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# COMPLEX OPIAL TYPE INEQUALITIES

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Abstract. We establish here complex Opial type inequalities for analytic functions from a complex numbers domain into the set of complex numbers.

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## 1. Introduction

This article is greatly motivated by the article of Z. Opial [4].

**Theorem 1.1.** *(Opial, 1960) Let $x(t) \in C^1([0,h])$ be such that $x(0) = x(h) = 0$, and $x(t) > 0$ in $(0,h)$. Then*

$$\int_0^h |x(t)\, x'(t)|\, dt \leq \frac{h}{4} \int_0^h (x'(t))^2\, dt. \tag{1}$$

In the last inequality the constant $\frac{h}{4}$ is the best possible.
Equality holds for the function

$$x(t) = t \quad \text{on} \quad \left[0, \frac{h}{2}\right]$$

and

$$x(t) = h - t \quad \text{on} \quad \left[\frac{h}{2}, h\right].$$

Opial type inequalities have applications in establishing uniqueness of solution to initial value problems in differential equations, see [5], also find upper bounds to such solutions.

We are also inspired by the author's monographs [1], [2], to continue our search for Opial type inequalities in the complex numbers setting.

## 2. Background

See also [3].

Let $\gamma$ be a smooth path parametrized by $z(t)$, $t \in [a,b]$ and $f$ is a complex function which is continuous on $\gamma$. Put $z(a) = u$ and $z(b) = w$ with $u, w \in \mathbb{C}$. We define the integral of $f$ on $\gamma_{u,w} = \gamma$ as

$$\int_\gamma f(z)\, dz = \int_{\gamma_{u,w}} f(z)\, dz := \int_a^b f(z(t))\, z'(t)\, dt.$$

We notice that the actual choice of parametrization of $\gamma$ does not matter.

This definition immediately extends to paths that are piecewise smooth. Suppose $\gamma$ is parametrized by $z(t)$, $t \in [a,b]$, which is differentiable on the intervals $[a,c]$ and $[c,b]$, then assuming that $f$ is continuous on $\gamma$ we define

$$\int_{\gamma_{u,w}} f(z)\,dz := \int_{\gamma_{u,v}} f(z)\,dz + \int_{\gamma_{v,w}} f(z)\,dz,$$

where $v := z(c)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z)\,|dz| := \int_a^b f(z(t))\,|z'(t)|\,dt$$

and the length of the curve $\gamma$ is then

$$l(\gamma) = \int_{\gamma_{u,w}} |dz| := \int_a^b |z'(t)|\,dt.$$

We mention also the triangle inequality for the complex integral, namely

$$\left|\int_\gamma f(z)\,dz\right| \le \int_\gamma |f(z)|\,|dz| \le \|f\|_{\gamma,\infty}\, l(\gamma), \tag{2}$$

where $\|f\|_{\gamma,\infty} := \sup_{z\in\gamma} |f(z)|$.

S. Dragomir in [3] proved the following useful complex Taylor's formula with remainder over a non-necessarily convex domain $D$.

**Theorem 2.1.** *Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the domain $D$ and $y, x \in D$. Suppose $\gamma$ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ with $z(a) = x$ and $z(b) = y$ then*

$$f(y) = \sum_{k=0}^{n} \frac{(y-x)^k}{k!} f^{(k)}(x) + \frac{1}{n!}\int_{\gamma_{x,y}} (y-z)^n f^{(n+1)}(z)\,dz, \tag{3}$$

*for $n \in \mathbb{Z}_+$.*

## 3. Main Results

A complex Opial type inequality follows

**Theorem 3.1.** *Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the domain $D$ and let $x, y, w \in D$. Suppose $\gamma$ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ with $z(a) = x$, $z(c) = y$, and $z(b) = w$, where $c \in [a,b]$ is floating. Assume that $f^{(k)}(x) = 0$, $k = 0, 1, \ldots, n$, $n \in \mathbb{Z}_+$, and $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then*

*1)*

$$\left|\int_a^b f(z(t))\, f^{(n+1)}(z(t))\, z'(t)\,dt\right| \le \int_a^b |f(z(t))| \left|f^{(n+1)}(z(t))\right| |z'(t)|\,dt \le \tag{4}$$

$$\frac{1}{2^{\frac{1}{q}} n!}\left[\int_a^b \left(\int_a^c |z(c) - z(t)|^{pn} |z'(t)|\,dt\right) |z'(c)|\,dc\right]^{\frac{1}{p}} \cdot$$

$$\left(\int_a^b \left|f^{(n+1)}(z(t))\right|^q |z'(t)|\,dt\right)^{\frac{2}{q}},$$

*equivalently it holds*

*2)*

$$\left|\int_{\gamma_{x,w}} f(z)\, f^{(n+1)}(z)\,dz\right| \le \int_{\gamma_{x,w}} |f(z)| \left|f^{(n+1)}(z)\right| |dz| \le$$



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$$(5)\qquad \frac{1}{2^{\frac{1}{q}} n!}\left[\int_a^b \left(\int_{\gamma_{x,y}} |z(c)-z|^{pn} |dz|\right) |z'(c)|\, dc\right]^{\frac{1}{p}} \left(\int_{\gamma_{x,w}} \left|f^{(n+1)}(z)\right|^q |dz|\right)^{\frac{2}{q}}.$$

*Proof.* By (3) we obtain

$$f(y) = \frac{1}{n!}\int_{\gamma_{x,y}} (y-z)^n f^{(n+1)}(z)\, dz, \quad n \in \mathbb{Z}_+.$$

Then by triangle's and Hölder's inequalities we have

$$|f(y)| \le \frac{1}{n!}\int_{\gamma_{x,y}} |y-z|^n \left|f^{(n+1)}(z)\right| |dz| =$$

$$(6)\qquad \frac{1}{n!}\int_a^c |y-z(t)|^n \left|f^{(n+1)}(z(t))\right| |z'(t)|\, dt \le$$

$$\frac{1}{n!}\left(\int_a^c |y-z(t)|^{pn} |z'(t)|\, dt\right)^{\frac{1}{p}} \left(\int_a^c \left|f^{(n+1)}(z(t))\right|^q |z'(t)|\, dt\right)^{\frac{1}{q}}.$$

We set

$$(7)\qquad \rho(c) := \int_a^c \left|f^{(n+1)}(z(t))\right|^q |z'(t)|\, dt, \quad a \le c \le b,$$

then $\rho(a) = 0$, and

$$\rho'(c) = \left|f^{(n+1)}(z(c))\right|^q |z'(c)| \ge 0.$$

That is

$$(8)\qquad \left|f^{(n+1)}(z(c))\right| |z'(c)|^{\frac{1}{q}} = (\rho'(c))^{\frac{1}{q}}.$$

Hence it holds

$$(9)\qquad |f(z(c))| \left|f^{(n+1)}(z(c))\right| |z'(c)|^{\frac{1}{q}} |z'(c)|^{\frac{1}{p}} \le$$

$$\frac{1}{n!}\left(\int_a^c |z(c)-z(t)|^{pn} |z'(t)|\, dt\right)^{\frac{1}{p}} (\rho(c)\rho'(c))^{\frac{1}{q}} |z'(c)|^{\frac{1}{p}}.$$

Integrating (9) and by Hölder's inequality we obtain

$$\int_a^b |f(z(c))| \left|f^{(n+1)}(z(c))\right| |z'(c)|\, dc \le$$

$$\frac{1}{n!}\int_a^b \left[\left(\int_a^c |z(c)-z(t)|^{pn} |z'(t)|\, dt\right) |z'(c)|\right]^{\frac{1}{p}} (\rho(c)\rho'(c))^{\frac{1}{q}}\, dc \le$$

$$(10)\qquad \frac{1}{n!}\left\{\int_a^b \left[\left(\int_a^c |z(c)-z(t)|^{pn} |z'(t)|\, dt\right) |z'(c)|\right] dc\right\}^{\frac{1}{p}} \left(\int_a^b \rho(c)\rho'(c)\, dc\right)^{\frac{1}{q}} =$$

$$\frac{1}{n!}\left\{\int_a^b \left[\left(\int_a^c |z(c)-z(t)|^{pn} |z'(t)|\, dt\right) |z'(c)|\right] dc\right\}^{\frac{1}{p}} \frac{\rho(b)^{\frac{2}{q}}}{2^{\frac{1}{q}}},$$

proving the claim. □

We continue with an extreme case

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**Proposition 3.2.** *All here are as in Theorem 3.1 but with $p = 1$, $q = \infty$. Then*

*1)*

$$\left|\int_a^b f(z(t)) f^{(n+1)}(z(t)) z'(t)\, dt\right| \leq \int_a^b |f(z(t))| \left|f^{(n+1)}(z(t))\right| |z'(t)|\, dt \leq$$

$$\left[\int_a^b \left(\int_{\gamma_{x,y}} |z(c) - z|^n |dz|\right) |z'(c)|\, dc\right] \left\|f^{(n+1)}\right\|_{\gamma_{x,y},\infty}^2, \tag{11}$$

*equivalently it holds*

*2)*

$$\left|\int_{\gamma_{x,w}} f(z) f^{(n+1)}(z)\, dz\right| \leq \int_{\gamma_{x,w}} |f(z)| \left|f^{(n+1)}(z)\right| |dz| \leq$$

$$\left[\int_a^b \left(\int_{\gamma_{x,y}} |z(c) - z|^n |dz|\right) |z'(c)|\, dc\right] \left\|f^{(n+1)}\right\|_{\gamma_{x,y},\infty}^2. \tag{12}$$

*Proof.* By (3) we obtain again

$$f(y) = \frac{1}{n!} \int_{\gamma_{x,y}} (y - z)^n f^{(n+1)}(z)\, dz, \quad n \in \mathbb{N}. \tag{13}$$

Hence it holds

$$|f(y)| \leq \frac{1}{n!} \int_{\gamma_{x,y}} |y - z|^n \left|f^{(n+1)}(z)\right| |dz| \leq$$

$$\left(\int_{\gamma_{x,y}} |y - z|^n |dz|\right) \left\|f^{(n+1)}\right\|_{\gamma_{x,y},\infty}. \tag{14}$$

Therefore we have

$$|f(y)| \left|f^{(n+1)}(y)\right| \leq \left(\int_{\gamma_{x,y}} |y - z|^n |dz|\right) \left\|f^{(n+1)}\right\|_{\gamma_{x,y},\infty}^2. \tag{15}$$

That is

$$|f(z(c))| \left|f^{(n+1)}(z(c))\right| |z'(c)| \leq \left(\int_{\gamma_{x,y}} |z(c) - z|^n |dz|\right) |z'(c)| \left\|f^{(n+1)}\right\|_{\gamma_{x,y},\infty}^2. \tag{16}$$

Consequently by integration of (16) we derive

$$\int_a^b |f(z(c))| \left|f^{(n+1)}(z(c))\right| |z'(c)|\, dc \leq$$

$$\left\{\int_a^b \left[\left(\int_{\gamma_{x,y}} |z(c) - z|^n |dz|\right) |z'(c)|\right] dc\right\} \left\|f^{(n+1)}\right\|_{\gamma_{x,y},\infty}^2, \tag{17}$$

proving the claim. □

A typical case follows:

**Corollary 3.3.** *(to Theorem 3.1 when $p = q = 2$) We have*

*1)*

$$\left|\int_a^b f(z(t)) f^{(n+1)}(z(t)) z'(t)\, dt\right| \leq \int_a^b |f(z(t))| \left|f^{(n+1)}(z(t))\right| |z'(t)|\, dt \leq \tag{18}$$

$$\frac{1}{\sqrt{2}n!}\left[\int_a^b\left(\int_a^c\left|z\left(c\right)-z\left(t\right)\right|^{2n}\left|z'\left(t\right)\right|dt\right)\left|z'\left(c\right)\right|dc\right]^{\frac{1}{2}}\cdot$$
$$\left(\int_a^b\left|f^{(n+1)}\left(z\left(t\right)\right)\right|^2\left|z'\left(t\right)\right|dt\right),$$

*equivalently it holds*

*2)*

$$\left|\int_{\gamma_{x,w}} f\left(z\right)f^{(n+1)}\left(z\right)dz\right|\leq\int_{\gamma_{x,w}}\left|f\left(z\right)\right|\left|f^{(n+1)}\left(z\right)\right|\left|dz\right|\leq$$

$$\frac{1}{\sqrt{2}n!}\left[\int_a^b\left(\int_{\gamma_{x,y}}\left|z\left(c\right)-z\right|^{2n}\left|dz\right|\right)\left|z'\left(c\right)\right|dc\right]^{\frac{1}{2}}\left(\int_{\gamma_{x,w}}\left|f^{(n+1)}\left(z\right)\right|^2\left|dz\right|\right). \tag{19}$$

We finish with

**Corollary 3.4.** *(to Theorem 3.1, $n=0$ case) Here we assume that $f\left(x\right)=0$. Then*

$$\left|\int_{\gamma_{x,w}} f\left(z\right)f'\left(z\right)dz\right|\leq\int_{\gamma_{x,w}}\left|f\left(z\right)\right|\left|f'\left(z\right)\right|\left|dz\right|\leq$$

$$2^{-\frac{1}{q}}\left(\int_{\gamma_{x,w}} l\left(\gamma_{x,z}\right)\left|dz\right|\right)^{\frac{1}{p}}\left(\int_{\gamma_{x,w}}\left|f'\left(z\right)\right|^q\left|dz\right|\right)^{\frac{2}{q}}. \tag{20}$$

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