

COMPLEX OPIAL TYPE INEQUALITIES

GEORGE A. ANASTASSIOU

ABSTRACT. We establish here complex Opial type inequalities for analytic functions from a complex numbers domain into the set of complex numbers.

Mathematics Subject Classification (2010): 26D10, 26D15, 30A10.

Key words: Opial's inequality, complex Taylor's formula.

Article history:

Received 20 February 2019

Accepted 24 October 2019

1. INTRODUCTION

This article is greatly motivated by the article of Z. Opial [4].

Theorem 1.1. (*Opial, 1960*) Let $x(t) \in C^1([0, h])$ be such that $x(0) = x(h) = 0$, and $x(t) > 0$ in $(0, h)$. Then

$$(1) \quad \int_0^h |x(t)x'(t)| dt \leq \frac{h}{4} \int_0^h (x'(t))^2 dt.$$

In the last inequality the constant $\frac{h}{4}$ is the best possible.
Equality holds for the function

$$x(t) = t \quad \text{on} \quad \left[0, \frac{h}{2}\right]$$

and

$$x(t) = h - t \quad \text{on} \quad \left[\frac{h}{2}, h\right].$$

Opial type inequalities have applications in establishing uniqueness of solution to initial value problems in differential equations, see [5], also find upper bounds to such solutions.

We are also inspired by the author's monographs [1], [2], to continue our search for Opial type inequalities in the complex numbers setting.

2. BACKGROUND

See also [3].

Let γ be a smooth path parametrized by $z(t)$, $t \in [a, b]$ and f is a complex function which is continuous on γ . Put $z(a) = u$ and $z(b) = w$ with $u, w \in \mathbb{C}$. We define the integral of f on $\gamma_{u,w} = \gamma$ as

$$\int_{\gamma} f(z) dz = \int_{\gamma_{u,w}} f(z) dz := \int_a^b f(z(t)) z'(t) dt.$$

We notice that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are piecewise smooth. Suppose γ is parametrized by $z(t)$, $t \in [a, b]$, which is differentiable on the intervals $[a, c]$ and $[c, b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z) dz := \int_{\gamma_{u,v}} f(z) dz + \int_{\gamma_{v,w}} f(z) dz,$$

where $v := z(c)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z) |dz| := \int_a^b f(z(t)) |z'(t)| dt$$

and the length of the curve γ is then

$$l(\gamma) = \int_{\gamma_{u,w}} |dz| := \int_a^b |z'(t)| dt.$$

We mention also the triangle inequality for the complex integral, namely

$$(2) \quad \left| \int_{\gamma} f(z) dz \right| \leq \int_{\gamma} |f(z)| |dz| \leq \|f\|_{\gamma, \infty} l(\gamma),$$

where $\|f\|_{\gamma, \infty} := \sup_{z \in \gamma} |f(z)|$.

S. Dragomir in [3] proved the following useful complex Taylor's formula with remainder over a non-necessarily convex domain D .

Theorem 2.1. *Let $f : D \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be an analytic function on the domain D and $y, x \in D$. Suppose γ is a smooth path parametrized by $z(t)$, $t \in [a, b]$ with $z(a) = x$ and $z(b) = y$ then*

$$(3) \quad f(y) = \sum_{k=0}^n \frac{(y-x)^k}{k!} f^{(k)}(x) + \frac{1}{n!} \int_{\gamma_{x,y}} (y-z)^n f^{(n+1)}(z) dz,$$

for $n \in \mathbb{Z}_+$.

3. MAIN RESULTS

A complex Opial type inequality follows

Theorem 3.1. *Let $f : D \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be an analytic function on the domain D and let $x, y, w \in D$. Suppose γ is a smooth path parametrized by $z(t)$, $t \in [a, b]$ with $z(a) = x$, $z(c) = y$, and $z(b) = w$, where $c \in [a, b]$ is floating. Assume that $f^{(k)}(x) = 0$, $k = 0, 1, \dots, n$, $n \in \mathbb{Z}_+$, and $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then*

1)

$$(4) \quad \left| \int_a^b f(z(t)) f^{(n+1)}(z(t)) z'(t) dt \right| \leq \int_a^b |f(z(t))| |f^{(n+1)}(z(t))| |z'(t)| dt \leq$$

$$\frac{1}{2^{\frac{1}{q}} n!} \left[\int_a^b \left(\int_a^c |z(c) - z(t)|^{pn} |z'(t)| dt \right) |z'(c)| dc \right]^{\frac{1}{p}} \cdot$$

$$\left(\int_a^b |f^{(n+1)}(z(t))|^q |z'(t)| dt \right)^{\frac{2}{q}},$$

equivalently it holds

2)

$$\left| \int_{\gamma_{x,w}} f(z) f^{(n+1)}(z) dz \right| \leq \int_{\gamma_{x,w}} |f(z)| |f^{(n+1)}(z)| |dz| \leq$$

$$(5) \quad \frac{1}{2^{\frac{1}{q}} n!} \left[\int_a^b \left(\int_{\gamma_{x,y}} |z(c) - z|^{pn} |dz| \right) |z'(c)| dc \right]^{\frac{1}{p}} \left(\int_{\gamma_{x,w}} |f^{(n+1)}(z)|^q |dz| \right)^{\frac{2}{q}}.$$

Proof. By (3) we obtain

$$f(y) = \frac{1}{n!} \int_{\gamma_{x,y}} (y-z)^n f^{(n+1)}(z) dz, \quad n \in \mathbb{Z}_+.$$

Then by triangle's and Hölder's inequalities we have

$$(6) \quad |f(y)| \leq \frac{1}{n!} \int_{\gamma_{x,y}} |y-z|^n |f^{(n+1)}(z)| |dz| =$$

$$\frac{1}{n!} \int_a^c |y-z(t)|^n |f^{(n+1)}(z(t))| |z'(t)| dt \leq$$

$$\frac{1}{n!} \left(\int_a^c |y-z(t)|^{pn} |z'(t)| dt \right)^{\frac{1}{p}} \left(\int_a^c |f^{(n+1)}(z(t))|^q |z'(t)| dt \right)^{\frac{1}{q}}.$$

We set

$$(7) \quad \rho(c) := \int_a^c |f^{(n+1)}(z(t))|^q |z'(t)| dt, \quad a \leq c \leq b,$$

then $\rho(a) = 0$, and

$$\rho'(c) = |f^{(n+1)}(z(c))|^q |z'(c)| \geq 0.$$

That is

$$(8) \quad |f^{(n+1)}(z(c))| |z'(c)|^{\frac{1}{q}} = (\rho'(c))^{\frac{1}{q}}.$$

Hence it holds

$$(9) \quad |f(z(c))| |f^{(n+1)}(z(c))| |z'(c)|^{\frac{1}{q}} |z'(c)|^{\frac{1}{p}} \leq$$

$$\frac{1}{n!} \left(\int_a^c |z(c) - z(t)|^{pn} |z'(t)| dt \right)^{\frac{1}{p}} (\rho(c) \rho'(c))^{\frac{1}{q}} |z'(c)|^{\frac{1}{p}}.$$

Integrating (9) and by Hölder's inequality we obtain

$$(10) \quad \int_a^b |f(z(c))| |f^{(n+1)}(z(c))| |z'(c)| dc \leq$$

$$\frac{1}{n!} \int_a^b \left[\left(\int_a^c |z(c) - z(t)|^{pn} |z'(t)| dt \right) |z'(c)| \right]^{\frac{1}{p}} (\rho(c) \rho'(c))^{\frac{1}{q}} dc \leq$$

$$\frac{1}{n!} \left\{ \int_a^b \left[\left(\int_a^c |z(c) - z(t)|^{pn} |z'(t)| dt \right) |z'(c)| \right] dc \right\}^{\frac{1}{p}} \left(\int_a^b \rho(c) \rho'(c) dc \right)^{\frac{1}{q}} =$$

$$\frac{1}{n!} \left\{ \int_a^b \left[\left(\int_a^c |z(c) - z(t)|^{pn} |z'(t)| dt \right) |z'(c)| \right] dc \right\}^{\frac{1}{p}} \frac{\rho(b)^{\frac{2}{q}}}{2^{\frac{1}{q}}},$$

proving the claim. □

We continue with an extreme case

Proposition 3.2. *All here are as in Theorem 3.1 but with $p = 1$, $q = \infty$. Then*

1)

$$(11) \quad \left| \int_a^b f(z(t)) f^{(n+1)}(z(t)) z'(t) dt \right| \leq \int_a^b |f(z(t))| |f^{(n+1)}(z(t))| |z'(t)| dt \leq \left[\int_a^b \left(\int_{\gamma_{x,y}} |z(c) - z|^n |dz| \right) |z'(c)| dc \right] \left\| f^{(n+1)} \right\|_{\gamma_{x,y}, \infty}^2,$$

equivalently it holds

2)

$$(12) \quad \left| \int_{\gamma_{x,w}} f(z) f^{(n+1)}(z) dz \right| \leq \int_{\gamma_{x,w}} |f(z)| |f^{(n+1)}(z)| |dz| \leq \left[\int_a^b \left(\int_{\gamma_{x,y}} |z(c) - z|^n |dz| \right) |z'(c)| dc \right] \left\| f^{(n+1)} \right\|_{\gamma_{x,y}, \infty}^2.$$

Proof. By (3) we obtain again

$$(13) \quad f(y) = \frac{1}{n!} \int_{\gamma_{x,y}} (y-z)^n f^{(n+1)}(z) dz, \quad n \in \mathbb{N}.$$

Hence it holds

$$(14) \quad |f(y)| \leq \frac{1}{n!} \int_{\gamma_{x,y}} |y-z|^n |f^{(n+1)}(z)| |dz| \leq \left(\int_{\gamma_{x,y}} |y-z|^n |dz| \right) \left\| f^{(n+1)} \right\|_{\gamma_{x,y}, \infty}.$$

Therefore we have

$$(15) \quad |f(y)| |f^{(n+1)}(y)| \leq \left(\int_{\gamma_{x,y}} |y-z|^n |dz| \right) \left\| f^{(n+1)} \right\|_{\gamma_{x,y}, \infty}^2.$$

That is

$$(16) \quad |f(z(c))| |f^{(n+1)}(z(c))| |z'(c)| \leq \left(\int_{\gamma_{x,y}} |z(c) - z|^n |dz| \right) |z'(c)| \left\| f^{(n+1)} \right\|_{\gamma_{x,y}, \infty}^2.$$

Consequently by integration of (16) we derive

$$(17) \quad \int_a^b |f(z(c))| |f^{(n+1)}(z(c))| |z'(c)| dc \leq \left\{ \int_a^b \left[\left(\int_{\gamma_{x,y}} |z(c) - z|^n |dz| \right) |z'(c)| dc \right] \right\} \left\| f^{(n+1)} \right\|_{\gamma_{x,y}, \infty}^2,$$

proving the claim. □

A typical case follows:

Corollary 3.3. *(to Theorem 3.1 when $p = q = 2$) We have*

1)

$$(18) \quad \left| \int_a^b f(z(t)) f^{(n+1)}(z(t)) z'(t) dt \right| \leq \int_a^b |f(z(t))| |f^{(n+1)}(z(t))| |z'(t)| dt \leq$$

$$\frac{1}{\sqrt{2n!}} \left[\int_a^b \left(\int_a^c |z(c) - z(t)|^{2n} |z'(t)| dt \right) |z'(c)| dc \right]^{\frac{1}{2}} \cdot \left(\int_a^b |f^{(n+1)}(z(t))|^2 |z'(t)| dt \right),$$

equivalently it holds

2)

$$(19) \quad \left| \int_{\gamma_{x,w}} f(z) f^{(n+1)}(z) dz \right| \leq \int_{\gamma_{x,w}} |f(z)| |f^{(n+1)}(z)| |dz| \leq \frac{1}{\sqrt{2n!}} \left[\int_a^b \left(\int_{\gamma_{x,y}} |z(c) - z|^{2n} |dz| \right) |z'(c)| dc \right]^{\frac{1}{2}} \left(\int_{\gamma_{x,w}} |f^{(n+1)}(z)|^2 |dz| \right).$$

We finish with

Corollary 3.4. (to Theorem 3.1, $n = 0$ case) Here we assume that $f(x) = 0$. Then

$$(20) \quad \left| \int_{\gamma_{x,w}} f(z) f'(z) dz \right| \leq \int_{\gamma_{x,w}} |f(z)| |f'(z)| |dz| \leq 2^{-\frac{1}{q}} \left(\int_{\gamma_{x,w}} l(\gamma_{x,z}) |dz| \right)^{\frac{1}{p}} \left(\int_{\gamma_{x,w}} |f'(z)|^q |dz| \right)^{\frac{2}{q}}.$$

REFERENCES

- [1] George A. Anastassiou, *Fractional Differentiation Inequalities*, Springer, Heidelberg, New York, 2009.
- [2] George A. Anastassiou, *Advanced Inequalities*, World Scientific, New Jersey, Singapore, 2011.
- [3] S.S. Dragomir, *An integral representation of the remainder in Taylor's expansion formula for analytic function on general domains*, RGMIA Res. Rep. Coll. 22 (2019), Art. 2, 14 pp., <http://rgmia.org/v22.php>.
- [4] Z. Opial, *Sur une inegalité*, Ann. Polon. Math., 8, 29-32, 1960.
- [5] D. Willett, *The existence-uniqueness theorems for an nth order linear ordinary differential equation*, Amer. Math. Monthly, 75, 174-178, 1968.

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF MEMPHIS, MEMPHIS, TN 38152, U.S.A.

Email address: ganastss@memphis.edu