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MULTIQUADRIC RADIAL BASIS FUNCTION METHOD FOR THE SOLUTION OF BRACHISTOCHRONE PROBLEM

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Abstract. In this paper we present a numerical approach for the solution of brachistochrone problem. The method is based on using collocation points and approximating the solution using multiquadric (MQ) radial basis functions (RBF). The approximate solution is calculated in the form of a series in which its components are computed easily. Error analysis and numerical results are included to demonstrate the capability of the proposed method, and a comparison is made with existing methods in the literature.

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1. Introduction

Finding the brachistochrone, or path of quickest decent, is a historically interesting problem that was posed by John Bernoulli in 1696. The solution of the brachistochrone problem is often cited as the origin of the calculus of variations as suggested in [23]. The classical brachistochrone problem deals with a mass moving along a smooth path in a uniform gravitational field. A mechanical analogy is the motion of a bead sliding down a frictionless wire.

The classical brachistochrone problem may be formulated as an optimal control problem as following. Minimize the performance index J

$$J[X] = \int_0^1 \left[\frac{1+U^2(t)}{1-X(t)}\right]^{\frac{1}{2}} dt, \tag{1.1}$$

subject to

$$U(t) = \dot{X}(t), \tag{1.2}$$

with

$$X(0) = 0, \quad X(1) = -0.5. \tag{1.3}$$

The minimal time transfer expression (1.1) is obtained from the law of conservation of energy. Here X and t are dimensionless and they represent, respectively, the vertical and horizontal coordinates of the sliding bead.

The exact solution to the brachistochrone problem is the cycloid defined by the parametric equations

$$X = 1 - \frac{c}{2}\big(1+\cos(2\alpha)\big), \quad t = t_0 + c\big(2\alpha + \sin(2\alpha)\big),$$

where

$$\tan(\alpha) = 2\frac{dX}{dt} = U(t).$$

With the given boundary conditions, the constants c and t_0 are

$$c = 1.6184891, \quad t_0 = 2.7300631.$$

A many of research work have been invested in recent years for the solution of equations (1.1)-(1.3) for instance, the gradient method [4], successive sweep algorithm [4], [5], the classic Chebshev method [15], multistage Monte Carlo method [24] and nonclassical pseudospectral method [3], [22] etc.

In this paper we use the idea of the interpolation by multiquadric (MQ) RBFs method to approximate the solution of equations (1.1)-(1.3). The collocation points for the interpolation and quadrature formula are Legendre-Gauss-Lobbato (LGL) nodes.

The outline of the rest of this paper organized as follows:

In Section 2, we will introduce positive definite or radial basis functions and its properties, also MQ radial basis function. In Section 3, we will introduce the approach of the approximation equations and error analysis of the MQ radial basis function method. Finally in Section 4, numerical results be given and compared with other methods given in literature.

2. Radial basis functions

In the last decade, the development of the radial basis functions (RBFs) as a truely meshless method, for approximation the solution of partial differential equations has drawn attention of many researcher in science and engineering. RBFs method is a method for interpolating multidimensional scattered data [20], [18]. The multiquaderic (MQ) radial basis function method was introduced in 1971 by Ronald Hardy [13], [14]. Hardy's MQ interpolation method went unnoticed until 1979. Then in [11], it has been shown that MQ method is the best method to solve the scattered data interpolation problems. Also in [19], it has been shown that MQ interpolation method has spectral convergence rate. RBFs were introduced by Hardy, form a primary tool for multivariate interpolation, and they are also receiving increased attention for solving PDE in irregular domains [10]. Hardy [14] showed that multiquadrics RBFs is related to a consistent solution of the biharmonic potential problem and thus has a physical foundation. The first use of the MQ method to solve differential equations [16], [17] was by Edward Kansa in 1990. After the MQ method was first used to solve PDEs [10], [18], the popularity of the method continued to grow rapidly and a large number of applications of the method appeared. Buhmann and Micchelli [7] and Chiu et al. [9] have shown that RBFs are related to prewavelets (wavelets that do not have orthogonality properties). Also Alipanah et al. [1], [2], used RBFs based on Legendre-Gauss-Lobatto nodes for solution of nonlinear integral equations.

2.1. Strictly positive definite functions.

Definition 2.1. A function ϕ on $X \subsetneq \mathbb{R}$ is said to be positive definite on X if for any set of distinct points $x_1, \cdots, x_N$ in X, then the $N \times N$ matrix $A_{ij} = \phi(x_i - x_j)$ is nonnegative definite, i.e.,

$$\mathbf{u}^T A \mathbf{u} = \sum_{i=1}^{N}\sum_{j=1}^{N} u_i u_j A_{ij} \geq 0,$$

for all $\mathbf{u} \in \mathbb{R}^N$. If $\mathbf{u}^T A \mathbf{u} > 0$ whenever the points x_i are distinct and $\mathbf{u} \neq 0$, then we say that $\phi(x)$ is strictly positive definite function ([7], [8]).

If $\phi(x)$ be strictly positive definite function on a linear space, then the eigenvalues of A are positive and its determinant is positive. Therefore we can use a linear combination translation of $\phi(x)$ for interpolation (see [8]).

Definition 2.2. A function $\phi(r)$ is said to be completely monotone on $[0, \infty)$ if for every $k \in \mathbb{N}$, we have

$$1. \quad \phi \in C^\infty[0, \infty), \qquad 2. \quad (-1)^k \phi^{(k)}(r) \geq 0.$$

A real-valued function F on an inner-product space is said [11], [19] to be radial if $F(x) = F(y)$ whenever $\|x\| = \|y\|$. Now we present a theorem that introduce a large number of strictly positive definite or radial basis functions.

Theorem 2.3. (Schoenberg's Theorem [8])
A function f is completely monotone but not constant on $[0, \infty)$, if and only if the function $x \to f(\|x\|^2)$ is strictly positive definite and radial basis function on any inner product space. Therefore the function $\phi(x) = f(\|x\|^2)$ is a strictly positive definite function.

We can find many strictly positive definite functions (RBFs) by using this theorem, i.e for any set of distinct points $x_1, x_2, \cdots, x_N$ on X, the matrix $A_{ij} = \phi(x_i - x_j)$ is strictly positive definite and nonsingular. In Table 1 we give some positive definite functions (RBFs).

Table 1. Some well-known strictly positive definite functions (RBFs)

Multiquadric (MQ)	$\phi(x) = \sqrt{c^2x^2 + 1}$
Generalized Multiquadric (GMQ)	$\phi(x) = (c^2x^2 + 1)^{\beta}$
Gaussian (GA)	$\phi(x) = e^{-x^2c^2}$
Hyperbolic secant (Sech)	$\phi(x) = \mathrm{sech}(cx)$

The standard radial basis functions are divided into two classes:
Class 1. Infinitely smooth RBFs, which are infinitely differentiable and depend on the shape parameter c, e.g. Hardy multiquadric (MQ), Gaussian (GA), inverse multiquadric (IQ) and inverse quadric (IQ) (see Table 1).
Class 2. Infinitely smooth (except in centers) RBFs, are not infinitely differentiable and are shape parameter free and have comparatively less accuracy than the basis functions (see [6], [7]).

For smooth functions, the MQ radial basis method is exponentially or spectrally accurate, i.e, the method has an error that decay at the rate $O(\eta^N)$ where $0 < \eta < 1$. The convergence of MQ radial basis method can be discussed in terms of two different types of approximation Stationary and non-stationary. In stationary approximation, the number of nodes N is fixed and the shape parameter c is refined toward zero and in nonstationary approximation fixes the value of shape papremeter (optimal or approximate optimal shape parameter) and N is increased. In this paper we use the nonstationary approximation method for the brachistochrone problem.

2.2. **Legendre-Gauss-Lobatto nodes and weights.** Let $L_N(x)$ be the shifted Legendre polynomial of order N on $[0, 1]$. Then the Legendre-Gauss-Lobatto nodes are

$$x_0 = 0 < x_1 < \cdots < x_{N-1} < x_N = 1, \tag{2.1}$$

and $x_m, \ \ 1 \le m \le N - 1$ are the zeros of $\dot{L}(x)$, where $\dot{L}(x)$ is the derivative of $L_N(x)$ with respect to $x \in [0, 1]$. No explicit formulas are known for the points x_m, and so they are computed numerically using subroutines [12].

Also the we approximate the integral of f on $[0, 1]$ as

$$\int_0^1 f(x)dx = \sum_{i=0}^{N} w_i f(x_i), \tag{2.2}$$

where x_i are Legendre-Guass-Lobatto nodes in (2.1) and the weights w_i given in [12]

$$w_i = \frac{2}{N(N+1)} \cdot \frac{1}{[L_N(x_i)]^2}, \quad i = 0, 1 \cdots, N. \tag{2.3}$$

It is well known [12] that the integration in (2.2) is exact whenever $f(x)$ is a polynomial of degree $\leq 2N+1$.

3. Solution of brachistochrone problem

Let $\phi(x) = \sqrt{1+c^2x^2}$ be a multiquadric radial basis function and we approximate $X(t)$ as follsows:

$$X(t) \simeq \sum_{j=0}^{N} a_j \phi(t - x_j) = A^T \Psi(t), \quad 0 \leq t \leq 1 \tag{3.1}$$

where $A^T = [a_0, a_1, \cdots, a_N]$, $\Psi^T(t) = [\phi(t-x_0), \phi(t-x_1), \cdots, \phi(t-x_N)]$ and $x_j, \quad j = 0, 1, \cdots, N$ are the Legendre Gauss-Lobatto nodes given in (2.1).

Also from (3.1) we have that

$$U(t) = \dot{X}(t) \simeq \sum_{j=0}^{N} a_j \phi'(t - x_j) = A^T \dot{\Psi}(t). \tag{3.2}$$

Substituting (3.1) and (3.2) in (1.1) we get

$$J = \int_0^1 \left(\frac{1 + \left(A^T \dot{\Psi}(t)\right)^2}{1 - A^T \Psi(t)} \right)^{\frac{1}{2}}, \tag{3.3}$$

with the boundary conditions:

$$A^T \Psi(0) = 0, \quad A^T \Psi(1) = 0.5. \tag{3.4}$$

Now we approximate the integral given in (3.3) by numerical integration method given in (2.2), we have that equation (3.3) can be restated as follows:

$$J = \sum_{k=1}^{N} w_k \left(\frac{1 + \left(A^T \dot{\Psi}(x_k)\right)^2}{1 - A^T \Psi(x_k)} \right)^{\frac{1}{2}}. \tag{3.5}$$

We now minimizing (3.5) subject to (3.4) using the Lagrange multiplier technique. So suppose that

$$\widehat{J} = J + \mu_1 A^T \Psi(0) + \mu_2 \left[A^T \Psi(1) - 0.5\right]. \tag{3.6}$$

The necessary conditions for minimum are

$$\frac{\partial \widehat{J}}{\partial a_k} = 0, \quad \frac{\partial \widehat{J}}{\partial \mu_1} = 0, \quad \frac{\partial \widehat{J}}{\partial \mu_2} = 0, \quad k = 0, 1, \cdots, N. \tag{3.7}$$

Equations (3.7) give $N+3$ nonlinear equations with $N+3$ unknowns, which can be solved for the unknowns $a_k, \quad k = 0, 1, \cdots, N$ and μ_1, μ_2 by using Newton's iterative method.

3.1. **Error Analysis.** We now discuss the convergence of MQ radial basis function method for the brachistochrone problem. As we know, the convergence of radial basis functions is exponential so, there are η_1, η_2 in $(0, 1)$ such that

$$E = O(\eta_1^N), \quad \hat{E} = O(\eta_2^N). \tag{3.8}$$

Lemma 3.1. *If* $F(X, \dot{X}) := \left[\frac{1+\dot{X}^2(t)}{1-X(t)}\right]^{\frac{1}{2}}$ *and* $X(0) = 0, \quad X(1) = -0.5$, *then*

$$\sqrt{\frac{2}{3}} \leq F(X, \dot{X}) \leq \sqrt{1 + \dot{X}^2(0) + g}. \tag{3.9}$$

where g is acceleration due to gravity.

Proof. As we know, the brachistochrone problem is determine a smooth path in which a mass moving along it from the point $(0,0)$ to $(1,-0.5)$ in shortest time. So it is clearly that $-0.5 \leq X(t) \leq 0$, for all $t \in [0,1]$. Therefore we have that

$$\frac{2}{3} \leq \frac{1}{1-X(t)} \leq 1. \tag{3.10}$$

Also we know that $\dot{X}(t)$ is the velocity of the mass in time t, and we have that

$$\dot{X}^2(t) \leq \dot{X}^2(0) + g, \tag{3.11}$$

So from (3.10) and (3.11) we conclude that

$$\sqrt{\frac{2}{3}} \leq F(X,\dot{X}) \leq \sqrt{1+\dot{X}^2(0)+g}.$$

□

Theorem 3.2. *If J is the functional given in (1.1), then*

$$\left|J[X]-J[A^T\Psi]\right| \leq \frac{\sqrt{3}}{2\sqrt{2}}\left(1+\dot{X}^2(0)+g\right)E + \frac{3\sqrt{3}}{2\sqrt{2}}\hat{E}\left[\sqrt{\dot{X}^2(0)+g}+\frac{\hat{E}}{2}\right]. \tag{3.12}$$

Proof. For simplicity, let that $Y(t) = A^T\Psi(t)$ and $\dot{Y}(t) = A^T\dot{\Psi}(t)$, so we can write

$$J[X]-J[Y] = \int_0^1 \left[\left(\frac{1+\dot{X}^2}{1-X}\right)^{\frac{1}{2}} - \left(\frac{1+\dot{Y}^2}{1-Y}\right)^{\frac{1}{2}}\right] dt$$

by simplify the above equation we have

$$\left|J[X]-J[Y]\right| = \int_0^1 \frac{1}{F(X,\dot{X})+F(Y,\dot{Y})}\left|\frac{(X-Y)+(\dot{X}-\dot{Y})(\dot{X}+\dot{Y})+\dot{Y}^2X-\dot{X}^2Y}{(1-X)(1-Y)}\right| dt \tag{3.13}$$

Now suppose that $X(t) = Y(t)+\varepsilon(t)$, where $\varepsilon(t)$ is the error of approximation by MQ radial basis function at t and it is clearly that

$$|\varepsilon(t)| \leq E, \quad |\dot{\varepsilon}(t)| \leq \hat{E}. \tag{3.14}$$

From (3.14) we have that

$$\dot{Y}^2(t)X(t) - \dot{X}^2(t)Y(t) = \varepsilon(t)\dot{X}^2(t) + X(t)\dot{\varepsilon}(t)\left(\dot{\varepsilon}(t)-2\dot{X}(t)\right). \tag{3.15}$$

Then from (3.9) and (3.10) and since $\varepsilon(t)$ is small, we have that

$$\frac{1}{\left(F(X,\dot{X})+F(Y,\dot{Y})\right)|(1-X)(1-Y)|} \leq \frac{\sqrt{3}}{2\sqrt{2}}. \tag{3.16}$$

So from (3.13)-(3.16) we have

$$\left|J[X]-J[Y]\right| \leq \frac{\sqrt{3}}{2\sqrt{2}}\int_0^1 \Big(|\varepsilon(t)|\big(1+\dot{X}^2(t)\big) + |\dot{\varepsilon}(t)|\big|(1-X)(2\dot{X}-\dot{\varepsilon}(t))\big|\Big)dt. \tag{3.17}$$

Then by (3.11) and (3.14) and as we know $1 \leq 1-X \leq \frac{3}{2}$, so we may write (3.17) in the form

$$\left|J[X]-J[Y]\right| \leq \frac{\sqrt{3}}{2\sqrt{2}}\left(1+\dot{X}^2(0)+g\right)E + \frac{3\sqrt{3}}{2\sqrt{2}}\hat{E}\left[\sqrt{\dot{X}^2(0)+g}+\frac{\hat{E}}{2}\right]. \tag{3.18}$$

□

4. Numerical results

To validate the application of the MQ radial basis function method to equations (1.1) and (1.3) we consider numerical results. We applied the method presented in this paper to the brachistochrone problem and evaluated J and $U(0)$. Also the numerical results obtained via other methods given in literatures are given in Table 2. All computational efforts in this work have been done by Maple software in 40 decimal digits.

Table 2. Numerical results of MQ radial basis method brachistochrone problem.

Methods	$U(0)$	J
Gradient Method [23]	-0.7832273	0.9984988
Successive sweep method [24]	-0.7834292	0.9984989
Chebyshev solution [24]	-0.7864406	0.9984981
Haar basis [21]	-0.7864408	0.99849814829
Nonclassical pseudospectral [3]	-0.78644079	0.998498148293
Present method		
$N = 6$	-0.78579955	0.998498199
$N = 8$	-0.78641095	0.99849814840
$N = 10$	-0.7864393819854	0.998498148293958
$N = 12$	-0.7864407310640	0.998498148293709
$N = 14$	-0.78644079588475961054	0.99849814829370853200
$N = 16$	-0.78644079845666653366	0.99849814829370853068
$N = 18$	-0.78644079917744183616	0.9984981482937085306776
$N = 20$	-0.78644079918492433886	0.9984981482937085306776

5. Conclusion.

In the present work, a new approach based on MQ radial basis function, has been introduced for the numerical solution of brachistochrone problem. Error analysis for this method show that our method is exponentially convergence and numerical results show this facts. From Table 2, it is obvious that numerical solution of the brachistochrone problem with MQ radial basis function are very accurate and also need less computational efforts with respect to methods given in [3], [15], [21]-[24].

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