domination*[11]. This new parameter is closely in relation with distance of each pair of vertices. The exponential domination number is the theoretical vulnerability parameters for a network that is represented by a graph [1, 11]. An exponential dominating set of graph G is a kind of distance domination subset $S \subseteq V(G)$ such that $\sum_{v \in S}(1/2)^{\overline{d}(u,v)-1} \geq 1, \forall v \in V(G)$, where $\overline{d}(u, v)$ is the length of a shortest path in $\langle V(G) - (S - \{u\})\rangle$ if such a path exist, and ∞ otherwise. The minimum exponential domination number, $\gamma_e(G)$ is the smallest cardinality of an exponential dominating set. We call such an edge set is a minimum exponential set which is denoted by γ_e-set.

Our aim in this paper is to define a new vulnerability parameter, so called exponential edge domination number. In Section 2, some well-known basic results are given for exponential domination number. In Section 3, we define a new parameter namely as exponential edge domination number denoted by $\gamma'_e(G)$. In Section 4, we determine upper bounds, lower bounds and exact solutions of the exponential edge domination number for any graph G. Finally, the exponential edge domination numbers of the popular well-known graphs are computed in Section 5.

2. Basic Results

In this section some well-known basic results are given with regard to exponential domination number.

Theorem 2.1. [11] *The exponential domination number of*

(a) *the path graph P_n of order $n \geq 1$ is $\gamma_e(P_n) = \lceil \frac{n+1}{4} \rceil$.*

(b) *the cycle graph C_n of order $n \geq 3$ is* $\gamma_e(C_n) = \begin{cases} 2 & , if\ n = 4; \\ \lceil \frac{n}{4} \rceil & , if\ n \neq 4. \end{cases}$

Theorem 2.2. [11] *For every graph G, $\gamma_e(G) \leq \gamma(G)$, and also $\gamma_e(G) = 1$ if and only if $\gamma(G) = 1$.*

Theorem 2.3. [1] *Let G be any connected graph with n vertices and $\exists v \in V(G)$ such that $deg(v) = n-1$. Then, $\gamma_e(G) = 1$*

Theorem 2.4. [11] *If G is a connected graph of diameter d, then $\gamma_e(G) \geq \lceil \frac{d+2}{4} \rceil$.*

Theorem 2.5. [11] *If G is a connected graph of order n, then $\gamma_e(G) \leq \frac{2}{5}(n + 2)$.*

Theorem 2.6. [11] *Let G be a connected graph of order n and T be a spanning tree of G. Then $\gamma_e(G) \leq \gamma_e(T)$.*

3. The Exponential Edge Domination Number

In this section, we introduce a new graph theoretical parameter: the *exponential edge domination number*. For this new parameter we are inspired by the notion of exponential domination number. Let G be a graph, and let $D \subseteq E(G)$. We denote by $\langle D \rangle$ the subgraph of G induced by D. For each edge $e \in D$ and for each $f \in E(G) - D$, we define $\overline{d}(e, f) = \overline{d}(f, e)$ to be the length of the shortest path in $\langle E(G) - (D - \{e\}) \rangle$ if such a path exist, and ∞ otherwise. Let $w_D(e)$ be the weight of D at the edge e. It is defined as follows:

$$w_D(e) = \begin{cases} \sum_{e \in S} (1/2)^{\overline{d}(e,f)} & ,\text{if } e \notin D; \\ 2 & ,\text{if } e \in D. \end{cases}$$

If, for each $e \in E(G)$, we have $w_D(e) \geq 1$, then D is an exponential edge set. The smallest cardinality of an exponential edge dominating set is the exponential edge domination number, $\gamma'_e(G)$, and such a set is a minimum exponential edge dominating set, or $\gamma'_e(G)$-set for short.If $e \in D$, $f \in E(G) - D$ and $(1/2)^{\overline{d}(e,f)} \geq 1$, then we say that e exponentially edge dominates f. Note that if D is an exponential edge dominating set, then every edge of $E(G) - D$ is exponentially edge dominated, but the converse is not true.

Example 3.1. *Let we calculate the exponential edge domination number of the graph P_8 in Figure 1.*

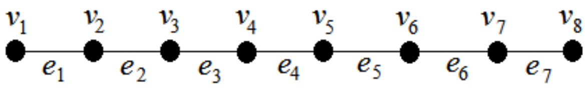

Figure 1. The graph P_8

The Tables 1 and 2 show us the weight of D_1 and D_2 at all edges of the graph P_8, where $D_1 = \{e_1, e_4, e_6\}$ and $D_2 = \{e_2, e_6\}$, respectively.

Table 1. Weight of D_1 at all edges of the graph P_8

e	$\overline{d}(e, e_1)$	$\overline{d}(e, e_4)$	$\overline{d}(e, e_6)$	$w_{D_1}(e)$
e_1	$-$	$-$	$-$	2
e_2	0	1	∞	1.5
e_3	1	0	∞	1.5
e_4	$-$	$-$	$-$	2
e_5	∞	0	0	2
e_6	$-$	$-$	$-$	2
e_7	∞	∞	0	1

For the sets D_1 and D_2, $\forall e \in E(P_8)$, $w_{D_1}(e) \geq 1$ and $w_{D_2}(e) \geq 1$ are satisfied. So, the sets D_1 and D_2 are two exponential edge dominating set.

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TABLE 2. Weight of D_2 at all edges of the graph P_8

e	$\overline{d}(e,e_2)$	$\overline{d}(e,e_6)$	$w_{D_2}(e)$
e_1	0	∞	1
e_2	–	–	2
e_3	0	2	1.25
e_4	1	1	1
e_5	2	0	1.25
e_6	–	–	2
e_7	∞	0	1

Similarly, we can get a lot of exponential edge dominating sets of the graph P_8, but for exponential edge domination number we need the minimum cardinality of among all exponential edge dominating sets. Then, we have

$$\gamma_e'(P_8) = min\{|D_1|, |D_2|\} = min\{3, 2\} = 2.$$

If we think a graph as a modeling of network, the exponential edge domination number may be more sensitive than other measures of vulnerability as like connectivity, edge connectivity, domination number, edge domination number and exponential domination number for distinguish two graphs whose number of the vertices and edges are the same. For example, consider two graphs G_1 and G_2 in Figure 2, where $|V(G_1)| = |V(G_2)| = 10$ and $|E(G_1)| = |E(G_2)| = 11$. They have not only equal connectivity but also equal edge connectivity, domination number, edge domination number and exponential domination number such as $k(G_1) = k(G_2) = 1$, $k'(G_1) = k'(G_2) = 1$, $\gamma(G_1) = \gamma(G_2) = 3$, $\gamma'(G_1) = \gamma'(G_2) = 3$ and $\gamma_e(G_1) = \gamma_e(G_2) = 2$.

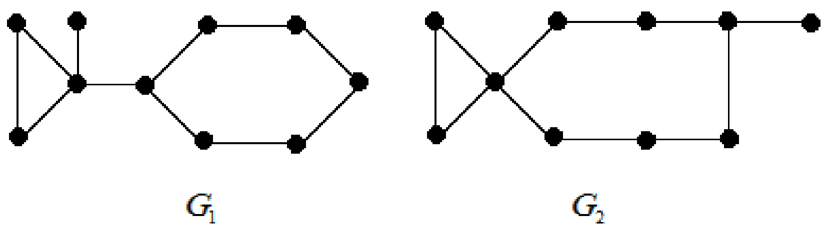

FIGURE 2. The graphs G_1 and G_2

These values could be easily checked by readers. So, how can we distinguish between the graphs G_1 and G_2?
When the exponential edge domination numbers of these two graphs G_1 and G_2 are computed, $\gamma_e'(G_1) = 3$ and $\gamma_e'(G_2) = 2$ are obtained. The results could be checked by readers. Thus, the exponential edge domination number may be used for distinguish between these two graphs G_1 and G_2.

4. Some General Results

Theorem 4.1. *For every graph G, $\gamma_e'(G) \leq \gamma'(G)$, and also $\gamma_e'(G) = 1$ if and only if $\gamma'(G) = 1$.*

Proof. Because of the definition of edge domination number and exponential edge domination number, the proof is clear. □

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Theorem 4.2. *Let G be a graph of order n. If G has at least one vertex with degree $(n-1)$, then $\gamma'_e(G) \leq 2$.*

Proof. Let v_c be center vertex with degree $n-1$, and let e_1 and v_1 be edge which is incident to v_c, also be vertex which incident to e_1, respectively. Clearly, the edge e_1 exponentially dominates all edges which are incident to the vertex v_c and all edges which are incident to vertex v_1. If all vertices of G are exponentially dominated with the edge e_1, clearly we have $\gamma'_e(G) = 1$. If all vertices of G are not exponentially dominated by the edge e_1, then e_1 contributes $1/2$ to edges which are not adjacent to the edge e_1. Let e_2 be any edge which is adjacent to v_c. If the edge e_2 is added to exponential edge dominating set, then clearly e_2 contributes $1/2$ to edges which are not adjacent to the edge e_1. As a result, we get $\gamma'_e(G) \leq 2$.
The proof is completed. □

Theorem 4.3. *Let G be a graph of order n. If G has at least two vertices whose degree $(n-1)$, the minimum degree $\delta(G) = n-2$ and $|E(G)| \geq \frac{n^2-3n+4}{2}$, then $\gamma'_e(G) = 2$.*

Proof. Let v_1 and v_2 be two vertices with degree $n-1$, and let e_x be edge between the vertices v_1 and v_2. If the edge e_x is added to the set D, then $2n-4$ edges are exponentially dominated by e_x, where D is a minimum exponentially edge dominating set. Due to the minimum degree $\delta(G) = n-2$ and $|E(G)| \geq \frac{n^2-3n+4}{2}$, all vertices exponentially edge dominated if any edge of the graph G is added to the set D. So, we get $\gamma'_e(G) = 2$.
The proof is completed. □

5. The Exponential Edge Domination Number of Some Well-Known Graphs

In this section we calculate the exponential edge domination number of some well known graphs such as the path graph P_n, the cycle graph C_n, the complete graph K_n, the star graph $S_{1,n}$ and the wheel graph $W_{1,n}$.

Theorem 5.1. *The exponential edge domination number of the path graph P_n of order $(n \geq 2)$ is given by $\gamma'_e(P_n) = \lceil \frac{n}{4} \rceil$.*

Proof. Let D be a minimum exponential dominating set of P_n, and also let $V(P_n) = \{v_0, v_1, ..., v_{n-1}\}$ and $E(P_n) = \{e_0, e_1, ..., e_{diam(P_n)-1}\}$, respectively. Let $D = \{e_{4i+1} | i \in \{0, ..., \lfloor \frac{diam(P_n)-2}{4} \rfloor\}$. Any edge e in D dominates all adjacent edges. Consider the edges $f \in (E(P_n) - N_{P_n}[e])$. These edges are at distance 1 to exactly two edges in D. This implies $w_D(e) >\geq 1$ for all $e \in E(P_n)$. Thus, the edges of D dominate either all edges of $E(P_n)$ or some edges of $E(P_n)$ which are not dominated, and then we have $\gamma'_e(P_n) \geq |D| = 1 + \lfloor \frac{diam(P_n)-2}{4} \rfloor$.

Let D^* be a minimum exponential edge dominating set of P_n, and also D^* contains all edges of D. We have four cases depending on n.

Case 1. $n \equiv 0(mod4)$.
Clearly, we get $\gamma'_e(P_n) \geq |D^*| = 1 + \frac{diam(P_n)-3}{4} = \frac{diam(P_n)+1}{4}$.

Case 2. $n \equiv 1(mod4)$.
An edge e_{n-2} is not dominated by the edges of D^*. Thus, this edge must be added to D^*, and then we have

$$\gamma'_e(P_n) \geq |D^*| = 1 + 1 + \frac{diam(P_n) - 4}{4} = \frac{diam(P_n) + 4}{4}.$$

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Case 3. $n \equiv 2(mod4)$.

Two edges e_{n-2} and e_{n-3} are not dominated by the edges of D^*. Thus, one of them must be added to D^*, and then we have

$$\gamma'_e(P_n) \geq |D^*| = 1 + 1 + \frac{diam(P_n) - 5}{4} = \frac{diam(P_n) + 3}{4}.$$

Case 4. $n \equiv 3(mod4)$.

Clearly, we get $\gamma'_e(P_n) \geq |D^*| = 1 + \frac{diam(P_n)-2}{4} = \frac{diam(P_n)+2}{4}$.

It is easy to say that from the Case 1, $\{e_1\}$ is minimum exponential edge dominating set of P_4 and $\gamma'_e(P_4) = 1$. Furthermore, if $n > 4$ and D^* is a minimum exponential edge dominating set of P_n, then $D^* \cap \{e_0, e_1, e_2\}$ is an exponential edge dominating set of P_4 and $D^* - \{e_0, e_1, e_2\}$ is an exponential edge dominating set of P_{n-4}. By an inductive argument, we obtain

$$\gamma'_e(P_n) \leq \gamma'_e(P_4) + \gamma'_e(P_{n-4}).$$

From the Case 1, we have

$$\frac{n}{4} \leq \gamma'_e(P_n) \leq 1 + \frac{n - 5 + 1}{4} = \frac{n}{4}.$$

Thus, we obtain $\gamma'_e(P_n) = \frac{n}{4}$.

Examining the other cases as above, we obtain $\gamma'_e(P_n) = \frac{n+3}{4}$ from the Case 2, $\gamma'_e(P_n) = \frac{n+2}{4}$ from the Case 3 and $\gamma'_e(P_n) = \frac{n+1}{4}$ from the Case 4.

Consequently, we obtain $\gamma'_e(P_n) = \lceil \frac{n}{4} \rceil$.

The proof is completed. □

Theorem 5.2. *The exponential edge domination number of the cycle graph C_n of order $(n \geq 5)$ is given by $\gamma'_e(C_n) = \lceil \frac{n}{4} \rceil$.*

Proof. The proof of Theorem 5.2 is very similar to the proof of Theorem 5.1. □

Theorem 5.3. *The exponential edge domination number of the wheel graph $W_{1,n}$ of order $(n + 1)$ is given by $\gamma'_e(W_{1,n}) = 2$.*

Proof. By the Theorem 4.2, we have $\gamma'_e(W_{1,n}) \leq 2$. On the other hand, by the Theorem 4.1, we have $\gamma'_e(W_{1,n}) \geq 2$. Because of the edge domination number of $W_{1,n}$ is not equal 1. So, $\gamma'_e(W_{1,n}) = 2$ is obtained.

The proof is completed. □

Theorem 5.4. *The exponential edge domination number of the star graph $S_{1,n}$ of order $(n+1)$ is given by $\gamma'_e(S_{1,n}) = 1$.*

Proof. By the Theorem 4.1, the proof of Theorem 5.4 is clear. □

Theorem 5.5. *The exponential edge domination number of the complete graph K_n of order n is given by $\gamma'_e(K_n) = 2$.*

Proof. By the Theorem 4.3, the proof of Theorem 5.5 is clear. □

6. Conclusion

In this study, a new graph theoretical parameter namely the exponential edge domination number has been presented for the network vulnerability. The stability of popular interconnection networks which are complete graphs, the path graphs, the cycle graphs, the star graphs and the wheel graphs has been studied and their exponential edge domination numbers have been computed. As a further study, exact formulas or bounds may be obtained for any graph G, graph operations and trees.

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