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ITERATED FRACTIONAL APPROXIMATION BY MAX-PRODUCT OPERATORS

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Abstract. Here we consider the approximation of functions by sublinear positive operators with applications to a large variety of Max-Product operators under iterated fractional differentiability. Our approach is based on our general fractional results about positive sublinear operators. We produce Jackson type inequalities under iterated fractional initial conditions. So our way is quantitative by producing inequalities with their right hand sides involving the modulus of continuity of iterated fractional derivative of the function under approximation.

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1. Introduction

The inspiring motivation here is the monograph by B. Bede, L. Coroianu and S. Gal [6], 2016.

Let $N \in \mathbb{N}$, the well-known Bernstein polynomials ([9]) are positive linear operators, defined by the formula

$$B_N(f)(x) = \sum_{k=0}^{N} \binom{N}{k} x^k (1-x)^{N-k} f\left(\frac{k}{N}\right), \quad x \in [0,1], \; f \in C([0,1]). \tag{1}$$

T. Popoviciu in [11], 1935, proved for $f \in C([0,1])$ that

$$|B_N(f)(x) - f(x)| \leq \frac{5}{4}\omega_1\left(f, \frac{1}{\sqrt{N}}\right), \quad \forall\, x \in [0,1], \tag{2}$$

where

$$\omega_1(f,\delta) = \sup_{\substack{x,y\in[a,b]:\\ |x-y|\leq\delta}} |f(x) - f(y)|, \quad \delta > 0, \tag{3}$$

is the first modulus of continuity, here $[a,b] = [0,1]$.

G.G. Lorentz in [9], 1986, p. 21, proved for $f \in C^1([0,1])$ that

$$|B_N(f)(x) - f(x)| \leq \frac{3}{4\sqrt{N}}\omega_1\left(f', \frac{1}{\sqrt{N}}\right), \quad \forall\, x \in [0,1], \tag{4}$$

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In [6], p. 10, the authors introduced the basic Max-product Bernstein operators,

$$B_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} p_{N,k}(x) f\left(\frac{k}{N}\right)}{\bigvee_{k=0}^{N} p_{N,k}(x)}, \quad N \in \mathbb{N}, \tag{5}$$

where $\bigvee$ stands for maximum, and $p_{N,k}(x) = \binom{N}{k} x^k (1-x)^{N-k}$ and $f : [0,1] \to \mathbb{R}_+ = [0,\infty)$.

These are nonlinear and piecewise rational operators.

The authors in [6] studied similar such nonlinear operators such as: the Max-product Favard-Szász-Mirakjan operators and their truncated version, the Max-product Baskakov operators and their truncated version, also many other similar specific operators. The study in [6] is based on presented there general theory of sublinear operators. These Max-product operators tend to converge faster to the on hand function.

So we mention from [6], p. 30, that for $f : [0,1] \to \mathbb{R}_+$ continuous, we have the estimate

$$\left| B_N^{(M)}(f)(x) - f(x) \right| \leq 12\omega_1\left(f, \frac{1}{\sqrt{N+1}}\right), \quad \text{for all } N \in \mathbb{N},\ x \in [0,1]. \tag{6}$$

In this paper we expand the study of [6] by considering iterated fractional smoothness of functions. So our inequalities are with respect to $\omega_1\left(D^{(n+1)\alpha} f, \delta\right)$, $\delta > 0$, where $D^{(n+1)\alpha} f$ with $\alpha > 0$, $n \in \mathbb{N}$, is the iterated fractional derivative.

2. Main Results

We make

Remark 2.1. *Let $f : [a,b] \to \mathbb{R}$ such that $f' \in L_\infty([a,b])$, $x_0 \in [a,b]$, $0 < \alpha < 1$, the left Caputo fractional derivative of order α is defined as follows*

$$\left(D_{*x_0}^{\alpha} f\right)(x) = \frac{1}{\Gamma(1-\alpha)} \int_{x_0}^{x} (x-t)^{-\alpha} f'(t)\, dt, \tag{7}$$

where Γ is the gamma function for all $x_0 \leq x \leq b$.

We observe that

$$\left|\left(D_{*x_0}^{\alpha} f\right)(x)\right| \leq \frac{1}{\Gamma(1-\alpha)} \int_{x_0}^{x} (x-t)^{-\alpha} |f'(t)|\, dt$$

$$\leq \frac{\|f'\|_\infty}{\Gamma(1-\alpha)} \int_{x_0}^{x} (x-t)^{-\alpha} dt = \frac{\|f'\|_\infty}{\Gamma(1-\alpha)} \frac{(x-x_0)^{1-\alpha}}{(1-\alpha)} = \frac{\|f'\|_\infty (x-x_0)^{1-\alpha}}{\Gamma(2-\alpha)}. \tag{8}$$

I.e.

$$\left|\left(D_{*x_0}^{\alpha} f\right)(x)\right| \leq \frac{\|f'\|_\infty (x-x_0)^{1-\alpha}}{\Gamma(2-\alpha)} \leq \frac{\|f'\|_\infty (b-x_0)^{1-\alpha}}{\Gamma(2-\alpha)} < +\infty, \tag{9}$$

$\forall\ x \in [x_0, b]$.

Clearly, then

$$\left(D_{*x_0}^{\alpha} f\right)(x_0) = 0. \tag{10}$$

We define $\left(D_{*x_0}^{\alpha} f\right)(x) = 0$, for $a \leq x < x_0$.

Let $n \in \mathbb{N}$, we denote the iterated fractional derivative $D_{*x_0}^{n\alpha} = D_{*x_0}^{\alpha} D_{*x_0}^{\alpha} \ldots D_{*x_0}^{\alpha}$ (n-times).

Let us assume that

$$D_{*x_0}^{k\alpha} f \in C([x_0, b]), \quad k = 0, 1, \ldots, n+1;\ n \in \mathbb{N},\ 0 < \alpha < 1.$$

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By [10], [4], *pp. 156-158, we have the following generalized fractional Caputo type Taylor's formula:*

$$f(x)=\sum_{i=0}^{n}\frac{(x-x_0)^{i\alpha}}{\Gamma(i\alpha+1)}\left(D_{*x_0}^{i\alpha}f\right)(x_0)+ \tag{11}$$

$$\frac{1}{\Gamma((n+1)\alpha)}\int_{x_0}^{x}(x-t)^{(n+1)\alpha-1}\left(D_{*x_0}^{(n+1)\alpha}f\right)(t)\,dt,$$

$\forall\ x\in[x_0,b]$.

Based on the above (10) and (11), we derive

$$f(x)-f(x_0)=\sum_{i=2}^{n}\frac{(x-x_0)^{i\alpha}}{\Gamma(i\alpha+1)}\left(D_{*x_0}^{i\alpha}f\right)(x_0)+ \tag{12}$$

$$\frac{1}{\Gamma((n+1)\alpha)}\int_{x_0}^{x}(x-t)^{(n+1)\alpha-1}\left(D_{*x_0}^{(n+1)\alpha}f\right)(t)\,dt,$$

$\forall\ x\in[x_0,b]$, $0<\alpha<1$.

In case of $\left(D_{*x_0}^{i\alpha}f\right)(x_0)=0$, $i=2,3,...,n+1$, *we get*

$$f(x)-f(x_0)=$$

$$\frac{1}{\Gamma((n+1)\alpha)}\int_{x_0}^{x}(x-t)^{(n+1)\alpha-1}\left(\left(D_{*x_0}^{(n+1)\alpha}f\right)(t)-\left(D_{*x_0}^{(n+1)\alpha}f\right)(x_0)\right)dt, \tag{13}$$

$\forall\ x\in[x_0,b]$, $0<\alpha<1$.

We make

Remark 2.2. *Let* $f:[a,b]\to\mathbb{R}$ *such that* $f'\in L_\infty([a,b])$, $x_0\in[a,b]$, $0<\alpha<1$, *the right Caputo fractional derivative of order* α *is defined as follows*

$$\left(D_{x_0-}^{\alpha}f\right)(x)=\frac{-1}{\Gamma(1-\alpha)}\int_{x}^{x_0}(z-x)^{-\alpha}f'(z)\,dz, \tag{14}$$

$\forall\ x\in[a,x_0]$.

We observe that

$$\left|\left(D_{x_0-}^{\alpha}f\right)(x)\right|\le\frac{1}{\Gamma(1-\alpha)}\int_{x}^{x_0}(z-x)^{-\alpha}|f'(z)|\,dz\le$$

$$\frac{\|f'\|_\infty}{\Gamma(1-\alpha)}\left(\int_{x}^{x_0}(z-x)^{-\alpha}dz\right)=\frac{\|f'\|_\infty}{\Gamma(1-\alpha)}\frac{(x_0-x)^{1-\alpha}}{(1-\alpha)}=\frac{\|f'\|_\infty}{\Gamma(2-\alpha)}(x_0-x)^{1-\alpha}. \tag{15}$$

That is

$$\left|\left(D_{x_0-}^{\alpha}f\right)(x)\right|\le\frac{\|f'\|_\infty}{\Gamma(2-\alpha)}(x_0-x)^{1-\alpha}\le\frac{\|f'\|_\infty}{\Gamma(2-\alpha)}(x_0-a)^{1-\alpha}<\infty, \tag{16}$$

$\forall\ x\in[a,x_0]$.

In particular we have

$$\left(D_{x_0-}^{\alpha}f\right)(x_0)=0. \tag{17}$$

We define $\left(D_{x_0-}^{\alpha}f\right)(x)=0$, *for* $x_0<x\le b$.

For $n\in\mathbb{N}$, *denote the iterated fractional derivative* $D_{x_0-}^{n\alpha}=D_{x_0-}^{\alpha}D_{x_0-}^{\alpha}...D_{x_0-}^{\alpha}$ *(n-times).*

In [1], *we proved the following right generalized fractional Taylor's formula: Suppose that*

$$D_{x_0-}^{k\alpha}f\in C([a,x_0]),\ \ for\ k=0,1,...,n+1,\ 0<\alpha<1.$$

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Then

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$$f(x) = \sum_{i=0}^{n} \frac{\quad}{\Gamma(i\alpha+1)} \left(D^{i\alpha}_{x_0-} f\right)(x_0) + \tag{18}$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_x^{x_0} (z-x)^{(n+1)\alpha-1} \left(D^{(n+1)\alpha}_{x_0-} f\right)(z)\, dz,$$

$\forall\ x \in [a, x_0]$.

Based on (17) and (18), we derive

$$f(x) - f(x_0) = \sum_{i=2}^{n} \frac{(x_0-x)^{i\alpha}}{\Gamma(i\alpha+1)} \left(D^{i\alpha}_{x_0-} f\right)(x_0) + \tag{19}$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_x^{x_0} (z-x)^{(n+1)\alpha-1} \left(D^{(n+1)\alpha}_{x_0-} f\right)(z)\, dz,$$

$\forall\ x \in [a, x_0]$, $0 < \alpha < 1$.

In case of $\left(D^{i\alpha}_{x_0-} f\right)(x_0) = 0$, *for* $i = 2, 3, ..., n+1$, *we get*

$$f(x) - f(x_0) =$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_x^{x_0} (z-x)^{(n+1)\alpha-1} \left(\left(D^{(n+1)\alpha}_{x_0-} f\right)(z) - \left(D^{(n+1)\alpha}_{x_0-} f\right)(x_0)\right) dz, \tag{20}$$

$\forall\ x \in [a, x_0]$, $0 < \alpha < 1$.

We need

Definition 2.3. *Let* $D^{(n+1)\alpha}_{x_0} f$ *denote any of* $D^{(n+1)\alpha}_{*x_0} f$, $D^{(n+1)\alpha}_{x_0-} f$, *and* $\delta > 0$. *We set*

$$\omega_1\left(D^{(n+1)\alpha}_{x_0} f, \delta\right) = \max\left\{\omega_1\left(D^{(n+1)\alpha}_{*x_0} f, \delta\right)_{[x_0,b]}, \omega_1\left(D^{(n+1)\alpha}_{x_0-} f, \delta\right)_{[a,x_0]}\right\}, \tag{21}$$

where $x_0 \in [a, b]$. *Here the moduli of continuity are considered over* $[x_0, b]$ *and* $[a, x_0]$, *respectively.*

We present

Theorem 2.4. *Let* $0 < \alpha < 1$, $f : [a, b] \to \mathbb{R}$, $f' \in L_\infty([a, b])$, $x_0 \in [a, b]$. *Assume that* $D^{k\alpha}_{*x_0} f \in C([x_0, b])$, $k = 0, 1, ..., n+1$; $n \in \mathbb{N}$, *and* $\left(D^{i\alpha}_{*x_0} f\right)(x_0) = 0$, $i = 2, 3, ..., n+1$. *Also, suppose that* $D^{k\alpha}_{x_0-} f \in C([a, x_0])$, *for* $k = 0, 1, ..., n+1$, *and* $\left(D^{i\alpha}_{x_0-} f\right)(x_0) = 0$, *for* $i = 2, 3, ..., n+1$. *Then*

$$|f(x) - f(x_0)| \leq \frac{\omega_1\left(D^{(n+1)\alpha}_{x_0} f, \delta\right)}{\Gamma((n+1)\alpha+1)} \left[|x-x_0|^{(n+1)\alpha} + \frac{|x-x_0|^{(n+1)\alpha+1}}{\delta((n+1)\alpha+1)}\right], \tag{22}$$

$\forall\ x \in [a, b]$, $\delta > 0$.

Proof. By (13) we have

$$|f(x) - f(x_0)| \leq$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_{x_0}^{x} (x-t)^{(n+1)\alpha-1} \left|\left(D^{(n+1)\alpha}_{*x_0} f\right)(t) - \left(D^{(n+1)\alpha}_{*x_0} f\right)(x_0)\right| dt$$

$(\delta > 0)$

$$\leq \frac{1}{\Gamma((n+1)\alpha)} \int_{x_0}^{x} (x-t)^{(n+1)\alpha-1} \omega_1\left(D^{(n+1)\alpha}_{*x_0} f, \frac{\delta(t-x_0)}{\delta}\right)_{[x_0,b]} dt$$

$$\leq \frac{\omega_1\left(D^{(n+1)\alpha}_{*x_0} f, \delta\right)_{[x_0,b]}}{\Gamma((n+1)\alpha)} \int_{x_0}^{x} (x-t)^{(n+1)\alpha-1} \left(1 + \frac{(t-x_0)}{\delta}\right) dt = \tag{23}$$

$$\frac{\omega_1\left(D_{*x_0}^{(n+1)\alpha}f,\delta\right)_{[x_0,b]}}{\Gamma\left((n+1)\alpha\right)}\left[\frac{(x-x_0)^{(n+1)\alpha}}{(n+1)\alpha}+\frac{1}{\delta}\int_{x_0}^{x}(x-t)^{(n+1)\alpha-1}(t-x_0)^{2-1}dt\right]=$$

$$\frac{\omega_1\left(D_{*x_0}^{(n+1)\alpha}f,\delta\right)_{[x_0,b]}}{\Gamma\left((n+1)\alpha\right)}\left[\frac{(x-x_0)^{(n+1)\alpha}}{(n+1)\alpha}+\frac{1}{\delta}\frac{\Gamma\left((n+1)\alpha\right)\Gamma(2)}{\Gamma\left((n+1)\alpha+2\right)}(x-x_0)^{(n+1)\alpha+1}\right]=$$

$$\frac{\omega_1\left(D_{*x_0}^{(n+1)\alpha}f,\delta\right)_{[x_0,b]}}{\Gamma\left((n+1)\alpha\right)}\left[\frac{(x-x_0)^{(n+1)\alpha}}{(n+1)\alpha}+\frac{(x-x_0)^{(n+1)\alpha+1}}{\delta(n+1)\alpha\left((n+1)\alpha+1\right)}\right]. \tag{24}$$

We have proved

$$|f(x)-f(x_0)|\leq\frac{\omega_1\left(D_{*x_0}^{(n+1)\alpha}f,\delta\right)_{[x_0,b]}}{\Gamma\left((n+1)\alpha+1\right)}\left[(x-x_0)^{(n+1)\alpha}+\frac{(x-x_0)^{(n+1)\alpha+1}}{\delta\left((n+1)\alpha+1\right)}\right], \tag{25}$$

$\forall\ x\in[x_0,b]$, $\delta>0$.

By (20) we get

$$|f(x)-f(x_0)|\leq$$

$$\frac{1}{\Gamma\left((n+1)\alpha\right)}\int_{x}^{x_0}(z-x)^{(n+1)\alpha-1}\left|\left(D_{x_0-}^{(n+1)\alpha}f\right)(z)-\left(D_{x_0-}^{(n+1)\alpha}f\right)(x_0)\right|dz$$

$$\leq\frac{1}{\Gamma\left((n+1)\alpha\right)}\int_{x}^{x_0}(z-x)^{(n+1)\alpha-1}\omega_1\left(D_{x_0-}^{(n+1)\alpha}f,\frac{\delta(x_0-z)}{\delta}\right)_{[a,x_0]}dz$$

$$\leq\frac{\omega_1\left(D_{x_0-}^{(n+1)\alpha}f,\delta\right)_{[a,x_0]}}{\Gamma\left((n+1)\alpha\right)}\left[\int_{x}^{x_0}(z-x)^{(n+1)\alpha-1}\left(1+\frac{x_0-z}{\delta}\right)dz\right]= \tag{26}$$

$$\frac{\omega_1\left(D_{x_0-}^{(n+1)\alpha}f,\delta\right)_{[a,x_0]}}{\Gamma\left((n+1)\alpha\right)}\left[\frac{(x_0-x)^{(n+1)\alpha}}{(n+1)\alpha}+\frac{1}{\delta}\int_{x}^{x_0}(x_0-z)^{2-1}(z-x)^{(n+1)\alpha-1}dz\right]=$$

$$\frac{\omega_1\left(D_{x_0-}^{(n+1)\alpha}f,\delta\right)_{[a,x_0]}}{\Gamma\left((n+1)\alpha\right)}\left[\frac{(x_0-x)^{(n+1)\alpha}}{(n+1)\alpha}+\frac{1}{\delta}\frac{\Gamma(2)\Gamma\left((n+1)\alpha\right)}{\Gamma\left((n+1)\alpha+2\right)}(x_0-x)^{(n+1)\alpha+1}\right]=$$

$$\frac{\omega_1\left(D_{x_0-}^{(n+1)\alpha}f,\delta\right)_{[a,x_0]}}{\Gamma\left((n+1)\alpha\right)}\left[\frac{(x_0-x)^{(n+1)\alpha}}{(n+1)\alpha}+\frac{(x_0-x)^{(n+1)\alpha+1}}{\delta(n+1)\alpha\left((n+1)\alpha+1\right)}\right]. \tag{27}$$

We have proved

$$|f(x)-f(x_0)|\leq\frac{\omega_1\left(D_{x_0-}^{(n+1)\alpha}f,\delta\right)_{[a,x_0]}}{\Gamma\left((n+1)\alpha+1\right)}\left[(x_0-x)^{(n+1)\alpha}+\frac{(x_0-x)^{(n+1)\alpha+1}}{\delta\left((n+1)\alpha+1\right)}\right], \tag{28}$$

$\forall\ x\in[a,x_0]$, $\delta>0$.

By (25) and (28) we derive (22). □

We need

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Definition 2.5. *Here* $C_+([a,b]) := \{f : [a,b] \to \mathbb{R}_+,\ \text{continuous functions}\}$. *Let* $L_N : C_+([a,b]) \to C_+([a,b])$, *operators,* $\forall\ N \in \mathbb{N}$, *such that*

(i)

$$L_N(\alpha f) = \alpha L_N(f), \ \forall \alpha \geq 0, \forall f \in C_+([a,b]), \tag{29}$$

(ii) if $f, g \in C_+([a,b]) : f \leq g$, *then*

$$L_N(f) \leq L_N(g), \ \forall N \in \mathbb{N}, \tag{30}$$

(iii)

$$L_N(f+g) \leq L_N(f) + L_N(g), \ \ \forall\ f, g \in C_+([a,b]). \tag{31}$$

We call $\{L_N\}_{N\in\mathbb{N}}$ *positive sublinear operators.*

We make

Remark 2.6. *By* [6], *p. 17, we get: let* $f, g \in C_+([a,b])$, *then*

$$|L_N(f)(x) - L_N(g)(x)| \leq L_N(|f-g|)(x), \ \ \forall\ x \in [a,b]. \tag{32}$$

Furthermore, we also have that

$$|L_N(f)(x) - f(x)| \leq L_N(|f(\cdot) - f(x)|)(x) + |f(x)|\,|L_N(e_0)(x) - 1|, \tag{33}$$

$\forall\ x \in [a,b]$; $e_0(t) = 1, \forall\ t \in [a,b]$.

From now on we assume that $L_N(1) = 1$. *Hence it holds*

$$|L_N(f)(x) - f(x)| \leq L_N(|f(\cdot) - f(x)|)(x), \ \ \forall\ x \in [a,b]. \tag{34}$$

In the assumption of Theorem 2.4 and by (22) and (34) we obtain

$$|L_N(f)(x_0) - f(x_0)| \leq \frac{\omega_1\left(D_{x_0}^{(n+1)\alpha} f, \delta\right)}{\Gamma((n+1)\alpha+1)} \cdot \tag{35}$$

$$\left[L_N\left(|\cdot - x_0|^{(n+1)\alpha}\right)(x_0) + \frac{L_N\left(|\cdot - x_0|^{(n+1)\alpha+1}\right)(x_0)}{((n+1)\alpha+1)\delta}\right], \ \ \delta > 0.$$

We have proved

Theorem 2.7. *Let* $\frac{1}{n+1} < \alpha < 1$, $n \in \mathbb{N}$, $f : [a,b] \to \mathbb{R}_+$, $f' \in L_\infty([a,b])$, $x_0 \in [a,b]$. *Assume that* $D_{*x_0}^{k\alpha} f \in C([x_0,b])$, $k = 0,1,...,n+1$, *and* $\left(D_{*x_0}^{i\alpha} f\right)(x_0) = 0$, $i = 2,3,...,n+1$. *Also, suppose that* $D_{x_0-}^{k\alpha} f \in C([a,x_0])$, *for* $k = 0,1,...,n+1$, *and* $\left(D_{x_0-}^{i\alpha} f\right)(x_0) = 0$, *for* $i = 2,3,...,n+1$. *Denote* $\lambda = (n+1)\alpha > 1$. *Let* $L_N : C_+([a,b]) \to C_+([a,b])$, $\forall\ N \in \mathbb{N}$, *be positive sublinear operators, such that* $L_N(1) = 1$, $\forall\ N \in \mathbb{N}$. *Then*

$$|L_N(f)(x_0) - f(x_0)| \leq \frac{\omega_1\left(D_{x_0}^{(n+1)\alpha} f, \delta\right)}{\Gamma(\lambda+1)} \cdot$$

$$\left[L_N\left(|\cdot - x_0|^{\lambda}\right)(x_0) + \frac{L_N\left(|\cdot - x_0|^{\lambda+1}\right)(x_0)}{(\lambda+1)\delta}\right], \tag{36}$$

$\delta > 0$, $\forall\ N \in \mathbb{N}$.

Note: Theorem 2.7 is also true when $0 < \alpha \leq \frac{1}{n+1}$.

Case of $(n+1)\alpha > 1$.

We give

Theorem 3.1. *Let* $\frac{1}{n+1} < \alpha < 1$, $n \in \mathbb{N}$, $f : [0,1] \to \mathbb{R}_+$, $f' \in L_\infty([0,1])$, $x \in [0,1]$. *Assume that* $D_{*x}^{k\alpha} f \in C([x,1])$, $k = 0,1,...,n+1$, *and* $\left(D_{*x}^{i\alpha} f\right)(x) = 0$, $i = 2,3,...,n+1$. *Also, suppose that* $D_{x-}^{k\alpha} f \in C([0,x])$, *for* $k = 0,1,...,n+1$, *and* $\left(D_{x-}^{i\alpha} f\right)(x) = 0$, *for* $i = 2,3,...,n+1$. *Denote* $\lambda := (n+1)\alpha > 1$. *Then*

$$\left|B_N^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_1\left(D_x^{(n+1)\alpha} f, \left(\frac{6}{\sqrt{N+1}}\right)^{\frac{1}{\lambda+1}}\right)}{\Gamma(\lambda+1)} \cdot$$

$$\left[\frac{6}{\sqrt{N+1}} + \frac{1}{(\lambda+1)}\left(\frac{6}{\sqrt{N+1}}\right)^{\frac{\lambda}{\lambda+1}}\right], \tag{37}$$

$\forall\ N \in \mathbb{N}$.

We get $\lim_{N \to +\infty} B_N^{(M)}(f)(x) = f(x)$.

Proof. By [3] we get that

$$B_N^{(M)}\left(|\cdot - x|^\lambda\right)(x) \leq \frac{6}{\sqrt{N+1}}, \quad \forall\ x \in [0,1], \tag{38}$$

$\forall\ N \in \mathbb{N}, \forall\ \lambda > 1$.

Also $B_N^{(M)}$ maps $C_+([0,1])$ into itself, $B_N^{(M)}(1) = 1$, and it is positive sublinear operator.

We apply Theorem 2.7 and (36), we get

$$\left|B_N^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_1\left(D_x^{(n+1)\alpha} f, \delta\right)}{\Gamma(\lambda+1)}\left[\frac{6}{\sqrt{N+1}} + \frac{\frac{6}{\sqrt{N+1}}}{(\lambda+1)\delta}\right]. \tag{39}$$

Choose $\delta = \left(\frac{6}{\sqrt{N+1}}\right)^{\frac{1}{\lambda+1}}$, then $\delta^{\lambda+1} = \frac{6}{\sqrt{N+1}}$, and apply it to (39). Clearly we derive (37). ∎

We continue with

Remark 3.2. *The truncated Favard-Szász-Mirakjan operators are given by*

$$T_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^N s_{N,k}(x) f\left(\frac{k}{N}\right)}{\bigvee_{k=0}^N s_{N,k}(x)}, \quad x \in [0,1], \ N \in \mathbb{N}, \ f \in C_+([0,1]), \tag{40}$$

$s_{N,k}(x) = \frac{(Nx)^k}{k!}$, *see also* [6], *p. 11.*

By [6], *p. 178-179, we get that*

$$T_N^{(M)}(|\cdot - x|)(x) \leq \frac{3}{\sqrt{N}}, \quad \forall\ x \in [0,1], \ \forall\ N \in \mathbb{N}. \tag{41}$$

Clearly it holds

$$T_N^{(M)}\left(|\cdot - x|^{1+\beta}\right)(x) \leq \frac{3}{\sqrt{N}}, \quad \forall\ x \in [0,1], \ \forall\ N \in \mathbb{N}, \forall\ \beta > 0. \tag{42}$$

The operators $T_N^{(M)}$ *are positive sublinear operators mapping* $C_+([0,1])$ *into itself, with* $T_N^{(M)}(1) = 1$.

We continue with

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Theorem 3.3. *Same assumptions as in Theorem 3.1. Then*

$$\left|T_N^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_1\left(D_x^{(n+1)\alpha} f, \left(\frac{3}{\sqrt{N}}\right)^{\frac{1}{\lambda+1}}\right)}{\Gamma(\lambda+1)} \cdot$$

(43) $$\left[\frac{3}{\sqrt{N}} + \frac{1}{(\lambda+1)}\left(\frac{3}{\sqrt{N}}\right)^{\frac{\lambda}{\lambda+1}}\right], \quad \forall\ N \in \mathbb{N}.$$

We get $\lim_{N\to+\infty} T_N^{(M)}(f)(x) = f(x)$.

Proof. Use of Theorem 2.7, similar to the proof of Theorem 3.1. □

We make

Remark 3.4. *Next we study the truncated Max-product Baskakov operators (see* [6]*, p. 11)*

(44) $$U_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} b_{N,k}(x) f\left(\frac{k}{N}\right)}{\bigvee_{k=0}^{N} b_{N,k}(x)}, \quad x \in [0,1], \ f \in C_+([0,1]), \ N \in \mathbb{N},$$

where

(45) $$b_{N,k}(x) = \binom{N+k-1}{k} \frac{x^k}{(1+x)^{N+k}}.$$

From [6]*, pp. 217-218, we get* $(x \in [0,1])$

(46) $$\left(U_N^{(M)}(|\cdot - x|)\right)(x) \leq \frac{2\sqrt{3}\left(\sqrt{2}+2\right)}{\sqrt{N+1}}, \quad N \geq 2, \ N \in \mathbb{N}.$$

Let $\lambda \geq 1$*, clearly then it holds*

(47) $$\left(U_N^{(M)}\left(|\cdot - x|^{\lambda}\right)\right)(x) \leq \frac{2\sqrt{3}\left(\sqrt{2}+2\right)}{\sqrt{N+1}}, \quad \forall\ N \geq 2, \ N \in \mathbb{N}.$$

Also it holds $U_N^{(M)}(1) = 1$*, and* $U_N^{(M)}$ *are positive sublinear operators from* $C_+([0,1])$ *into itself.*

We give

Theorem 3.5. *Same assumptions as in Theorem 3.1. Then*

(48) $$\left|U_N^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_1\left(D_x^{(n+1)\alpha} f, \left(\frac{2\sqrt{3}(\sqrt{2}+2)}{\sqrt{N+1}}\right)^{\frac{1}{\lambda+1}}\right)}{\Gamma(\lambda+1)} \cdot$$

$$\left[\frac{2\sqrt{3}\left(\sqrt{2}+2\right)}{\sqrt{N+1}} + \frac{1}{(\lambda+1)}\left(\frac{2\sqrt{3}\left(\sqrt{2}+2\right)}{\sqrt{N+1}}\right)^{\frac{\lambda}{\lambda+1}}\right], \quad \forall\ N \geq 2, \ N \in \mathbb{N}.$$

We get $\lim_{N\to+\infty} U_N^{(M)}(f)(x) = f(x)$.

Proof. Use of Theorem 2.7, similar to the proof of Theorem 3.1. □

We continue with

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Remark 3.6. *Here we study the Max-product Meyer-Köning and Zeller operators (see* [6], *p. 11) defined by*

$$Z_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{\infty} s_{N,k}(x) f\left(\frac{k}{N+k}\right)}{\bigvee_{k=0}^{\infty} s_{N,k}(x)}, \quad \forall\ N \in \mathbb{N},\ f \in C_+([0,1]), \tag{49}$$

$s_{N,k}(x) = \begin{pmatrix} N+k \\ k \end{pmatrix} x^k$, $x \in [0,1]$.

By [6], *p. 253, we get that*

$$Z_N^{(M)}(|\cdot - x|)(x) \leq \frac{8\left(1+\sqrt{5}\right)}{3} \frac{\sqrt{x}(1-x)}{\sqrt{N}}, \quad \forall\ x \in [0,1],\ \forall\ N \geq 4,\ N \in \mathbb{N}. \tag{50}$$

We have that (for $\lambda \geq 1$*)*

$$Z_N^{(M)}\left(|\cdot - x|^{\lambda}\right)(x) \leq \frac{8\left(1+\sqrt{5}\right)}{3} \frac{\sqrt{x}(1-x)}{\sqrt{N}} := \rho(x), \tag{51}$$

$\forall\ x \in [0,1]$, $N \geq 4$, $N \in \mathbb{N}$.

Also it holds $Z_N^{(M)}(1) = 1$, *and* $Z_N^{(M)}$ *are positive sublinear operators from* $C_+([0,1])$ *into itself.*

We give

Theorem 3.7. *Same assumptions as in Theorem 3.1. Then*

$$\left|Z_N^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_1\left(D_x^{(n+1)\alpha} f, (\rho(x))^{\frac{1}{\lambda+1}}\right)}{\Gamma(\lambda+1)} \cdot \tag{52}$$

$$\left[\rho(x) + \frac{1}{(\lambda+1)}(\rho(x))^{\frac{\lambda}{\lambda+1}}\right], \forall\ N \in \mathbb{N},\ N \geq 4.$$

We get $\lim_{N\to+\infty} Z_N^{(M)}(f)(x) = f(x)$, *where* $\rho(x)$ *is as in (51).*

Proof. Use of Theorem 2.7, similar to the proof of Theorem 3.1. ∎

We continue with

Remark 3.8. *Here we deal with the Max-product truncated sampling operators (see* [6], *p. 13) defined by*

$$W_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} \frac{\sin(Nx-k\pi)}{Nx-k\pi} f\left(\frac{k\pi}{N}\right)}{\bigvee_{k=0}^{N} \frac{\sin(Nx-k\pi)}{Nx-k\pi}}, \tag{53}$$

and

$$K_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} \frac{\sin^2(Nx-k\pi)}{(Nx-k\pi)^2} f\left(\frac{k\pi}{N}\right)}{\bigvee_{k=0}^{N} \frac{\sin^2(Nx-k\pi)}{(Nx-k\pi)^2}}, \tag{54}$$

$\forall\ x \in [0,\pi]$, $f : [0,\pi] \to \mathbb{R}_+$ *a continuous function.*

Following [6], *p. 343, and making the convention* $\frac{\sin(0)}{0} = 1$ *and denoting* $s_{N,k}(x) = \frac{\sin(Nx-k\pi)}{Nx-k\pi}$, *we get that* $s_{N,k}\left(\frac{k\pi}{N}\right) = 1$, *and* $s_{N,k}\left(\frac{j\pi}{N}\right) = 0$, *if* $k \neq j$, *furthermore* $W_N^{(M)}(f)\left(\frac{j\pi}{N}\right) = f\left(\frac{j\pi}{N}\right)$, *for all* $j \in \{0, ..., N\}$.

Clearly $W_N^{(M)}(f)$ *is a well-defined function for all* $x \in [0,\pi]$, *and it is continuous on* $[0,\pi]$, *also* $W_N^{(M)}(1) = 1$.

By [6], *p. 344,* $W_N^{(M)}$ *are positive sublinear operators.*

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Call $I_N^+(x) = \{k \in \{0,1,...,N\}; s_{N,k}(x) > 0\}$, *and set* $x_{N,k} := \frac{k\pi}{N}$, $k \in \{0,1,...,N\}$.

We see that

$$W_N^{(M)}(f)(x) = \frac{\bigvee_{k\in I_N^+(x)} s_{N,k}(x) f(x_{N,k})}{\bigvee_{k\in I_N^+(x)} s_{N,k}(x)}. \tag{55}$$

By [6], *p. 346, we have*

$$W_N^{(M)}(|\cdot - x|)(x) \le \frac{\pi}{2N}, \quad \forall\, N \in \mathbb{N},\ \forall\, x \in [0,\pi]. \tag{56}$$

Notice also $|x_{N,k} - x| \le \pi$, $\forall\, x \in [0,\pi]$.

Therefore ($\lambda \ge 1$) *it holds*

$$W_N^{(M)}\left(|\cdot - x|^{\lambda}\right)(x) \le \frac{\pi^{\lambda-1}\pi}{2N} = \frac{\pi^{\lambda}}{2N}, \quad \forall\, x \in [0,\pi], \forall\, N \in \mathbb{N}. \tag{57}$$

We continue with

Theorem 3.9. *Let* $\frac{1}{n+1} < \alpha < 1$, $n \in \mathbb{N}$, $f : [0,\pi] \to \mathbb{R}_+$, $f' \in L_\infty([0,\pi])$, $x \in [0,\pi]$. *Assume that* $D_{*x}^{k\alpha} f \in C([x,\pi])$, $k = 0,1,...,n+1$, *and* $\left(D_{*x}^{i\alpha} f\right)(x) = 0$, $i = 2,3,...,n+1$. *Also, suppose that* $D_{x-}^{k\alpha} f \in C([0,x])$, *for* $k = 0,1,...,n+1$, *and* $\left(D_{x-}^{i\alpha} f\right)(x) = 0$, *for* $i = 2,3,...,n+1$. *Denote* $\lambda = (n+1)\alpha > 1$. *Then*

$$\left|W_N^{(M)}(f)(x) - f(x)\right| \le \frac{\omega_1\left(D_x^{(n+1)\alpha} f, \left(\frac{\pi^{\lambda+1}}{2N}\right)^{\frac{1}{\lambda+1}}\right)}{\Gamma(\lambda+1)} \cdot$$

$$\left[\frac{\pi^{\lambda}}{2N} + \frac{1}{(\lambda+1)}\left(\frac{\pi^{\lambda+1}}{2N}\right)^{\frac{\lambda}{\lambda+1}}\right], \quad \forall\, N \in \mathbb{N}. \tag{58}$$

It holds $\lim_{N\to+\infty} W_N^{(M)}(f)(x) = f(x)$.

Proof. Applying (36) for $W_N^{(M)}$ and using (57), we get

$$\left|W_N^{(M)}(f)(x) - f(x)\right| \le \frac{\omega_1\left(D_x^{(n+1)\alpha} f, \delta\right)}{\Gamma(\lambda+1)}\left[\frac{\pi^{\lambda}}{2N} + \frac{\frac{\pi^{\lambda+1}}{2N}}{(\lambda+1)\delta}\right]. \tag{59}$$

Choose $\delta = \left(\frac{\pi^{\lambda+1}}{2N}\right)^{\frac{1}{\lambda+1}}$, then $\delta^{\lambda+1} = \frac{\pi^{\lambda+1}}{2N}$, and $\delta^{\lambda} = \left(\frac{\pi^{\lambda+1}}{2N}\right)^{\frac{\lambda}{\lambda+1}}$. We use the last into (59) and we obtain (58). ☐

We make

Remark 3.10. *Here we continue with the Max-product truncated sampling operators (see* [6], *p. 13) defined by*

$$K_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} \frac{\sin^2(Nx-k\pi)}{(Nx-k\pi)^2} f\left(\frac{k\pi}{N}\right)}{\bigvee_{k=0}^{N} \frac{\sin^2(Nx-k\pi)}{(Nx-k\pi)^2}}, \tag{60}$$

$\forall\, x \in [0,\pi]$, $f : [0,\pi] \to \mathbb{R}_+$ *a continuous function.*

Following [6], *p. 350, and making the convention* $\frac{\sin(0)}{0} = 1$ *and denoting* $s_{N,k}(x) = \frac{\sin^2(Nx-k\pi)}{(Nx-k\pi)^2}$, *we get that* $s_{N,k}\left(\frac{k\pi}{N}\right) = 1$, *and* $s_{N,k}\left(\frac{j\pi}{N}\right) = 0$, *if* $k \ne j$, *furthermore* $K_N^{(M)}(f)\left(\frac{j\pi}{N}\right) = f\left(\frac{j\pi}{N}\right)$, *for all* $j \in \{0,...,N\}$.

Since $s_{N,j}\left(\frac{j\pi}{N}\right)=1$ *it follows that* $\bigvee_{k=0}^{N} s_{N,k}\left(\frac{j\pi}{N}\right)\geq 1>0$, *for all* $j\in\{0,1,...,N\}$. *Hence* $K_N^{(M)}(f)$ *is well-defined function for all* $x\in[0,\pi]$, *and it is continuous on* $[0,\pi]$, *also* $K_N^{(M)}(1)=1$. *By* [6], *p. 350,* $K_N^{(M)}$ *are positive sublinear operators.*

Denote $x_{N,k}:=\frac{k\pi}{N}$, $k\in\{0,1,...,N\}$.

By [6], *p. 352, we have*

$$K_N^{(M)}(|\cdot-x|)(x)\leq\frac{\pi}{2N},\quad\forall\, N\in\mathbb{N},\ \forall\, x\in[0,\pi]. \tag{61}$$

Notice also $|x_{N,k}-x|\leq\pi$, $\forall\, x\in[0,\pi]$.

Therefore $(\lambda\geq 1)$ *it holds*

$$K_N^{(M)}\left(|\cdot-x|^{\lambda}\right)(x)\leq\frac{\pi^{\lambda-1}\pi}{2N}=\frac{\pi^{\lambda}}{2N},\quad\forall\, x\in[0,\pi],\ \forall\, N\in\mathbb{N}. \tag{62}$$

We give

Theorem 3.11. *All as in Theorem 3.9. Then*

$$\left|K_N^{(M)}(f)(x)-f(x)\right|\leq\frac{\omega_1\left(D_x^{(n+1)\alpha}f,\left(\frac{\pi^{\lambda+1}}{2N}\right)^{\frac{1}{\lambda+1}}\right)}{\Gamma(\lambda+1)}\cdot$$

$$\left[\frac{\pi^{\lambda}}{2N}+\frac{1}{(\lambda+1)}\left(\frac{\pi^{\lambda+1}}{2N}\right)^{\frac{\lambda}{\lambda+1}}\right],\quad\forall\, N\in\mathbb{N}. \tag{63}$$

We have that $\lim_{N\to+\infty}K_N^{(M)}(f)(x)=f(x)$.

Proof. As in Theorem 3.9. ∎

We make

Remark 3.12. *We mention the interpolation Hermite-Fejer polynomials on Chebyshev knots of the first kind (see* [6], *p. 4): Let* $f:[-1,1]\to\mathbb{R}$ *and based on the knots* $x_{N,k}=\cos\left(\frac{(2(N-k)+1)}{2(N+1)}\pi\right)\in(-1,1)$, $k\in\{0,...,N\}$, $-1<x_{N,0}<x_{N,1}<...<x_{N,N}<1$, *which are the roots of the first kind Chebyshev polynomial* $T_{N+1}(x)=\cos((N+1)\arccos x)$, *we define (see Fejér* [8]*)*

$$H_{2N+1}(f)(x)=\sum_{k=0}^{N}h_{N,k}(x)\,f(x_{N,k}), \tag{64}$$

where

$$h_{N,k}(x)=(1-x\cdot x_{N,k})\left(\frac{T_{N+1}(x)}{(N+1)(x-x_{N,k})}\right)^2, \tag{65}$$

the fundamental interpolation polynomials.

The Max-product interpolation Hermite-Fejér operators on Chebyshev knots of the first kind (see p. 12 of [6]*) are defined by*

$$H_{2N+1}^{(M)}(f)(x)=\frac{\bigvee_{k=0}^{N}h_{N,k}(x)\,f(x_{N,k})}{\bigvee_{k=0}^{N}h_{N,k}(x)},\quad\forall\, N\in\mathbb{N}, \tag{66}$$

where $f:[-1,1]\to\mathbb{R}_+$ *is continuous.*

Call

$$E_N(x):=H_{2N+1}^{(M)}(|\cdot-x|)(x)=\frac{\bigvee_{k=0}^{N}h_{N,k}(x)\,|x_{N,k}-x|}{\bigvee_{k=0}^{N}h_{N,k}(x)},\quad x\in[-1,1]. \tag{67}$$

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Then by [6], *p. 287 we obtain that*

$$E_N(x) \leq \frac{2\pi}{N+1}, \ \forall x \in [-1,1], \ N \in \mathbb{N}. \tag{68}$$

For $m > 1$, *we get*

$$H_{2N+1}^{(M)}\left(|\cdot - x|^m\right)(x) = \frac{\bigvee_{k=0}^{N} h_{N,k}(x)\left|x_{N,k} - x\right|^m}{\bigvee_{k=0}^{N} h_{N,k}(x)} =$$

$$\frac{\bigvee_{k=0}^{N} h_{N,k}(x)\left|x_{N,k} - x\right|\left|x_{N,k} - x\right|^{m-1}}{\bigvee_{k=0}^{N} h_{N,k}(x)} \leq 2^{m-1}\frac{\bigvee_{k=0}^{N} h_{N,k}(x)\left|x_{N,k} - x\right|}{\bigvee_{k=0}^{N} h_{N,k}(x)} \tag{69}$$

$$\leq \frac{2^m\pi}{N+1}, \ \forall x \in [-1,1], \ N \in \mathbb{N}.$$

Hence it holds

$$H_{2N+1}^{(M)}\left(|\cdot - x|^m\right)(x) \leq \frac{2^m\pi}{N+1}, \ \forall x \in [-1,1], \ m > 1, \ \forall N \in \mathbb{N}. \tag{70}$$

Furthermore we have

$$H_{2N+1}^{(M)}(1)(x) = 1, \ \forall x \in [-1,1], \tag{71}$$

and $H_{2N+1}^{(M)}$ *maps continuous functions to continuous functions over* $[-1,1]$ *and for any* $x \in \mathbb{R}$ *we have* $\bigvee_{k=0}^{N} h_{N,k}(x) > 0$.

We also have $h_{N,k}(x_{N,k}) = 1$, *and* $h_{N,k}(x_{N,j}) = 0$, *if* $k \neq j$, *furthermore it holds* $H_{2N+1}^{(M)}(f)(x_{N,j}) = f(x_{N,j})$, *for all* $j \in \{0,...,N\}$, *see* [6], *p. 282.*

$H_{2N+1}^{(M)}$ *are positive sublinear operators,* [6], *p. 282.*

We give

Theorem 3.13. *Let* $\frac{1}{n+1} < \alpha < 1$, $n \in \mathbb{N}$, $f : [-1,1] \to \mathbb{R}_+$, $f' \in L_\infty([-1,1])$, $x \in [-1,1]$. *Assume that* $D_{*x}^{k\alpha} f \in C([x,1])$, $k = 0,1,...,n+1$, *and* $\left(D_{*x}^{i\alpha} f\right)(x) = 0$, $i = 2,3,...,n+1$. *Also, suppose that* $D_{x-}^{k\alpha} f \in C([-1,x])$, *for* $k = 0,1,...,n+1$, *and* $\left(D_{x-}^{i\alpha} f\right)(x) = 0$, *for* $i = 2,3,...,n+1$. *Denote* $\lambda = (n+1)\alpha > 1$. *Then*

$$\left|H_{2N+1}^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_1\left(D_x^{(n+1)\alpha} f, \left(\frac{2^{\lambda+1}\pi}{N+1}\right)^{\frac{1}{\lambda+1}}\right)}{\Gamma(\lambda+1)} \cdot \tag{72}$$

$$\left[\frac{2^\lambda\pi}{N+1} + \frac{1}{(\lambda+1)}\left(\frac{2^{\lambda+1}\pi}{N+1}\right)^{\frac{\lambda}{\lambda+1}}\right], \ \forall N \in \mathbb{N}.$$

Furthermore it holds $\lim_{N \to +\infty} H_{2N+1}^{(M)}(f)(x) = f(x)$.

Proof. Use of Theorem 2.7, (36) and (70). Choose $\delta := \left(\frac{2^{\lambda+1}\pi}{N+1}\right)^{\frac{1}{\lambda+1}}$, etc. □

We continue with

Remark 3.14. *Here we deal with Lagrange interpolation polynomials on Chebyshev knots of second kind plus the endpoints* ± 1 *(see* [6], *p. 5). These polynomials are linear operators attached to* $f : [-1,1] \to \mathbb{R}$ *and to the knots* $x_{N,k} = \cos\left(\left(\frac{N-k}{N-1}\right)\pi\right) \in [-1,1]$, $k = 1,...,N$, $N \in \mathbb{N}$, *which are the roots* of $\omega_N(x) = \sin(N-1)t \sin t$, $x = \cos t$. Notice that $x_{N,1} = -1$ and $x_{N,N} = 1$. Their formula is given by ([6], p. 377)

$$L_N(f)(x) = \sum_{k=1}^{N} l_{N,k}(x) f(x_{N,k}), \tag{73}$$

where

$$l_{N,k}(x) = \frac{(-1)^{k-1}\omega_N(x)}{(1+\delta_{k,1}+\delta_{k,N})(N-1)(x-x_{N,k})}, \tag{74}$$

$N \geq 2$, $k = 1,...,N$, and $\omega_N(x) = \prod_{k=1}^{N}(x - x_{N,k})$ and $\delta_{i,j}$ denotes the Kronecher's symbol, that is $\delta_{i,j} = 1$, if $i = j$, and $\delta_{i,j} = 0$, if $i \neq j$.

The Max-product Lagrange interpolation operators on Chebyshev knots of second kind, plus the endpoints ± 1, are defined by ([6], p. 12)

$$L_N^{(M)}(f)(x) = \frac{\bigvee_{k=1}^{N} l_{N,k}(x) f(x_{N,k})}{\bigvee_{k=1}^{N} l_{N,k}(x)}, \quad x \in [-1,1], \tag{75}$$

where $f : [-1,1] \to \mathbb{R}_+$ continuous.

First we see that $L_N^{(M)}(f)(x)$ is well defined and continuous for any $x \in [-1,1]$. Following [6], p. 289, because $\sum_{k=1}^{N} l_{N,k}(x) = 1$, $\forall\, x \in \mathbb{R}$, for any x there exists $k \in \{1,...,N\} : l_{N,k}(x) > 0$, hence $\bigvee_{k=1}^{N} l_{N,k}(x) > 0$. We have that $l_{N,k}(x_{N,k}) = 1$, and $l_{N,k}(x_{N,j}) = 0$, if $k \neq j$. Furthermore it holds $L_N^{(M)}(f)(x_{N,j}) = f(x_{N,j})$, all $j \in \{1,...,N\}$, and $L_N^{(M)}(1) = 1$.

Call $I_N^+(x) = \{k \in \{1,...,N\}; l_{N,k}(x) > 0\}$, then $I_N^+(x) \neq \emptyset$.

So for $f \in C_+([-1,1])$ we get

$$L_N^{(M)}(f)(x) = \frac{\bigvee_{k \in I_N^+(x)} l_{N,k}(x) f(x_{N,k})}{\bigvee_{k \in I_N^+(x)} l_{N,k}(x)} \geq 0. \tag{76}$$

Notice here that $|x_{N,k} - x| \leq 2$, $\forall\, x \in [-1,1]$.

By [6], p. 297, we get that

$$L_N^{(M)}(|\cdot - x|)(x) = \frac{\bigvee_{k=1}^{N} l_{N,k}(x) |x_{N,k} - x|}{\bigvee_{k=1}^{N} l_{N,k}(x)} =$$

$$\frac{\bigvee_{k \in I_N^+(x)} l_{N,k}(x) |x_{N,k} - x|}{\bigvee_{k \in I_N^+(x)} l_{N,k}(x)} \leq \frac{\pi^2}{6(N-1)}, \tag{77}$$

$N \geq 3$, $\forall\, x \in (-1,1)$, N is odd.

We get that $(m > 1)$

$$L_N^{(M)}(|\cdot - x|^m)(x) = \frac{\bigvee_{k \in I_N^+(x)} l_{N,k}(x) |x_{N,k} - x|^m}{\bigvee_{k \in I_N^+(x)} l_{N,k}(x)} \leq \frac{2^{m-1}\pi^2}{6(N-1)}, \tag{78}$$

$N \geq 3$ odd, $\forall\, x \in (-1,1)$.

$L_N^{(M)}$ are positive sublinear operators, [6], p. 290.

We give

Theorem 3.15. *Same assumptions as in Theorem 3.13. Then*

$$\left| L_N^{(M)}(f)(x) - f(x) \right| \leq \frac{\omega_1\left(D_x^{(n+1)\alpha} f, \left(\frac{2^\lambda \pi^2}{6(N-1)}\right)^{\frac{1}{\lambda+1}}\right)}{\Gamma(\lambda+1)}. \tag{79}$$

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$$\left[\frac{2^{\lambda-1}\pi^2}{6(N-1)}+\frac{1}{(\lambda+1)}\left(\frac{2^{\lambda}\pi^2}{6(N-1)}\right)^{\frac{\lambda}{\lambda+1}}\right], \quad \forall\, N\in\mathbb{N}: N\geq 3,\ odd.$$

It holds $\lim_{N\to+\infty} L_N^{(M)}(f)(x)=f(x)$.

Proof. By Theorem 2.7, choose $\delta:=\left(\frac{2^{\lambda}\pi^2}{6(N-1)}\right)^{\frac{1}{\lambda+1}}$, use of (36) and (78). At ±1 the left hand side of (79) is zero, thus (79) is trivially true. □

We make

Remark 3.16. *Let* $f\in C_+([-1,1])$, $N\geq 4$, $N\in\mathbb{N}$, N *even.*

By [6], *p. 298, we get*

$$L_N^{(M)}(|\cdot-x|)(x)\leq\frac{4\pi^2}{3(N-1)}=\frac{2^2\pi^2}{3(N-1)}, \quad \forall x\in(-1,1). \tag{80}$$

Hence $(m>1)$

$$L_N^{(M)}(|\cdot-x|^m)(x)\leq\frac{2^{m+1}\pi^2}{3(N-1)}, \quad \forall\, x\in(-1,1). \tag{81}$$

We present

Theorem 3.17. *Same assumptions as in Theorem 3.13. Then*

$$\left|L_N^{(M)}(f)(x)-f(x)\right|\leq\frac{\omega_1\left(D_x^{(n+1)\alpha}f,\left(\frac{2^{\lambda+2}\pi^2}{3(N-1)}\right)^{\frac{1}{\lambda+1}}\right)}{\Gamma(\lambda+1)}\cdot \tag{82}$$

$$\left[\frac{2^{\lambda+1}\pi^2}{3(N-1)}+\frac{1}{(\lambda+1)}\left(\frac{2^{\lambda+2}\pi^2}{3(N-1)}\right)^{\frac{\lambda}{\lambda+1}}\right], \quad \forall\, N\in\mathbb{N},\ N\geq 4,\ N\ is\ even.$$

It holds $\lim_{N\to+\infty} L_N^{(M)}(f)(x)=f(x)$.

Proof. By Theorem 2.7, use of (36) and (81). Choose $\delta=\left(\frac{2^{\lambda+2}\pi^2}{3(N-1)}\right)^{\frac{1}{\lambda+1}}$, etc. □

We make

Remark 3.18. *Let* $f:\mathbb{R}\to\mathbb{R}$ *such that* $f'\in L_\infty(\mathbb{R})$, $x_0\in\mathbb{R}$, $0<\alpha<1$. *The left Caputo fractional derivative* $\left(D_{*x_0}^{\alpha}f\right)(x)$ *is given by (7) for* $x\geq x_0$. *Clearly it holds* $\left(D_{*x_0}^{\alpha}f\right)(x_0)=0$, *and we define* $\left(D_{*x_0}^{\alpha}f\right)(x)=0$, *for* $x<x_0$.

Let us assume that $D_{*x_0}^{k\alpha}f\in C([x_0,+\infty))$, $k=0,1,...,n+1$; $n\in\mathbb{N}$.

Still (11)-(13) are valid $\forall\, x\in[x_0,+\infty)$.

The right Caputo fractional derivative $\left(D_{x_0-}^{\alpha}f\right)(x)$ *is given by (14) for* $x\leq x_0$. *Clearly it holds* $\left(D_{x_0-}^{\alpha}f\right)(x_0)=0$, *and define* $\left(D_{x_0-}^{\alpha}f\right)(x)=0$, *for* $x>x_0$.

Let us assume that $D_{x_0-}^{k\alpha}f\in C((-\infty,x_0])$, $k=0,1,...,n+1$.

Still (18)-(20) are valid $\forall\, x\in(-\infty,x_0]$.

Here we restrict again ourselves to $\frac{1}{n+1}<\alpha<1$, *that is* $\lambda:=(n+1)\alpha>1$. *We denote* $D_{*x_0}^{\lambda}f:=D_{*x_0}^{(n+1)\alpha}f$, *and* $D_{x_0-}^{\lambda}f:=D_{x_0-}^{(n+1)\alpha}f$.

We need

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Definition 3.19. *([7], p. 41) Let $I \subset \mathbb{R}$ be an interval of finite or infinite length, and $f : I \to \mathbb{R}$ a bounded or uniformly continuous function. We define the first modulus of continuity*

$$\omega_1 (f, \delta)_I = \sup_{\substack{x,y \in I \\ |x-y| \leq \delta}} |f(x) - f(y)|, \quad \delta > 0. \tag{83}$$

Clearly, it holds $\omega_1 (f, \delta)_I < +\infty$.

We also have

$$\omega_1 (f, r\delta)_I \leq (r+1) \omega_1 (f, \delta)_I, \quad \text{any } r \geq 0. \tag{84}$$

Convention 3.20. *We assume that $D^{\lambda}_{x_0-} f$ is either bounded or uniformly continuous function on $(-\infty, x_0]$, similarly we assume that $D^{\lambda}_{*x_0} f$ is either bounded or uniformly continuous function on $[x_0, +\infty)$.*

We need

Definition 3.21. *Let $D^{\lambda}_{x_0} f$ denote any of $D^{\lambda}_{x_0-} f$, $D^{\lambda}_{*x_0} f$ and $\delta > 0$. We set*

$$\omega_1 \left(D^{\lambda}_{x_0} f, \delta\right)_{\mathbb{R}} := \max \left\{ \omega_1 \left(D^{\lambda}_{x_0-} f, \delta\right)_{(-\infty, x_0]}, \omega_1 \left(D^{\lambda}_{*x_0} f, \delta\right)_{[x_0, +\infty)} \right\}, \tag{85}$$

where $x_0 \in \mathbb{R}$. Notice that $\omega_1 \left(D^{\lambda}_{x_0} f, \delta\right)_{\mathbb{R}} < +\infty$.

We give

Theorem 3.22. *Let $\frac{1}{n+1} < \alpha < 1$, $n \in \mathbb{N}$, $\lambda := (n+1)\alpha > 1$, $f : \mathbb{R} \to \mathbb{R}$, $f' \in L_{\infty}(\mathbb{R})$, $x_0 \in \mathbb{R}$. Assume that $D^{k\alpha}_{*x_0} f \in C([x_0, +\infty))$, $k = 0, 1, ..., n+1$, and $\left(D^{i\alpha}_{*x_0} f\right)(x_0) = 0$, $i = 2, 3, ..., n+1$. Suppose that $D^{k\alpha}_{x_0-} f \in C((-\infty, x_0])$, for $k = 0, 1, ..., n+1$, and $\left(D^{i\alpha}_{x_0-} f\right)(x_0) = 0$, for $i = 2, 3, ..., n+1$. Then*

$$|f(x) - f(x_0)| \leq \frac{\omega_1 \left(D^{\lambda}_{x_0} f, \delta\right)_{\mathbb{R}}}{\Gamma(\lambda+1)} \left[|x - x_0|^{\lambda} + \frac{|x - x_0|^{\lambda+1}}{(\lambda+1)\delta} \right], \tag{86}$$

$\forall\ x \in \mathbb{R}$, $\delta > 0$.

Proof. Similar to Theorem 2.4. ∎

Remark 3.23. *Let $b : \mathbb{R} \to \mathbb{R}_+$ be a centered (it takes a global maximum at 0) bell-shaped function, with compact support $[-T, T]$, $T > 0$ (that is $b(x) > 0$ for all $x \in (-T, T)$) and $I = \int_{-T}^{T} b(x)\, dx > 0$.*

The Cardaliaguet-Euvrard neural network operators are defined by (see [5])

$$C_{N,\alpha}(f)(x) = \sum_{k=-N^2}^{N^2} \frac{f\left(\frac{k}{n}\right)}{I N^{1-\alpha}} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right), \tag{87}$$

$0 < \alpha < 1$, $N \in \mathbb{N}$ and typically here $f : \mathbb{R} \to \mathbb{R}$ is continuous and bounded or uniformly continuous on $\mathbb{R}$.

$CB(\mathbb{R})$ denotes the continuous and bounded function on $\mathbb{R}$, and

$$CB_+(\mathbb{R}) = \{f : \mathbb{R} \to [0, \infty);\ f \in CB(\mathbb{R})\}.$$

The corresponding max-product Cardaliaguet-Euvrard neural network operators will be given by

$$C^{(M)}_{N,\alpha}(f)(x) = \frac{\bigvee_{k=-N^2}^{N^2} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right) f\left(\frac{k}{N}\right)}{\bigvee_{k=-N^2}^{N^2} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right)}, \tag{88}$$

$x \in \mathbb{R}$, typically here $f \in CB_+(\mathbb{R})$, see also [5].

Next we follow [5].

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For any $x \in \mathbb{R}$, *denoting*

$$J_{T,N}(x) = \left\{ k \in \mathbb{Z};\ -N^2 \leq k \leq N^2,\ N^{1-\alpha}\left(x - \frac{k}{N}\right) \in (-T,T) \right\},$$

we can write

$$C_{N,\alpha}^{(M)}(f)(x) = \frac{\bigvee_{k \in J_{T,N}(x)} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right) f\left(\frac{k}{N}\right)}{\bigvee_{k \in J_{T,N}(x)} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right)}, \tag{89}$$

$x \in \mathbb{R}$, $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$, *where* $J_{T,N}(x) \neq \emptyset$. *Indeed, we have* $\bigvee_{k \in J_{T,N}(x)} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right) > 0$, $\forall\ x \in \mathbb{R}$ *and* $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$.

We have that $C_{N,\alpha}^{(M)}(1)(x) = 1$, $\forall\ x \in \mathbb{R}$ *and* $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$.

See in [5] there: Lemma 2.1, Corollary 2.2 and Remarks.

We need

Theorem 3.24. *([5]) Let* $b(x)$ *be a centered bell-shaped function, continuous and with compact support* $[-T,T]$, $T > 0$, $0 < \alpha < 1$ *and* $C_{N,\alpha}^{(M)}$ *be defined as in (88).*

(i) If $|f(x)| \leq c$ *for all* $x \in \mathbb{R}$ *then* $\left|C_{N,\alpha}^{(M)}(f)(x)\right| \leq c$, *for all* $x \in \mathbb{R}$ *and* $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$ *and* $C_{N,\alpha}^{(M)}(f)(x)$ *is continuous at any point* $x \in \mathbb{R}$, *for all* $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$;

(ii) If $f, g \in CB_+(\mathbb{R})$ *satisfy* $f(x) \leq g(x)$ *for all* $x \in \mathbb{R}$, *then* $C_{N,\alpha}^{(M)}(f)(x) \leq C_{N,\alpha}^{(M)}(g)(x)$ *for all* $x \in \mathbb{R}$ *and* $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$;

(iii) $C_{N,\alpha}^{(M)}(f+g)(x) \leq C_{N,\alpha}^{(M)}(f)(x) + C_{N,\alpha}^{(M)}(g)(x)$ *for all* $f, g \in CB_+(\mathbb{R})$, $x \in \mathbb{R}$ *and* $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$;

(iv) For all $f, g \in CB_+(\mathbb{R})$, $x \in \mathbb{R}$ *and* $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$, *we have*

$$\left|C_{N,\alpha}^{(M)}(f)(x) - C_{N,\alpha}^{(M)}(g)(x)\right| \leq C_{N,\alpha}^{(M)}(|f-g|)(x);$$

(v) $C_{N,\alpha}^{(M)}$ *is positive homogeneous, that is* $C_{N,\alpha}^{(M)}(\lambda f)(x) = \lambda C_{N,\alpha}^{(M)}(f)(x)$ *for all* $\lambda \geq 0$, $x \in \mathbb{R}$, $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$ *and* $f \in CB_+(\mathbb{R})$.

We make

Remark 3.25. *We have that*

$$E_{N,\alpha}(x) := C_{N,\alpha}^{(M)}(|\cdot - x|)(x) = \frac{\bigvee_{k \in J_{T,N}(x)} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right)\left|x - \frac{k}{N}\right|}{\bigvee_{k \in J_{T,N}(x)} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right)}, \tag{90}$$

$\forall\ x \in \mathbb{R}$, *and* $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$.

We mention from [5] the following:

Theorem 3.26. *([5]) Let* $b(x)$ *be a centered bell-shaped function, continuous and with compact support* $[-T,T]$, $T > 0$ *and* $0 < \alpha < 1$. *In addition, suppose that the following requirements are fulfilled:*

(i) There exist $0 < m_1 \leq M_1 < \infty$ *such that* $m_1(T - x) \leq b(x) \leq M_1(T - x)$, $\forall\ x \in [0,T]$;

(ii) There exist $0 < m_2 \leq M_2 < \infty$ *such that* $m_2(x + T) \leq b(x) \leq M_2(x + T)$, $\forall\ x \in [-T,0]$.

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Then for all $f \in CB_+(\mathbb{R})$, $x \in \mathbb{R}$ *and for all* $N \in \mathbb{N}$ *satisying* $N > \max\left\{T + |x|, \left(\frac{2}{T}\right)^{\frac{1}{\alpha}}\right\}$, *we have the estimate*

$$\left|C_{N,\alpha}^{(M)}(f)(x) - f(x)\right| \leq c\omega_1\left(f, N^{\alpha-1}\right)_{\mathbb{R}}, \tag{91}$$

where

$$c := 2\left(\max\left\{\frac{TM_2}{2m_2}, \frac{TM_1}{2m_1}\right\} + 1\right),$$

and

$$\omega_1(f,\delta)_{\mathbb{R}} := \sup_{\substack{x,y\in\mathbb{R}:\\ |x-y|\leq\delta}} |f(x) - f(y)|. \tag{92}$$

We make

Remark 3.27. *In* [5], *was proved that*

$$E_{N,\alpha}(x) \leq \max\left\{\frac{TM_2}{2m_2}, \frac{TM_1}{2m_1}\right\} N^{\alpha-1}, \quad \forall\, N > \max\left\{T + |x|, \left(\frac{2}{T}\right)^{\frac{1}{\alpha}}\right\}. \tag{93}$$

That is

$$C_{N,\alpha}^{(M)}(|\cdot - x|)(x) \leq \max\left\{\frac{TM_2}{2m_2}, \frac{TM_1}{2m_1}\right\} N^{\alpha-1}, \quad \forall\, N > \max\left\{T + |x|, \left(\frac{2}{T}\right)^{\frac{1}{\alpha}}\right\}. \tag{94}$$

From (90) we have that $\left|x - \frac{k}{N}\right| \leq \frac{T}{N^{1-\alpha}}$.

Hence $(\lambda > 1)$ $(\forall\, x \in \mathbb{R}$ *and* $N > \max\left\{T + |x|, \left(\frac{2}{T}\right)^{\frac{1}{\alpha}}\right\})$

$$C_{N,\alpha}^{(M)}\left(|\cdot - x|^{\lambda}\right)(x) = \frac{\bigvee_{k\in J_{T,N}(x)} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right)\left|x - \frac{k}{N}\right|^{\lambda}}{\bigvee_{k\in J_{T,N}(x)} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right)} \leq \tag{95}$$

$$\left(\frac{T}{N^{1-\alpha}}\right)^{\lambda-1} \max\left\{\frac{TM_2}{2m_2}, \frac{TM_1}{2m_1}\right\} N^{\alpha-1}, \quad \forall\, N > \max\left\{T + |x|, \left(\frac{2}{T}\right)^{\frac{1}{\alpha}}\right\}.$$

Then $(\lambda > 1)$ *it holds*

$$C_{N,\alpha}^{(M)}\left(|\cdot - x|^{\lambda}\right)(x) \leq$$

$$T^{\lambda-1} \max\left\{\frac{TM_2}{2m_2}, \frac{TM_1}{2m_1}\right\} \frac{1}{N^{\lambda(1-\alpha)}}, \quad \forall\, N > \max\left\{T + |x|, \left(\frac{2}{T}\right)^{\frac{1}{\alpha}}\right\}. \tag{96}$$

Call

$$\theta := \max\left\{\frac{TM_2}{2m_2}, \frac{TM_1}{2m_1}\right\} > 0. \tag{97}$$

Consequently $(\lambda > 1)$ *we derive*

$$C_{N,\alpha}^{(M)}\left(|\cdot - x|^{\lambda}\right)(x) \leq \frac{\theta T^{\lambda-1}}{N^{\lambda(1-\alpha)}}, \quad \forall\, N > \max\left\{T + |x|, \left(\frac{2}{T}\right)^{\frac{1}{\alpha}}\right\}. \tag{98}$$

We need

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Theorem 3.28. *All here as in Theorem 3.22, where $x = x_0 \in \mathbb{R}$ is fixed. Let b be a centered bell-shaped function, continuous and with compact support $[-T, T]$, $T > 0$, $0 < \alpha < 1$ and $C_{N,\alpha}^{(M)}$ be defined as in (88). Then*

$$\left| C_{N,\alpha}^{(M)}(f)(x) - f(x) \right| \leq$$

$$\frac{\omega_1\left(D_x^\lambda f, \delta\right)_{\mathbb{R}}}{\Gamma(\lambda+1)} \left[C_{N,\alpha}^{(M)}\left(|\cdot - x|^\lambda\right)(x) + \frac{C_{N,\alpha}^{(M)}\left(|\cdot - x|^{\lambda+1}\right)(x)}{(\lambda+1)\delta} \right], \tag{99}$$

$\forall\ N \in \mathbb{N} : N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$.

Proof. By Theorem 3.22 and (86) we get

$$|f(\cdot) - f(x)| \leq \frac{\omega_1\left(D_x^\lambda f, \delta\right)_{\mathbb{R}}}{\Gamma(\lambda+1)} \left[|\cdot - x|^\lambda + \frac{|\cdot - x|^{\lambda+1}}{(\lambda+1)\delta} \right], \quad \delta > 0, \tag{100}$$

true over $\mathbb{R}$.

As in Theorem 3.24 and using similar reasoning and $C_{N,\alpha}^{(M)}(1) = 1$, we get

$$\left| C_{N,\alpha}^{(M)}(f)(x) - f(x) \right| \leq C_{N,\alpha}^{(M)}(|f(\cdot) - f(x)|)(x) \overset{(100)}{\leq}$$

$$\frac{\omega_1\left(D_x^\lambda f, \delta\right)_{\mathbb{R}}}{\Gamma(\lambda+1)} \left[C_{N,\alpha}^{(M)}\left(|\cdot - x|^\lambda\right)(x) + \frac{C_{N,\alpha}^{(M)}\left(|\cdot - x|^{\lambda+1}\right)(x)}{(\lambda+1)\delta} \right], \tag{101}$$

$\forall\ N \in \mathbb{N} : N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$. ∎

We continue with

Theorem 3.29. *Here all as in Theorem 3.22, where $x = x_0 \in \mathbb{R}$ is fixed. Also the same assumptions as in Theorem 3.26. Then*

$$\left| C_{N,\alpha}^{(M)}(f)(x) - f(x) \right| \leq \frac{1}{\Gamma(\lambda+1)} \omega_1\left(D_x^\lambda f, \left(\frac{\theta T^\lambda}{N^{(\lambda+1)(1-\alpha)}} \right)^{\frac{1}{\lambda+1}} \right)_{\mathbb{R}} \cdot$$

$$\left[\frac{\theta T^{\lambda-1}}{N^{\lambda(1-\alpha)}} + \frac{1}{(\lambda+1)} \left(\frac{\theta T^\lambda}{N^{(\lambda+1)(1-\alpha)}} \right)^{\frac{\lambda}{\lambda+1}} \right], \tag{102}$$

$\forall\ N \in \mathbb{N} : N > \max\left\{T + |x|, \left(\frac{2}{T}\right)^{\frac{1}{\alpha}}\right\}$.

We have that $\lim_{N \to +\infty} C_{N,\alpha}^{(M)}(f)(x) = f(x)$.

Proof. We apply Theorem 3.28. In (99) we choose

$$\delta := \left(\frac{\theta T^\lambda}{N^{(\lambda+1)(1-\alpha)}} \right)^{\frac{1}{\lambda+1}},$$

thus $\delta^{\lambda+1} = \frac{\theta T^\lambda}{N^{(\lambda+1)(1-\alpha)}}$, and

$$\delta^\lambda = \left(\frac{\theta T^\lambda}{N^{(\lambda+1)(1-\alpha)}} \right)^{\frac{\lambda}{\lambda+1}}. \tag{103}$$

Therefore we have

$$\left|C_{N,\alpha}^{(M)}(f)(x)-f(x)\right| \overset{(98)}{\leq} \frac{1}{\Gamma(\lambda+1)}\omega_1\left(D_x^{\lambda}f,\left(\frac{\theta T^{\lambda}}{N^{(\lambda+1)(1-\alpha)}}\right)^{\frac{1}{\lambda+1}}\right)_{\mathbb{R}} \cdot \tag{104}$$

$$\left[\frac{\theta T^{\lambda-1}}{N^{\lambda(1-\alpha)}}+\frac{1}{(\lambda+1)\delta}\frac{\theta T^{\lambda}}{N^{(\lambda+1)(1-\alpha)}}\right]=$$

$$\frac{1}{\Gamma(\lambda+1)}\omega_1\left(D_x^{\lambda}f,\left(\frac{\theta T^{\lambda}}{N^{(\lambda+1)(1-\alpha)}}\right)^{\frac{1}{\lambda+1}}\right)\left[\frac{\theta T^{\lambda-1}}{N^{\lambda(1-\alpha)}}+\frac{1}{(\lambda+1)\delta}\delta^{\lambda+1}\right]\overset{(103)}{=}$$

$$\frac{1}{\Gamma(\lambda+1)}\omega_1\left(D_x^{\lambda}f,\left(\frac{\theta T^{\lambda}}{N^{(\lambda+1)(1-\alpha)}}\right)^{\frac{1}{\lambda+1}}\right)_{\mathbb{R}} \cdot$$

$$\left[\frac{\theta T^{\lambda-1}}{N^{\lambda(1-\alpha)}}+\frac{1}{(\lambda+1)}\left(\frac{\theta T^{\lambda}}{N^{(\lambda+1)(1-\alpha)}}\right)^{\frac{\lambda}{\lambda+1}}\right], \tag{105}$$

$\forall\ N\in\mathbb{N}: N>\max\left\{T+|x|,\left(\frac{2}{T}\right)^{\frac{1}{\alpha}}\right\}$, proving the inequality (102). □

It follows an interesting application to Theorem 3.1 when $\alpha=\frac{1}{2}$, $n=2$.

Corollary 3.30. *Let $f:[0,1]\to\mathbb{R}_+$, $f'\in L_\infty([0,1])$, $x\in[0,1]$. Assume that $D_{*x}^{k\frac{1}{2}}f\in C([x,1])$, $k=0,1,2,3$, and $\left(D_{*x}^{i\frac{1}{2}}f\right)(x)=0$, $i=2,3$. Suppose that $D_{x-}^{k\frac{1}{2}}f\in C([0,x])$, for $k=0,1,2,3$, and $\left(D_{x-}^{i\frac{1}{2}}f\right)(x)=0$, for $i=2,3$. Then*

$$\left|B_N^{(M)}(f)(x)-f(x)\right|\leq\frac{4\omega_1\left(D_x^{3\cdot\frac{1}{2}}f,\left(\frac{6}{\sqrt{N+1}}\right)^{\frac{2}{5}}\right)}{3\sqrt{\pi}}$$

$$\left[\frac{6}{\sqrt{N+1}}+\frac{2}{5}\left(\frac{6}{\sqrt{N+1}}\right)^{\frac{3}{5}}\right],\quad\forall\ N\in\mathbb{N}. \tag{106}$$

We get $\lim_{N\to+\infty}B_N^{(M)}(f)(x)=f(x)$.

4. Applications, Part B

Case of $(n+1)\alpha\leq 1$.

We need

Theorem 4.1. *([2]) Let $L:C_+([a,b])\to C_+([a,b])$, be a positive sublinear operator and $f,g\in C_+([a,b])$, furthermore let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Assume that $L((f(\cdot))^p)(s_*)$, $L((g(\cdot))^q)(s_*)>0$ for some $s_*\in[a,b]$. Then*

$$L(f(\cdot)g(\cdot))(s_*)\leq\left(L((f(\cdot))^p)(s_*)\right)^{\frac{1}{p}}\left(L((g(\cdot))^q)(s_*)\right)^{\frac{1}{q}}. \tag{107}$$

We give

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Theorem 4.2. *Let $0 < \alpha \leq \frac{1}{n+1}$, $n \in \mathbb{N}$, $f : [a,b] \to \mathbb{R}_+$, $f' \in L_\infty([a,b])$, $x_0 \in [a,b]$. Assume that $D_{*x_0}^{k\alpha} f \in C([x_0,b])$, $k = 0,1,...,n+1$, and $\left(D_{*x_0}^{i\alpha} f\right)(x_0) = 0$, $i = 2,3,...,n+1$. Also, suppose that $D_{x_0-}^{k\alpha} f \in C([a,x_0])$, for $k = 0,1,...,n+1$, and $\left(D_{x_0-}^{i\alpha} f\right)(x_0) = 0$, for $i = 2,3,...,n+1$. Denote $\lambda := (n+1)\alpha \leq 1$. Let $L_N : C_+([a,b]) \to C_+([a,b])$, $\forall\, N \in \mathbb{N}$, be positive sublinear operators, such that $L_N\left(|\cdot - x_0|^{\lambda+1}\right)(x_0) > 0$ and $L_N(1) = 1$, $\forall\, N \in \mathbb{N}$. Then*

$$|L_N(f)(x_0) - f(x_0)| \leq \frac{\omega_1\left(D_{x_0}^{(n+1)\alpha} f, \delta\right)}{\Gamma(\lambda+1)} \cdot \tag{108}$$

$$\left[\left(L_N\left(|\cdot - x_0|^{\lambda+1}\right)(x_0)\right)^{\frac{\lambda}{\lambda+1}} + \frac{L_N\left(|\cdot - x_0|^{\lambda+1}\right)(x_0)}{(\lambda+1)\delta}\right],$$

$\delta > 0$, $\forall\, N \in \mathbb{N}$.

Proof. By Theorems 2.7, 4.1. □

We give

Theorem 4.3. *Let $0 < \alpha \leq \frac{1}{n+1}$, $n \in \mathbb{N}$, $f : [a,b] \to \mathbb{R}_+$, $f' \in L_\infty([a,b])$, $x_0 \in [a,b]$. Assume that $D_{*x_0}^{k\alpha} f \in C([x_0,b])$, $k = 0,1,...,n+1$, and $\left(D_{*x_0}^{i\alpha} f\right)(x_0) = 0$, $i = 2,3,...,n+1$. Also, suppose that $D_{x_0-}^{k\alpha} f \in C([a,x_0])$, for $k = 0,1,...,n+1$, and $\left(D_{x_0-}^{i\alpha} f\right)(x_0) = 0$, for $i = 2,3,...,n+1$. Denote $\lambda := (n+1)\alpha \leq 1$. Let $L_N : C_+([a,b]) \to C_+([a,b])$, $\forall\, N \in \mathbb{N}$, be positive sublinear operators, such that $L_N\left(|\cdot - x_0|^{\lambda+1}\right)(x_0) > 0$ and $L_N(1) = 1$, $\forall\, N \in \mathbb{N}$. Then*

$$|L_N(f)(x_0) - f(x_0)| \leq \frac{(\lambda+2)\,\omega_1\left(D_{x_0}^{(n+1)\alpha} f, \left(L_N\left(|\cdot - x_0|^{\lambda+1}\right)(x_0)\right)^{\frac{1}{\lambda+1}}\right)}{\Gamma(\lambda+2)} \cdot \tag{109}$$

$$\left(L_N\left(|\cdot - x_0|^{\lambda+1}\right)(x_0)\right)^{\frac{\lambda}{\lambda+1}}, \quad \forall\, N \in \mathbb{N}.$$

Proof. In (108) choose $\delta := \left(L_N\left(|\cdot - x_0|^{\lambda+1}\right)(x_0)\right)^{\frac{1}{\lambda+1}}$. □

Note: From (109) we get that: if $L_N\left(|\cdot - x_0|^{\lambda+1}\right)(x_0) \to 0$, as $N \to +\infty$, then $L_N(f)(x_0) \to f(x_0)$, as $N \to +\infty$.

We present

Theorem 4.4. *Let $0 < \alpha \leq \frac{1}{n+1}$, $n \in \mathbb{N}$, $f : [0,1] \to \mathbb{R}_+$, $f' \in L_\infty([0,1])$, $x \in (0,1)$. Assume that $D_{*x}^{k\alpha} f \in C([x,1])$, $k = 0,1,...,n+1$, and $\left(D_{*x}^{i\alpha} f\right)(x) = 0$, $i = 2,3,...,n+1$. Also, suppose that $D_{x-}^{k\alpha} f \in C([0,x])$, for $k = 0,1,...,n+1$, and $\left(D_{x-}^{i\alpha} f\right)(x) = 0$, for $i = 2,3,...,n+1$. Denote $\lambda := (n+1)\alpha \leq 1$. Then*

$$\left|B_N^{(M)}(f)(x) - f(x)\right| \leq \frac{(\lambda+2)\,\omega_1\left(D_x^{\lambda} f, \left(\frac{6}{\sqrt{N+1}}\right)^{\frac{1}{\lambda+1}}\right)}{\Gamma(\lambda+2)} \left(\frac{6}{\sqrt{N+1}}\right)^{\frac{\lambda}{\lambda+1}}, \tag{110}$$

$\forall\, N \in \mathbb{N}$.

See that $\lim_{N \to +\infty} B_N^{(M)}(f)(x) = f(x)$.

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Proof. The Max-product Bernstein operators $B_N^{(M)}(f)(x)$ are defined by (5), see also [6], p. 10; here $f : [0,1] \to \mathbb{R}_+$ is a continuous function.

We have $B_N^{(M)}(1) = 1$, and

$$B_N^{(M)}(|\cdot - x|)(x) \leq \frac{6}{\sqrt{N+1}}, \quad \forall\, x \in [0,1], \forall\, N \in \mathbb{N}, \tag{111}$$

see [6], p. 31.

$B_N^{(M)}$ are positive sublinear operators and thus they possess the monotonicity property, also since $|\cdot - x| \leq 1$, then $|\cdot - x|^{\beta} \leq 1$, $\forall\, x \in [0,1]$, $\forall\, \beta > 0$.

Therefore it holds

$$B_N^{(M)}\left(|\cdot - x|^{1+\beta}\right)(x) \leq \frac{6}{\sqrt{N+1}}, \quad \forall\, x \in [0,1], \forall\, N \in \mathbb{N}, \forall\, \beta > 0. \tag{112}$$

Furthermore, clearly it holds that

$$B_N^{(M)}\left(|\cdot - x|^{1+\beta}\right)(x) > 0, \forall\, N \in \mathbb{N}, \forall\, \beta \geq 0 \text{ and any } x \in (0,1). \tag{113}$$

The operator $B_N^{(M)}$ maps $C_+([0,1])$ into itself. We apply (109). □

We continue with

Remark 4.5. *The truncated Favard-Szász-Mirakjan operators are given by*

$$T_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} s_{N,k}(x) f\left(\frac{k}{N}\right)}{\bigvee_{k=0}^{N} s_{N,k}(x)}, \quad x \in [0,1], \ N \in \mathbb{N}, \ f \in C_+([0,1]), \tag{114}$$

$s_{N,k}(x) = \frac{(Nx)^k}{k!}$, *see also* [6], *p. 11.*

By [6], *p. 178-179, we get that*

$$T_N^{(M)}(|\cdot - x|)(x) \leq \frac{3}{\sqrt{N}}, \quad \forall\, x \in [0,1], \ \forall\, N \in \mathbb{N}. \tag{115}$$

Clearly it holds

$$T_N^{(M)}\left(|\cdot - x|^{1+\beta}\right)(x) \leq \frac{3}{\sqrt{N}}, \quad \forall\, x \in [0,1], \ \forall\, N \in \mathbb{N}, \forall\, \beta > 0. \tag{116}$$

The operators $T_N^{(M)}$ are positive sublinear operators mapping $C_+([0,1])$ into itself, with $T_N^{(M)}(1) = 1$.

Furthermore it holds

$$T_N^{(M)}\left(|\cdot - x|^{\lambda}\right)(x) = \frac{\bigvee_{k=0}^{N} \frac{(Nx)^k}{k!} \left|\frac{k}{N} - x\right|^{\lambda}}{\bigvee_{k=0}^{N} \frac{(Nx)^k}{k!}} > 0, \quad \forall\, x \in (0,1], \ \forall\, \lambda \geq 1, \forall\, N \in \mathbb{N}. \tag{117}$$

We give

Theorem 4.6. *All as in Theorem 4.4, with $x \in (0,1]$. Then*

$$\left|T_N^{(M)}(f)(x) - f(x)\right| \leq \frac{(\lambda+2)\,\omega_1\left(D_x^{\lambda} f, \left(\frac{3}{\sqrt{N}}\right)^{\frac{1}{\lambda+1}}\right)}{\Gamma(\lambda+2)} \left(\frac{3}{\sqrt{N}}\right)^{\frac{\lambda}{\lambda+1}}, \quad \forall\, N \in \mathbb{N}. \tag{118}$$

As $N \to +\infty$, we get $T_N^{(M)}(f)(x) \to f(x)$.

Proof. We apply (109). □

We make

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Remark 4.7. *Next we study the truncated Max-product Baskakov operators (see [6], p. 11)*

$$U_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} b_{N,k}(x) f\left(\frac{k}{N}\right)}{\bigvee_{k=0}^{N} b_{N,k}(x)}, \quad x \in [0,1], \ f \in C_+([0,1]), \ N \in \mathbb{N}, \tag{119}$$

where

$$b_{N,k}(x) = \binom{N+k-1}{k} \frac{x^k}{(1+x)^{N+k}}. \tag{120}$$

From [6], pp. 217-218, we get ($x \in [0,1]$)

$$\left(U_N^{(M)}(|\cdot - x|)\right)(x) \le \frac{2\sqrt{3}\left(\sqrt{2}+2\right)}{\sqrt{N+1}}, \quad N \ge 2, \ N \in \mathbb{N}. \tag{121}$$

Let $\beta \ge 1$, clearly then it holds

$$\left(U_N^{(M)}\left(|\cdot - x|^{\beta}\right)\right)(x) \le \frac{2\sqrt{3}\left(\sqrt{2}+2\right)}{\sqrt{N+1}}, \quad \forall \ N \ge 2, \ N \in \mathbb{N}. \tag{122}$$

Also it holds $U_N^{(M)}(1) = 1$, and $U_N^{(M)}$ are positive sublinear operators from $C_+([0,1])$ into itself. Furthermore it holds

$$U_N^{(M)}\left(|\cdot - x|^{\beta}\right)(x) > 0, \quad \forall \ x \in (0,1], \ \forall \ \beta \ge 1, \ \forall \ N \in \mathbb{N}. \tag{123}$$

We give

Theorem 4.8. *All as in Theorem 4.4, with $x \in (0,1]$. Then*

$$\left|U_N^{(M)}(f)(x) - f(x)\right| \le \frac{(\lambda+2)\,\omega_1\left(D_x^{\lambda} f, \left(\frac{2\sqrt{3}\left(\sqrt{2}+2\right)}{\sqrt{N+1}}\right)^{\frac{1}{\lambda+1}}\right)}{\Gamma(\lambda+2)}.$$

$$\left(\frac{2\sqrt{3}\left(\sqrt{2}+2\right)}{\sqrt{N+1}}\right)^{\frac{\lambda}{\lambda+1}}, \quad \forall \ N \ge 2, \ N \in \mathbb{N}. \tag{124}$$

As $N \to +\infty$, we get $U_N^{(M)}(f)(x) \to f(x)$.

Proof. By Theorem 4.3. □

We continue with

Remark 4.9. *Here we study the Max-product Meyer-Köning and Zeller operators (see [6], p. 11) defined by*

$$Z_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{\infty} s_{N,k}(x) f\left(\frac{k}{N+k}\right)}{\bigvee_{k=0}^{\infty} s_{N,k}(x)}, \quad \forall \ N \in \mathbb{N}, \ f \in C_+([0,1]), \tag{125}$$

$s_{N,k}(x) = \binom{N+k}{k} x^k$, $x \in [0,1]$.

By [6], p. 253, we get that

$$Z_N^{(M)}(|\cdot - x|)(x) \le \frac{8\left(1+\sqrt{5}\right)}{3} \frac{\sqrt{x}(1-x)}{\sqrt{N}}, \quad \forall \ x \in [0,1], \ \forall \ N \ge 4, \ N \in \mathbb{N}. \tag{126}$$

We have that (for $\beta \ge 1$)

$$Z_N^{(M)}\left(|\cdot - x|^{\beta}\right)(x) \le \frac{8\left(1+\sqrt{5}\right)}{3} \frac{\sqrt{x}(1-x)}{\sqrt{N}} := \rho(x), \tag{127}$$

$\forall\ x \in [0,1],\ N \geq 4,\ N \in \mathbb{N}.$

Also it holds $Z_N^{(M)}(1) = 1$, *and* $Z_N^{(M)}$ *are positive sublinear operators from* $C_+([0,1])$ *into itself. lso it holds*

$$Z_N^{(M)}\left(|\cdot - x|^{\beta}\right)(x) > 0,\quad \forall\ x \in (0,1),\ \forall\ \beta \geq 1,\ \forall\ N \in \mathbb{N}. \tag{128}$$

We give

Theorem 4.10. *All as in Theorem 4.4. Then*

$$\left|Z_N^{(M)}(f)(x) - f(x)\right| \leq \frac{(\lambda+2)\,\omega_1\left(D_x^{\lambda} f, (\rho(x))^{\frac{1}{\lambda+1}}\right)}{\Gamma(\lambda+2)}(\rho(x))^{\frac{\lambda}{\lambda+1}} \tag{129}$$

$\forall\ N \geq 4,\ N \in \mathbb{N}.$

As $N \to +\infty$, *we get* $Z_N^{(M)}(f)(x) \to f(x)$.

Proof. By Theorem 4.3. ∎

We continue with

Remark 4.11. *Here we deal with the Max-product truncated sampling operators (see* [6]*, p. 13) defined by*

$$W_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} \frac{\sin(Nx-k\pi)}{Nx-k\pi} f\left(\frac{k\pi}{N}\right)}{\bigvee_{k=0}^{N} \frac{\sin(Nx-k\pi)}{Nx-k\pi}}, \tag{130}$$

$\forall\ x \in [0,\pi]$, $f : [0,\pi] \to \mathbb{R}_+$ *a continuous function. See also Remark 3.8.*

By [6]*, p. 346, we have*

$$W_N^{(M)}(|\cdot - x|)(x) \leq \frac{\pi}{2N},\quad \forall\ N \in \mathbb{N},\ \forall\ x \in [0,\pi]. \tag{131}$$

Furthermore it holds $(\beta \geq 1)$

$$W_N^{(M)}\left(|\cdot - x|^{\beta}\right)(x) \leq \frac{\pi^{\beta}}{2N},\quad \forall\ N \in \mathbb{N}, \forall\ x \in [0,\pi]. \tag{132}$$

Also it holds $(\beta \geq 1)$

$$W_N^{(M)}\left(|\cdot - x|^{\beta}\right)(x) > 0,\quad \forall\ x \in [0,\pi], \tag{133}$$

such that $x \neq \frac{k\pi}{N}$, *for any* $k \in \{0,1,...,N\}$, *see* [3].

We present

Theorem 4.12. *Let* $0 < \alpha \leq \frac{1}{n+1}$, $n \in \mathbb{N}$, $x \in [0,\pi]$ *be such that* $x \neq \frac{k\pi}{N}$, $k \in \{0,1,...,N\}$, $\forall\ N \in \mathbb{N}$; $f : [0,\pi] \to \mathbb{R}_+$, $f' \in L_{\infty}([0,\pi])$. *Assume that* $D_{*x}^{k\alpha} f \in C([x,\pi])$, $k = 0,1,...,n+1$, *and* $\left(D_{*x}^{i\alpha} f\right)(x) = 0$, $i = 2,3,...,n+1$. *Also, suppose that* $D_{x-}^{k\alpha} f \in C([0,x])$, *for* $k = 0,1,...,n+1$, *and* $\left(D_{x-}^{i\alpha} f\right)(x) = 0$, *for* $i = 2,3,...,n+1$. *Denote* $\lambda := (n+1)\alpha \leq 1$. *Then*

$$\left|W_N^{(M)}(f)(x) - f(x)\right| \leq \frac{(\lambda+2)\,\omega_1\left(D_x^{\lambda} f, \left(\frac{\pi^{\lambda+1}}{2N}\right)^{\frac{1}{\lambda+1}}\right)}{\Gamma(\lambda+2)}\left(\frac{\pi^{\lambda+1}}{2N}\right)^{\frac{\lambda}{\lambda+1}},\quad \forall\ N \in \mathbb{N}. \tag{134}$$

As $N \to +\infty$, *we get* $W_N^{(M)}(f)(x) \to f(x)$.

Proof. By (132), (133) and Theorem 4.3. ∎

We make

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Remark 4.13. *Here we continue with the Max-product truncated sampling operators (see* [6]*, p. 13) defined by*

$$K_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} \frac{\sin^2(Nx-k\pi)}{(Nx-k\pi)^2} f\left(\frac{k\pi}{N}\right)}{\bigvee_{k=0}^{N} \frac{\sin^2(Nx-k\pi)}{(Nx-k\pi)^2}}, \tag{135}$$

$\forall\, x \in [0,\pi]$, $f : [0,\pi] \to \mathbb{R}_+$ a continuous function.

See also Remark 3.10.

It holds ($\beta \geq 1$)

$$K_N^{(M)}\left(|\cdot - x|^{\beta}\right)(x) \leq \frac{\pi^{\beta}}{2N}, \quad \forall\, N \in \mathbb{N}, \forall\, x \in [0,\pi]. \tag{136}$$

By [3]*, we get that ($\beta \geq 1$)*

$$K_N^{(M)}\left(|\cdot - x|^{\beta}\right)(x) > 0, \quad \forall\, x \in [0,\pi], \tag{137}$$

such that $x \neq \frac{k\pi}{N}$, for any $k \in \{0,1,...,N\}$.

We continue with

Theorem 4.14. *All as in Theorem 4.12. Then*

$$\left|K_N^{(M)}(f)(x) - f(x)\right| \leq \frac{(\lambda+2)\,\omega_1\left(D_x^{\lambda} f, \left(\frac{\pi^{\lambda+1}}{2N}\right)^{\frac{1}{\lambda+1}}\right)}{\Gamma(\lambda+1)} \left(\frac{\pi^{\lambda+1}}{2N}\right)^{\frac{\lambda}{\lambda+1}}, \quad \forall\, N \in \mathbb{N}. \tag{138}$$

As $N \to +\infty$, we get $K_N^{(M)}(f)(x) \to f(x)$.

Proof. By (136), (137) and Theorem 4.3. □

We finish with

Corollary 4.15. *(to Theorem 4.4, $\alpha = \frac{1}{4}$, $n = 2$, $\lambda = \frac{3}{4}$) Let $f : [0,1] \to \mathbb{R}_+$, $f' \in L_{\infty}([0,1])$, $x \in (0,1)$. Assume that $D_{*x}^{k\frac{1}{4}} f \in C([x,1])$, $k = 0,1,2,3$, and $\left(D_{*x}^{i\frac{1}{4}} f\right)(x) = 0$, $i = 2,3$. Suppose that $D_{x-}^{k\frac{1}{4}} f \in C([0,x])$, for $k = 0,1,2,3$, and $\left(D_{x-}^{i\frac{1}{4}} f\right)(x) = 0$, for $i = 2,3$. Then*

$$\left|B_N^{(M)}(f)(x) - f(x)\right| \leq \tag{139}$$

$$(1.709)\,\omega_1\left(D_x^{3\cdot\frac{1}{4}} f, \left(\frac{6}{\sqrt{N+1}}\right)^{\frac{4}{7}}\right)\left(\frac{6}{\sqrt{N+1}}\right)^{\frac{3}{7}}, \quad \forall\, N \in \mathbb{N}.$$

And $\lim_{N \to +\infty} B_N^{(M)}(f)(x) = f(x)$.

Proof. Use of (110). □

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