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On δ-Primary Ideals of Commutative Semirings

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Abstract In this paper we introduce the notion of a δ-zero-divisor of a commutative semiring and we also study some of it properties. Here δ is a mapping that assigns to each ideal I an ideal $\delta(I)$ of the same semiring. We analyze possible structures of δ-semidomain and relationships between semirings that share some properties with δ-semidomains, but whose definitions are less restrictive. We also investigate δ-primary ideals of a commutative semiring R which unify prime ideals and primary ideals of R.

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1 Introduction

As a generalization of rings, the representation theory of semirings has developed greatly in the recent years. The structure of semirings have proven to be useful tools in various disciplines. They are used to test conjectures by searching large lists for counterexamples, but also to find structures with some specified properties, like e.g. some molecules for a given chemical formula or networks with certain properties [9].

The study of the set of zero-divisor elements of a commutative ring can often be a frustrating one. Almost immediately one runs into the ugly issue of a profound lack of algebraic structure, highlighted by (typically) a lack of closure under addition. This unfortunate lack of algebraic structure is most disturbing in such an important subset within a ring. In recent years, however, the study of zero divisors has been energized by a lovely collaboration with the tools and methods of graph theory. We are motivated in this regard by the recent success of studies of the notion of domainlike commutative rings in papers [2, 3, 7] because they are useful rings for studying factorization in commutative rings with zero-divisors. The factorization of nonunits into atoms is a central theme in algebra. Classically the theory has concentrated on integral domains. Much of this theory generalizes to the case of rings with zero-divisors, but important differences remain (see [2, 3]). Therefore, zero-divisor elements of a commutative ring (resp. a commutative semiring) are very important. One point of this paper is to introduce δ-zero-divisor element of a commutative semiring.

δ-primary ideals of a commutative ring were introduced by Dongsheng in [8], where δ is a mapping with some additional properties. Such δ-primary ideals unify the prime and primary ideals under one frame. The aim of the present paper, in Section 2, is to generalize the results in the paper [8], from commutative ring theory to commutative semiring theory. In Section 3, we introduce δ-zero-divisor element of a commutative semiring, and we focus on a class of semirings, representable by the property that every δ-zero-divisor of the semiring is δ-nilpotent. In fact, we establish a connection between δ-semidomainlike semiring and δ-semidomain (see Section 2 and Section 3).

For the sake of completeness, we state some definitions and notation used throughout. By a commutative semiring, we mean a commutative semigroup $(R, \cdot)$ and a commutative monoid $(R, +, 0)$ in which 0 is the additive identity and $r \cdot 0 = 0 \cdot r = 0$ for all $r \in R$, both are being connected by ring-like distributivity. In this paper, all semirings considered will be assumed to be commutative semirings with non-zero identity. A semiring R is said to be a semidomain if $ab = 0$ $(a, b \in R)$, then either $a = 0$ or $b = 0$.

Definition 1.1 *Let R be a commutative semiring with non-zero identity.*

(1) A subset I of R will be called an ideal if $a, b \in I$ and $r \in R$ implies $a + b \in I$ and $ra \in I$.

(2) A subtractive ideal (= k-ideal) I is an ideal such that if $x, x + y \in I$ then $y \in I$ (so $\{0\}$ is a k-ideal of R).

(3) The k-closure $\mathrm{cl}(I)$ of I is defined by $\mathrm{cl}(I) = \{a \in R : a + c = d \text{ for some } c, d \in I\}$ is an ideal of R satisfying $I \subseteq \mathrm{cl}(I)$ and $\mathrm{cl}(\mathrm{cl}(I)) = \mathrm{cl}(I)$.

(4) If I is an ideal of R, then the radical of I, denoted by $\sqrt{I}$, is the set of all $x \in R$ for which $x^n \in I$ for some positive integer n. This is an ideal of R contains I.

(5) A prime ideal of R is a proper ideal I of R in which $x \in I$ or $y \in I$ whenever $xy \in I$. A proper ideal I of R is called primary if $ab \in I$, then $a \in I$ or $b \in \sqrt{I}$.

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(6) A proper ideal I of R is said to be maximal (resp k-maximal) if for any ideal J (resp. k-ideal) in R with $I \subsetneq J$, one has that $J = R$.

(7) An ideal I of a semiring R is called a partitioning ideal (= Q-ideal) if there exists a subset Q of R such that $R = \cup\{q+I : q \in Q\}$ and if $q_1, q_2 \in Q$, then $(q_1 + I) \cap (q_2 + I) \neq \emptyset$ if and only if $q_1 = q_2$.

Let I be a Q-ideal of R and let $R/I = \{q+I : q \in Q\}$. Then R/I forms a semiring under the operations $\oplus$ and $\odot$ defined as follows: $(q_1+I)\oplus(q_2+I) = q_3 + I$, where $q_3 \in Q$ is the unique element such that $q_1 + q_2 + I \subseteq q_3 + I$ and $(q_1 + I) \odot (q_2 + I) = q_4 + I$, where $q_4 \in Q$ is the unique element such that $q_1q_2 + I \subseteq q_4 + I$. This semiring R/I is called the quotient semiring of R by I [1, 4].

2 Notation and basic structure

Let R be a commutative semiring with $\mathrm{Id}(R)$ its set of ideals, $\mathrm{Id}_k(R)$ its set of k-ideals, and $\mathrm{Id}_q(R)$ its set of Q-ideals. Since every Q-ideal is a k-ideal, $\mathrm{Id}_q(R) \subseteq \mathrm{Id}_k(R) \subseteq \mathrm{Id}(R)$. Our starting point is the following definition.

Definition 2.1 *(1) An ideal expansion is a function δ which assigns to each ideal I of a semiring R another ideal $\delta(I)$ of the same semiring, such that $I \subseteq \delta(I)$, and $J \subseteq L$ implies $\delta(J) \subseteq \delta(L)$ for all ideals I, J and L of R [8].*

(2) A Q-ideal expansion is a function δ which assigns to each Q-ideal I of a semiring R another Q-ideal $\delta(I)$ of the same semiring, such that $I \subseteq \delta(I)$, and $J \subseteq L$ implies $\delta(J) \subseteq \delta(L)$ for all Q-ideals I, J and L of R.

Definition 2.2 *Given an expansion δ of ideals, a proper ideal I of a semiring R is called δ-primary if $ab \in I$ and $a \notin I$, then $b \in \delta(I)$ [8].*

It is clear that the definition of δ-primary ideals can be also stated as: If $ab \in I$ and $a \notin \delta(I)$, then $b \in I$.

Remark 2.3 *Let R be a commutative semiring.*

(1) The identity function δ_0, where $\delta_0(I) = I$ (resp. $\delta_0^q(I) = I$) for every $I \in \mathrm{Id}(R)$ (resp. for every $I \in \mathrm{Id}_q(R)$), is an expansion of ideals. So an ideal I is δ_0-primary (resp. δ_0^q-primary) if and only if it is a prime ideal (resp. it is a prime Q-ideal).

(2) For each $I \in \mathrm{Id}(R)$ (resp. for each $I \in \mathrm{Id}_q(R)$) define $\delta_1(I) = \sqrt{I}$ (resp. $\delta_1^q(I) = \sqrt{I}$). Then δ_1 (resp. δ_1^q) is an expansion of ideals. So an ideal

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I is δ_1-primary (resp. δ_1^q-primary) if and only if it is a primary ideal (resp. it is a primary Q-ideal); hence the intersection of two δ-primary ideals is not δ-primary, in general.

(3) For each $I \in \mathrm{Id}(R)$ define $\delta_2(I) = \mathrm{cl}(I)$, the k-closure of I. Then δ_2 is an expansion of ideals.

(4) The function δ_3 that assigns the biggest ideal of R to each ideal is an expansion of ideals. So every ideal I is δ_3-primary.

(5) By [6, Theorem 1], if I is a proper Q-ideal of R , then there exists a maximal k-ideal M of R such that $I \subseteq M$. Now for each proper Q-ideal I, let $\delta_4^q(I)$ be the intersections of all maximal k-ideals containing I, and $\delta_4^q(R) = R$. Then δ_4^q is an expansion of Q-ideals.

(6) Let I be an ideal of R and set

$$\bar{I} = \{x \in R : a + nx = (n+1)x \ for\ some\ positive\ integer\ n\ and\ a \in I\}.$$

Let $x, y \in \bar{I}$ and $r \in R$; so $a + nx = (n+1)x$ and $b + my = (m+1)y$ for some positive integers m, n and $a, b \in I$. Then $a + b + (m+n)(x+y) = (m+n+1)(x+y)$ and $ra + n(rx) = (n+1)rx$ gives $\bar{I}$ is an ideal of R with $I \subseteq \bar{I}$. For each $I \in \mathrm{Id}(R)$ define $\delta_5(I) = \bar{I}$. Then δ_5 is an expasion of ideals.

(7) If δ and γ are two ideal expansions and $\delta(I) \subseteq \gamma(I)$ for each ideal I, then every δ-primary ideal is also γ-primary. In particular, a prime ideal is δ-primary for every δ.

(8) An inspection will show that the intersection of any collection of ideal expansions is an ideal expansion.

(9) Given an expansion δ of ideals. Define $E_\delta : \mathrm{Id}(R) \to \mathrm{Id}(R)$ by $E_\delta(I) = \bigcap\{J \in \mathrm{Id}(R) : I \subseteq J,\ J$ is δ-primary $\}$. Then E_δ is an ideal expansion. Clearly, $E_{\delta_0} = \delta_1$, $E_{\delta_1} = \delta_1$ and $E_{\delta_4^q} = \delta_4^q$.

The proof of the following proposition is straightforward, but we give the details for convenience.

Proposition 2.4 *Assume that R is a commutative semiring and let δ be an ideal expansion. Then the following hold:*

(1) An ideal I is δ-primary if and only if for any two ideals J and L, if $JL \subseteq I$ and $J \nsubseteq I$, then $L \subseteq \delta(I)$.

(2) If I is δ-primary and T is a subset of R, then $(I : T) = \{r \in R : rT \subseteq I\}$ is δ-primary. Moreover, if T is an ideal of R with $T \nsubseteq \delta(I)$, then $(I : T) = I$.

(3) If $\delta(I) \subseteq \sqrt{I}$ for every δ-primary ideal I, then $\delta(I) = \sqrt{I}$

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(4) if $\{I_i : i \in \Lambda\}$ is a directed collection of δ-primary ideals of R, then $I = \cup_{i\in\Lambda} I_i$ is δ-primary.

Proof. (1) Suppose that I is a δ-primary ideal and let $J \nsubseteq I$ and $L \nsubseteq \delta(I)$. Then we can chose $x \in J - I$ and $y \in L - \delta(I)$. By assumption, $xy \in I$; so either $x \in I$ or $y \in \delta(I)$ which is a contradiction. The other implication can similarily be proved.

(2) Let $xy \in (I : T)$ such that $x \notin (I : T)$. Then there exists $z \in T$ such that $xz \notin I$. As $xyz \in I$ and $I \subseteq (I : T)$, I δ-primary gives $y \in \delta(I) \subseteq \delta((I : T))$. Thus $(I : T)$ is δ-primary. Finally, it suffices to show that $(I : T) \subseteq I$. Since $T(I : T) \subseteq I$ and $T \nsubseteq \delta(I)$, we must have $(I : T) \subseteq I$ by (1), as required.

(3) Let $x \in \sqrt{I}$. Then $x^n \in I$ for some the least positive integer n. We may assume that $n > 1$. Now $x^n \in I$ with $x^{n-1} \notin I$ gives $x \in \delta(I)$, and so we have equality.

(4) Let $xy \in I$ and $x \notin I$. Then there exists a I_i such that $xy \in I_i$ with $x \notin I_i$. So $y \in \delta(I_i)$; hence $\delta(I_i) \subseteq \delta(I)$ gives $y \in \delta(I)$, as needed. □

Let R be a semiring. An ideal expansion δ is said to be intersection preserving if $\delta(I \cap J) = \delta(I) \cap \delta(J)$ for all ideals I and J of R [8]. An expansion δ is said to be global if for any semiring homomorphism $f : R \to R'$, $\delta(f^{-1}(I)) = f^{-1}(\delta(I))$ for all ideal I of R [8]. It is clear that δ_0, δ_1 and δ_3 are both intersection preserving and global. By an argument like that in [8, Lemma 2.2, Lemma 2.4, Lemma 2.6 and Proposition 2.7], we have the following proposition:

Proposition 2.5 *Let R be a semiring. Then the following hold:*

(1) The ideal expansion δ_4^q is intersection preserving.

(2) Assume that δ is an intersection preserving ideal expansion and let $I_1, ..., I_n$ be δ-primary ideals of R with $\delta(I_i) = \delta(I_j)$ for all i, j. Then $\cap_{i=1}^n I_i$ is δ-primary.

(3) If δ is global and $f : R \to S$ is a surjective semiring homomorphism, then an ideal I of R that contains $\ker(f)$ is δ-primary if and only if $f(I)$ is a δ-primary ideal of S.

3 Some basic properties of δ-zero-divisors

Let R be a commutative semiring with an ideal expansion δ. An element x of R is called δ-nilpotent if $x \in \delta(\{0\})$ [8]. The set of all δ-nilpotent elements of R is denoted by $\mathrm{nil}_\delta(R)$.

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Theorem 3.1 *Let I be a Q-deal of a semiring R with δ^q a global expansion, and let q_0 be the unique element in Q such that $q_0 + I$ is the zero in R/I. Then the following hold:*

(1) $\delta^q(I)/I = \delta^q(\{q_0 + I\})$.

(2) I is δ^q-primary if and only if every zero-divisor of R/I is δ^q-nilpotent.

Proof. (1) Let $v : R \to R/I$ be the natural homomorphism of R onto R/I. One can easily show that $v^{-1}(\{q_0 + I\}) = \{q_0 + a : a \in I\} = q_0 + I = I$. As δ^q is global, we have $\delta^q(I) = \delta^q(v^{-1}(\{q_0 + I\})) = v^{-1}(\delta^q(\{q_0 + I\}))$; hence $\delta^q(I)/I = v(\delta^q(I)) = v(v^{-1}(\delta^q(\{q_0 + I\}))) = \delta^q(\{q_0 + I\})$ since v is onto.

(2) Assume that I is δ^q-primary and let $q_1 + I$ is a zero-divisor of R/I. Then there exists $q_0 + I \neq q_2 + I$ (so $q_2 \notin I$) such that $(q_1 + I) \odot (q_2 + I) = q_0 + I$, where $q_1 q_2 + I \subseteq q_0 + I = I$; hence $q_1 q_2 \in I$ since I is a k-ideal. Now I is a δ^q-primary gives $q_1 \in \delta^q(I)$; so $q_1 + I \in \delta^q(I)/I = \delta^q(\{q_0 + I\})$ by (1). Thus $q_1 + I$ is δ^q-nilpotent. Conversely, let $r = q + a, s = q' + b \in R$ (where $q, q' \in Q$ and $a, b \in I$) with $rs \in I$ and $r \notin I$ (so $q \notin I$). Then $rs \in I$ gives $qq' \in I$ since I is a k-ideal. There exists $t \in Q$ such that $(q + I) \odot (q' + I) = t + I$, where $qq' + I \subseteq t + I$, thus $t \in I$. Therefore $q_0 + I = t + I$ by [5, Lemma 2.3 (ii)]. It follows that $q' + I$ is a zero-divisor of R/I; so $q' + I \in \delta^q(\{q_0 + I\}) = \delta^q(I)/I$. Then there exists $u \in \delta^q(I) \cap Q$ such that $q' + I = u + I$, so $q' = u \in \delta^q(I)$. Thus $b = q' + b \in \delta^q(I)$, as required. $\square$

Let I be a proper Q-ideal of a semiring R. We will now provide necessary and sufficient conditions for ensuring the set of zero-divisors of R/I is an ideal.

Theorem 3.2 *Let I be a proper Q-ideal of a semiring R. Then I is δ^q-primary if and only if $Z(R/I) \subseteq \{q + I : q \in Q \cap \delta^q(I)\} = \delta^q(I)/I$.*

Proof. Let q_0 be the unique element in Q such that $q_0 + I$ is the zero in R/I, and let $M = \{q + I : q \in Q \cap \delta^q(I)\}$. Let $q + I$ be a non-zero element of $Z(R/I)$ (so $q \notin I$ since every Q-ideal is a k-ideal). Then there exists $q' \in Q$ with $q' \notin I$ such that $(q + I) \odot (q' + I) = q_0 + I$, where $qq' + I \subseteq q_0 + I$. Since I is a k-ideal, $qq' \in I$ by [5, Lemma 2.3 (i)]; hence $q \in \delta(I)$ since I is δ^q-primary. Therefore, $Z(R/I) \subseteq M$. Conversely, let $a, b \in R$ such that $ab \in I$ with $a \notin I$. Since I is a Q-ideal, there exist $q_1 + I$ and $q_2 + I$ such that $a \in q_1 + I$ and $b \in q_2 + I$; thus $a = q_1 + c$ (so $q_1 \notin I$) and $b = q_2 + d$ for some $c, d \in I$. Then I is a k-ideal and $ab = cq_2 + dq_1 + cd + q_1 q_2$ gives $q_1 q_2 \in I$. Let q_3 be the unique element in Q such that $(q_1 + I) \odot (q_2 + I) = q_3 + I$,

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where $q_1q_2 + I \subseteq q_3 + I$; hence $q_3 \in I$. Now by [5, Lemma 2.3 (ii)], $q_3 = q_0$, so $q_2 + I \in Z(R/I)$, and therefore $q_2 \in \delta^q(I)$. Thus $b = q_2 + d \in \delta^q(I)$ since it is a k-ideal. Thus I is δ^q-primary. $\square$

Theorem 3.3 *Let I be a proper Q-ideal of a semiring R. Then the following hold:*

(1) I is δ_1^q-primary if and only if $Z(R/I) = \{q + I : q \in Q \cap \delta_1^q(I)\} = \delta_1^q(I)/I$.

(2) If I is a δ_1^q-primary ideal of R, then $Z(R/I)$ is an ideal of R/I.

Proof. (1) By Theorem 3.2, it is sufficient to show that $M = \{q + I : q \in Q \cap \delta_1^q(I)\} \subseteq Z(R/I)$. Let $s + I \in M$, where $s \in Q \cap \delta_1^q(I)$. Since $\delta_1^q(I) = \sqrt{I}$, there exists n which is the least positive integer n with $s^n \in I$. If $n = 1$, then $s + I = q_0 + I \in Z(R/I)$ by [5, Lemma 2.3 (ii)]. If $n > 1$, then $ss^{n-1} \in I$ with $s^{n-1} \notin I$. As I is a Q-ideal, $s^{n-1} = t + a$ for some $t \in Q - I$ and $a \in I$, and so $ss^{n-1} = st + sa$; thus $st \in I$ since I is a k-ideal. Let u be the unique element in Q such that $(s + I) \odot (t + I) = u + I$, where $st + I \subseteq u + I$; so $u \in I$. Now by [5, Lemma 2.3 (ii)], $u = q_0$; hence $s + I \in Z(R/I)$, we have equality.

(2) Let q_0 be the unique element in Q such that $q_0 + I$ is the zero in R/I, and let $t_1 + I, t_2 + I \in Z(R/I)$, $z + I \in R/I$, where $t_1, t_2 \in Q \cap \delta_1^q(I)$ and $z \in Q$. Then there exists $u_1 \in Q - I$ such that $(t_1 + I) \odot (u_1 + I) = q_0 + I$, where $t_1u_1 + I \subseteq q_0 + I$; so $t_1u_1 \in I$ with $u_1 \notin I$. It follows that $t_1 \in \delta_1^q(I)$. Similarly, $t_2 \in \delta_1^q(I)$. Thus $t_1 + t_2 \in \delta_1^q(I)$. There exists n which is the least positive integer n with $(t_1 + t_2)^n \in I$. Let q be the unique element in Q such that $(t_1 + I) \oplus (t_2 + I) = q + I$, where $t_1 + t_2 + I \subseteq q + I$. If $n = 1$, then $t_1 + t_2 \in I$; so $q \in I \subseteq \delta_1^q(I)$. If $n > 1$, then $(t_1 + t_2)(t_1 + t_2)^{n-1} \in I$ with $(t_1 + t_2)^{n-1} \notin I$. As I is a Q-ideal, $(t_1 + t_2))^{n-1} = t + a$ for some $t \in Q - I$ and $a \in I$, and so $(t_1 + t_2)(t_1 + t_2)^{n-1} = t(t_1 + t_2) + a(t_1 + t_2)$; thus $t(t_1 + t_2) \in I$ since I is a k-ideal; hence $t(t_1 + t_2) + I \subseteq tq + I$ gives $tq \in I$ with $t \notin I$. Thus $q \in \delta_1^q(I)$ since I is a δ_1^q-primary. Therefore, $(t_1 + I) \oplus (t_2 + I) \in Z(R/I)$. Similarly, $(t_1 + I) \odot (z + I) \in Z(R/I)$, and this completes the proof. $\square$

Corollary 3.4 *Let I be a proper ideal of a ring R. Then the following hold:*

(1) If δ is an ideal expansion, then I is δ-primary if and only if $Z(R/I) \subseteq \{r + I : r \in \delta(I)\} = \delta(I)/I$.

(2) I is δ_1-primary if and only if $Z(R/I) = \{r + I : r \in \delta_1(I)\} = \delta_1(I)/I$.

(3) If I is a δ_1-primary ideal of R, then $Z(R/I)$ is an ideal of R/I.

Proof. Apply Theorem 3.3 and Theorem 3.2. $\square$

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Definition 3.5 *(1) Let R be a semiring with an ideal expansion δ. A δ-zero-divisors in R is an element $x \in R$ for which there exists $y \in R$ with $y \notin \delta(\{0\})$ such that $xy \in \delta(\{0\})$.*

(2) Let R be a ring with an ideal expansion δ. A δ-zero-divisors in R is an element $x \in R$ for which there exists $y \in R$ with $y \notin \delta(\{0\})$ such that $xy \in \delta(\{0\})$.

Clearly, δ_0-zero-divisor elements are exactly the ordinary zero-divisor elements. The set of δ-zero-divisors in R will be denoted by $Z_\delta(R)$.

Proposition 3.6 *Assume that R is a semiring with non-zero identity, and let δ be an ideal expansion such that $\delta(\{0\}) \neq R$. Then the following hold:*

(1) $\mathrm{nil}_\delta(R)$ is an ideal of R with $\mathrm{nil}_\delta(R) \subseteq Z_\delta(R)$.

(2) If $Z_\delta(R)$ is an ideal of R, then $Z_\delta(R)$ is δ-primary.

Proof. (1) Let $x, y \in \mathrm{nil}_\delta(R)$ and $r \in R$. Then $x, y \in \delta(\{0\})$, so $x + y, rx \in \delta(\{0\})$ since $\delta(\{0\})$ is an ideal of R; hence $x + y, rx \in \mathrm{nil}_\delta(R)$. Thus $\mathrm{nil}_\delta(R)$ is an ideal. Finally, let $a \in \mathrm{nil}_\delta(R)$. Since $a = a1_R \in \delta(\{0\})$ and $1 \notin \delta(\{0\})$, we have $a \in Z_\delta(R)$, as needed.

(2) Let $x, y \in R$ be such that $xy \in Z_\delta(R)$. Then there exists $z \in R$ such that $z \notin \delta(\{0\})$ and $xyz \in \delta(\{0\})$. Therefore, if $yz \in \delta(\{0\})$, then $y \in Z_\delta(R)$. If $yz \notin \delta(\{0\})$, then $x \in Z_\delta(R)$. Thus $Z_\delta(R)$ is a δ-primary ideal of R. $\square$

Theorem 3.7 *Let I be a Q-ideal of a semiring R with δ^q a global expansion. Then $\delta^q(I)$ is δ^q-primary if and only if $Z_\delta(R/I) \subseteq \delta^q(\{q_0 + I\})$*

Proof. Let q_0 be the unique element in Q such that $q_0 + I$ is the zero in R/I, and let $q + I$ be an element of $Z_\delta(R/I)$. Then there exists $q' \in Q$ with $q' + I \notin \delta^q(\{q_0 + I\})$ (so $q' \notin \delta^q(I)$) such that $(q + I) \odot (q' + I) = q_1 + I \in \delta^q(\{q_0 + I\}) = \delta^q(I)/I$, where $qq' + I \subseteq q_1 + I$ and $q_1 \in Q \cap \delta^q(I)$. Since $\delta^q(I)$ is a k-ideal, $qq' \in \delta^q(I)$; hence $q \in \delta^q(I)$ since $\delta^q(I)$ is δ^q-primary and δ^q is a global expansion. Therefore, $Z(R/I) \subseteq \delta^q(I)/I$. Conversely, let $a, b \in R$ such that $ab \in \delta^q(I)$ with $a \notin \delta^q(I)$. Since I is a Q-ideal, there exist $q_1 + I$ and $q_2 + I$ such that $a \in q_1 + I$ and $b \in q_2 + I$; thus $a = q_1 + c$ (so $q_1 \notin \delta^q(I)$; hence $q_1 + I \notin \delta^q(I)/I = \delta^q(\{q_0 + I\})$) and $b = q_2 + d$ for some $c, d \in I \subseteq \delta^q(I)$. Then $\delta^q(I)$ is a k-ideal and $ab = cq_2 + dq_1 + cd + q_1q_2$ gives $q_1q_2 \in \delta^q(I)$. Let q_3 be the unique element in Q such that $(q_1 + I) \odot (q_2 + I) = q_3 + I$, where $q_1q_2 + I \subseteq q_3 + I$; hence $q_3 \in \delta^q(I)$. It follows that $q_3 + I \in \delta^q(\{q_0 + I\})$, and so $q_2 + I \in Z_\delta(R/I)$, and therefore $q_2 \in \delta^q(I)$. Hence $b = q_2 + d \in \delta^q(I)$ since it is a k-ideal. Thus $\delta^q(I)$ is δ^q-primary. $\square$

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Corollary 3.8 *Let I be an ideal of a ring R with δ a global expansion. Then $\delta(I)$ is δ-primary if and only if $Z_\delta(R/I) \subseteq \delta(\{I\})$.*

Proof. Apply Theorem 3.7. $\square$

Definition 3.9 *(1) A semiring R with an ideal expansion δ is called δ-semidomainlike semiring, if $Z_\delta(R) \subseteq \mathrm{nil}_\delta(R)$.*

(2) A ring R with an ideal expansion δ is called δ-domainlike ring, if $Z_\delta(R) \subseteq \mathrm{nil}_\delta(R)$.

Note that some of the expansion ideals satisfy the property $\delta(\delta(I)) = \delta(I)$ for every ideal I of a semiring R, say δ_0, δ_1 and δ_2.

Theorem 3.10 *Let R be a semiring with an ideal expansion δ such that $\delta(\delta(I)) = \delta(I)$ for every ideal I of R. Then the following hold:*

(1) $\delta(\{0\})$ is a δ-primary ideal of R if and only if $Z_\delta(R) = \mathrm{nil}_\delta(R)$. In particular, if $\delta(\{0\})$ is δ-primary, then $Z_\delta(R)$ is a δ-primary ideal of R.

(2) $\delta(\{0\})$ is δ-primary if and only if R is δ-semidomainlike semiring.

(3) If R is δ-semidomainlike semiring, then $Z_\delta(R)$ is the unique minimal δ-primary ideal of R.

Proof. (1) Let $\delta(\{0\})$ be a δ-primary ideal of R. By Proposition 3.6, it is sufficient to show that $Z_\delta(R) \subseteq \mathrm{nil}_\delta(R)$. Let $x \in Z_\delta(R)$. Then $xy \in \delta(\{0\})$ for some $y \notin \delta(\{0\})$. Since $\delta(\{0\})$ is a δ-primary, $y \in \delta(\delta(\{0\})) = \delta(\{0\})$, and so we have equality. Conversely, let $a, b \in R$ such that $ab \in \delta(\{0\})$ but $b \notin \delta(\{0\})$. Then $a \in Z_\delta(R) = \mathrm{nil}_\delta(R)$. So $\delta(\{0\})$ must be a δ-primary ideal of R. (2) follows from (1). To prove (3), as $Z_\delta(R) = \mathrm{nil}_\delta(R)$ by (1), we have that $Z_\delta(R)$ is a δ-primary ideal of R since $\delta(\{0\})$ is δ-primary. Now if J is a δ-primary ideal, then $Z_\delta(R) = \mathrm{nil}_\delta(R) \subseteq J$, as required. $\square$

Definition 3.11 *(1) A commutative semiring R with an ideal expansion δ is called a δ-semidomain if $ab \in \delta(\{0\})$ $(a, b \in R)$, then either $a \in \delta(\{0\})$ or $b \in \delta(\{0\})$.*

(2) A commutative ring R with an ideal expansion δ is called δ-domain if $ab \in \delta(\{0\})$ $(a, b \in R)$, then either $a \in \delta(\{0\})$ or $b \in \delta(\{0\})$.

A classical result of commutative semiring theory is that a Q-ideal I is prime if and only if R/I is a semidomain (see [5, Theorem 2.6]). The following theorem is a corresponding result for δ^q-semidomainlike semirings.

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Theorem 3.12 *Let I be a Q-ideal of a semiring R with δ^q a global expansion such that $\delta^q(\delta^q(I)) = \delta^q(I)$ for every ideal I of R. If $\delta^q(I)$ is δ^q-primary, then R/I is a δ^q-semidomainlike semiring if and only if R/I is δ^q-semidomain.*

Proof. Assume that q_0 is the unique element in Q such that $q_0 + I$ is the zero in R/I and let R/I be a δ^q-semidomainlike semiring; we show that R/I is δ-semidomain. Let $(q_1 + I) \odot (q_2 + I) = q_3 + I \in \delta^q(\{0\}) = \delta^q(I)/I$, where $q_1q_2 + I \subseteq q_3 + I$ and $q_3 \in Q \cap \delta^q(I)$, so $q_1q_2 \in \delta^q(I)$ since $\delta^q(I)$ is a k-ideal. Now $\delta^q(I)$ is a δ^q-primary gives either $q_1 \in \delta^q(I)$ or $q_2 \in \delta^q(\delta^q(I)) = \delta^q(I)$; hence $q_1 + I \in \delta^q(I)/I$ or $q_2 + I \in \delta^q(I)/I$. Conversely, by Proposition 3.5, it is sufficient to show that $Z_\delta(R/I) \subseteq \mathrm{nil}_\delta(R/I)$. Let $t + I \in Z_\delta(R/I)$. Then there exists $u + I \in R/I$ with $u + I \notin \delta^q(\{0\})$ such that $(t + I) \odot (u + I) \in \delta^q(\{0\})$. Now R/I is a δ^q-semidomain, one has that $t + I \in \mathrm{nil}_\delta(R/I)$. Thus R/I is a δ^q-semidomainlike semiring. $\square$

Corollary 3.13 *Let I be an ideal of a ring R with δ a global expansion such that $\delta(\delta(I)) = \delta(I)$ for every ideal I of R. If $\delta(I)$ is δ-primary, then R/I is a δ-domainlike ring if and only if R/I is δ-domain.*

Proof. Apply Theorem 3.12. $\square$

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