ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2015, VOLUME 5, ISSUE 2, p.130-137

SOME RESULTS ON A GENERALIZED SASAKIAN-SPACE-FORM ADMITTING TRANS-SASAKIAN STRUCTURE WITH RESPECT TO A GENERALIZED TANAKA WEBSTER OKUMURA CONNECTION

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Abstract. The object of the present paper is to study generalized Sasakian-space-forms admitting trans-Sasakian structure with respect to a generalized Tanaka Webster Okumura connection. Locally ϕ-symmetric as well as η- recurrent generalized Sasakian-space-forms admitting trans-Sasakian structure with respect to a generalized Tanaka Webster Okumura connection have also been studied in the present paper.

Mathematics Subject Classification (2010): 53C25, 53D15.
Keywords: Locally ϕ-symmetric, generalized Tanaka Webster Okumura connection, generalized Sasakian-space-forms.

Article history:
Received 5 April 2014
Received in revised form 28 March 2015
Accepted 31 August 2015

1. Introduction

The notion of generalized Sasakian-space-forms was introduced by P. Alerge, D. Blair and A. Carriazo (see [1]). These space-forms are defined as follows:

Given an almost contact metric manifold $M(\phi, \xi, \eta, g)$, we say that M is generalized Sasakian-space-forms if there exist three functions f_1, f_2, f_3 on M such that the curvature tensor R of M is given by

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &+ f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \end{aligned} \tag{1.1}$$

for any vector fields X, Y, Z on M. In such a case we denote the manifold as $M(f_1, f_2, f_3)$. Generalized sasakian-space-forms have been studied in the papers [8], [9].

In 1985 J. A. Oubina (see [17]) introduced a new class of almost contact metric manifolds, called trans-Sasakian manifolds, which includes Sasakian, Kenmotso and Cosymplectic structures. The authors in the papers [3], [5] and [6] studied such manifolds and obtained some interesting results. In the paper [16] the author studied conformally flat ϕ-recurrent trans-Sasakian manifolds. It is known that (see [12]) trans-Sasakian structure of type (0, 0), (0, β) and (α, 0) are Cosymplectic, β-Kenmotsu and α-Sasakian respectively, where $\alpha, \beta \in R$. In [15] J. C. Marrero has shown that a trans-Sasakian manifold of dimension n ≥ 5 is either Cosymplectic or α-Sasakian or β-Kenmotsu manifold. In the paper [2], contact metric and trans-Sasakian generalized Sasakian-space-forms have been studied. The notion of generalized Tanaka Webster Okumura connection was introduced and studied by the authors in the paper [11]. In the present paper we have studied generalized Sasakian-space-forms admitting trans-Sasakian structure with respect to a generalized Tanaka Webster Okumura connection. The present paper is organized as follows.

After introduction in Section 1 we give some preliminaries in Section 2. Section 3 is concern with the study of locally ϕ-symmetric generalized Sasakian-space-forms admitting trans-Sasakian structure with respect to a generalized Tanaka Webster Okumura connection. Section 4 is devoted to the study of η- recurrent generalized Sasakian-space-forms admitting trans-Sasakian structure with respect to a generalized Tanaka Webster Okumura connection.

2. Preliminaries

Let M be a $(2n+1)$-dimensional connected differentiable manifold endowed with an almost contact metric structure (ϕ, ξ, η, g), where ϕ is a tensor field of type $(1,1)$, ξ is a vector field, η is an 1-form and g is a Riemannian metric on M such that (see [4])

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1. \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad X, Y \in T(M). \tag{2.2}$$

Then also

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(X) = g(X, \xi). \tag{2.3}$$

$$g(\phi X, X) = 0. \tag{2.4}$$

An almost contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is said to be a trans-Sasakian manifold (see [17]) if $(M^{2n+1} \times R, J, G)$ belongs to the class W_4 (see [10]) of the Hermitian manifolds, where J is the almost complex structure on $M^{2n+1} \times R$ defined by (see [7])

$$J(Z, f\frac{d}{dt}) = (\phi Z - f\xi, \eta(Z)\frac{d}{dt}), \tag{2.5}$$

for any vector field Z on M^{2n+1} and smooth function f on $M^{2n+1} \times R$ and G is the Hermitian metric on the product $M^{2n+1} \times R$. This may be expressed by the condition (see [17])

$$(\nabla_X \phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.6}$$

for some smooth functions α and β on M^{2n+1}, and we say that the trans-Sasakian structure is of type (α, β).

From equation (2.6) it follows that (see [6])

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \tag{2.7}$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{2.8}$$

In a $(2n+1)$-dimensional trans-Sasakian manifold from (2.6), (2.7) and (2.8) we can write (see [6])

$$\begin{aligned} R(X,Y)\xi &= (\alpha^2 - \beta^2)\{\eta(Y)X - \eta(X)Y\} + 2\alpha\beta\{\eta(Y)\phi X - \eta(X)\phi Y\} \\ &- (X\alpha)\phi Y + (Y\alpha)\phi X - (X\beta)\phi^2 Y + (Y\beta)\phi^2 X. \end{aligned} \tag{2.9}$$

$$S(X,\xi) = \{2n(\alpha^2 - \beta^2) - \xi\beta\}\eta(X) - (2n-1)X\beta - (\phi X)\alpha, \tag{2.10}$$

where S is the Ricci tensor.

Further we have

$$2\alpha\beta + \xi\alpha = 0. \tag{2.11}$$

Again we know that (see [1]) in a generalized Sasakian-space-form

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &+ f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \end{aligned} \tag{2.12}$$

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for any vector fields X, Y, Z on M, where R denotes the curvature tensor of M and f_1, f_2, f_3 are smooth functions on the manifold. The Ricci operator Q, Ricci tensor S and the scalar curvature r of the manifold of dimension $(2n+1)$ are respectively given by (see [13])

$$QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n-1)f_3)\eta(X)\xi, \tag{2.13}$$

$$S(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y), \tag{2.14}$$

$$r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3. \tag{2.15}$$

The generalized Tanaka Webster Okumura connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ are related by (see [11])

$$\tilde{\nabla}_X Y = \nabla_X Y + A(X,Y) \tag{2.16}$$

for all vectors fields X, Y on M and

$$A(X,Y) = \alpha\{g(X,\phi Y)\xi + \eta(Y)\phi X\} + \beta\{g(X,Y)\xi - \eta(Y)X\} - l\eta(X)\phi Y, \tag{2.17}$$

where l is a real constant.

We suppose that the vector fields X, Y, Z and W are orthogonal to ξ. Then from relations (2.16) and (2.17) we get

$$\tilde{\nabla}_X Y = \nabla_X Y + \{\alpha g(X,\phi Y) + \beta g(X,Y)\}\xi. \tag{2.18}$$

We can write from (2.18)

$$\tilde{\nabla}_Y Z = \nabla_Y Z + \{\alpha g(Y,\phi Z) + \beta g(Y,Z)\}\xi. \tag{2.19}$$

Applying $\tilde{\nabla}_X$ on both side of (2.19) we get

$$\tilde{\nabla}_X\tilde{\nabla}_Y Z = \tilde{\nabla}_X(\nabla_Y Z) + \{\alpha g(Y,\phi Z) + \beta g(Y,Z)\}\tilde{\nabla}_X\xi + \alpha g(Y,\tilde{\nabla}_X\phi Z)\xi. \tag{2.20}$$

Using (2.19) in (2.20) we obtain

$$\begin{aligned} \tilde{\nabla}_X\tilde{\nabla}_Y Z &= \nabla_X\nabla_Y Z + \{\alpha g(X,\phi\nabla_Y Z) + \beta g(X,\nabla_Y Z)\}\xi \\ &+ \{\alpha g(Y,\phi Z) + \beta g(Y,Z)\}\nabla_X\xi + \alpha g(Y,\nabla_X\phi Z)\xi \\ &= \nabla_X\nabla_Y Z + \{\alpha g(X,\phi\nabla_Y Z) + \beta g(X,\nabla_Y Z) + \alpha g(Y,\nabla_X\phi Z)\}\xi \\ &+ \{\alpha g(Y,\phi Z) + \beta g(Y,Z)\}\nabla_X\xi. \end{aligned} \tag{2.21}$$

Interchanging X and Y in (2.21) we get

$$\begin{aligned} \tilde{\nabla}_Y\tilde{\nabla}_X Z &= \nabla_Y\nabla_X Z + \{\alpha g(X,\phi\nabla_Y Z) + \beta g(X,\nabla_Y Z) + \alpha g(Y,\nabla_X\phi Z)\}\xi \\ &+ \{\alpha g(X,\phi Z) + \beta g(X,Z)\}\nabla_Y\xi. \end{aligned} \tag{2.22}$$

Also by using (2.18) we get

$$\tilde{\nabla}_{[X,Y]}Z = \nabla_{[X,Y]}Z + \alpha\{g([X,Y],\phi Z) + \beta g([X,Y],Z)\}\xi. \tag{2.23}$$

We know that

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{2.24}$$

and

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X\tilde{\nabla}_Y Z - \tilde{\nabla}_Y\tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]}Z. \tag{2.25}$$

In view of (2.21), (2.22), (2.23) and (2.24) in (2.25) we get

$$\begin{aligned} &\tilde{R}(X,Y)Z = R(X,Y)Z + \{\alpha g(X,\phi\nabla_Y Z) + \beta g(X,\nabla_Y Z) + \alpha g(Y,\nabla_X\phi Z)\}\xi \\ -\ &\{\alpha g(X,\phi\nabla_Y Z) + \beta g(X,\nabla_Y Z) + \alpha g(Y,\nabla_X\phi Z)\}\xi \\ -\ &\alpha\{g([X,Y],\phi Z) + \beta g([X,Y],Z)\}\xi \\ +\ &\{\alpha g(Y,\phi Z) + \beta g(Y,Z)\}\nabla_X\xi - \{\alpha g(X,\phi Z) + \beta g(X,Z)\}\nabla_Y\xi. \end{aligned} \tag{2.26}$$

This is the relation between the curvature tensors $\tilde{R}$ and R with respect to a generalized Tanaka Webster Okumura connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ respectively when the vector fields X, Y and Z are orthogonal to ξ.

3. Locally ϕ-symmetric generalized Sasakian-space-forms admitting trans-Sasakian structure with respect to a generalized Tanaka Webster Okumura connection

Definition 3.1. A Sasakian manifold M^n is said to be locally ϕ-symmetric if

$$\phi^2((\nabla_W R)(X,Y)Z) = 0, \tag{3.1}$$

for all vector fields X, Y, Z, W orthogonal to ξ. This notion was introduced by T. Takahashi for Sasakian manifolds (see [18]).

Analogous to the definition of locally ϕ-symmetric Sasakian manifolds with respect to Levi-Civita connection we define locally ϕ- symmetric generalized Sasakian-space-forms admitting trans-sasakian structure with respect to a generalized Tanaka Webster Okumura connection by

$$\phi^2((\tilde{\nabla}_W R)(X,Y)Z) = 0, \tag{3.2}$$

for all vector fields X, Y, Z and W orthogonal to ξ.

In view of (1.1) and (2.7) we obtain from (2.26)

$$\begin{aligned} \tilde{R}(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &+ f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} \\ &+ \{\alpha g(X,\phi\nabla_Y Z) + \beta g(X,\nabla_Y Z) + \alpha g(Y,\nabla_X \phi Z)\}\xi \\ &- \{\alpha g(X,\phi\nabla_Y Z) + \beta g(X,\nabla_Y Z) + \alpha g(Y,\nabla_X \phi Z)\}\xi \\ &- \alpha\{g([X,Y],\phi Z) + \beta g([X,Y],Z)\}\xi \\ &+ \{\alpha g(Y,\phi Z) + \beta g(Y,Z)\}(-\alpha\phi X + \beta X) \\ &- \{\alpha g(X,\phi Z) + \beta g(X,Z)\}(-\alpha\phi Y + \beta Y). \end{aligned} \tag{3.3}$$

Differentiating both side of (3.3) covariantly by W with respect to the Levi-Civita connection ∇ we get

$$\begin{aligned} (\nabla_W \tilde{R})(X,Y)Z &= df_1(W)\{g(Y,Z)X - g(X,Z)Y\} \\ &+ df_2(W)\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &+ f_2\{g(X,\phi Z)(\nabla_W\phi)Y + g(X,(\nabla_W\phi)Z)\phi Y \\ &- g(Y,\phi Z)(\nabla_W\phi)X - g(Y,(\nabla_W\phi)Z)\phi X \\ &+ 2g(X,\phi Y)(\nabla_W\phi)Z + 2g(X,(\nabla_W\phi)Y)\phi Z\} \\ &+ df_3(W)\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} \\ &+ f_3\{(\nabla_W\eta)(X)\eta(Z)Y + \eta(X)(\nabla_W\eta)(Z)Y \\ &- (\nabla_W\eta)(Y)\eta(Z)X - \eta(Y)(\nabla_W\eta)(Z)X \\ &+ g(X,Z)(\nabla_W\eta)(Y)\xi + g(X,Z)\eta(Y)(\nabla_W\xi) \\ &- g(Y,Z)(\nabla_W\eta)(X)\xi - g(Y,Z)\eta(X)(\nabla_W\xi)\} \\ &+ \{\alpha g(X,\phi\nabla_Y Z) + \beta g(X,\nabla_Y Z) + \alpha g(Y,\nabla_X\phi Z)\}\nabla_W\xi \\ &- \{\alpha g(X,\phi\nabla_Y Z) + \beta g(X,\nabla_Y Z) + \alpha g(Y,\nabla_X\phi Z)\}\nabla_W\xi \\ &- \alpha\{g([X,Y],\phi Z) + \beta g([X,Y],Z)\}\nabla_W\xi \\ &+ \{\alpha g(Y,\phi Z) + \beta g(Y,Z)\}(-\alpha(\nabla_W\phi)X + \beta\nabla_W X) \\ &- \{\alpha g(X,\phi Z) + \beta g(X,Z)\}(-\alpha(\nabla_W\phi)Y + \beta\nabla_W Y) \\ &+ \{\alpha g(X,(\nabla_W\phi)\nabla_Y Z) + \beta g(X,\nabla_W\nabla_Y Z) + \alpha g(Y,\nabla_W\nabla_X\phi Z)\}\xi \\ &- \{\alpha g(X,(\nabla_W\phi)\nabla_Y Z) + \beta g(X,\nabla_W\nabla_Y Z) + \alpha g(Y,\nabla_W\nabla_X\phi Z)\}\xi \\ &- \alpha\{g([X,Y],(\nabla_W\phi)Z) + \beta g(\nabla_W[X,Y],Z)\}\xi \\ &+ \alpha g(Y,(\nabla_W\phi)Z)(-\alpha\phi X + \beta X) \\ &- \alpha g(X,(\nabla_W\phi)Z)(-\alpha\phi Y + \beta Y). \end{aligned} \tag{3.4}$$

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Using (2.19) we can write

$$(\tilde{\nabla}_W \tilde{R})(X,Y)Z = (\nabla_W \tilde{R})(X,Y)Z + \{\alpha g(W, \phi\tilde{R}(X,Y)Z) + \beta g(W, \tilde{R}(X,Y)Z)\}\xi. \tag{3.5}$$

Using relation (3.5) in equation (3.4) we obtain

$$\begin{aligned}
(\tilde{\nabla}_W \tilde{R})(X,Y)Z &= df_1(W)\{g(Y,Z)X - g(X,Z)Y\} \\
&+ df_2(W)\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\
&+ f_2\{g(X,\phi Z)(\nabla_W\phi)Y + g(X,(\nabla_W\phi)Z)\phi Y \\
&- g(Y,\phi Z)(\nabla_W\phi)X - g(Y,(\nabla_W\phi)Z)\phi X \\
&+ 2g(X,\phi Y)(\nabla_W\phi)Z + 2g(X,(\nabla_W\phi)Y)\phi Z\} \\
&+ df_3(W)\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\
&+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} \\
&+ f_3\{(\nabla_W\eta)(X)\eta(Z)Y + \eta(X)(\nabla_W\eta)(Z)Y \\
&- (\nabla_W\eta)(Y)\eta(Z)X - \eta(Y)(\nabla_W\eta)(Z)X \\
&+ g(X,Z)(\nabla_W\eta)(Y)\xi + g(X,Z)\eta(Y)(\nabla_W\xi) \\
&- g(Y,Z)(\nabla_W\eta)(X)\xi - g(Y,Z)\eta(X)(\nabla_W\xi)\} \\
&+ \{\alpha g(X,\phi\nabla_Y Z) + \beta g(X,\nabla_Y Z) + \alpha g(Y,\nabla_X\phi Z)\}\nabla_W\xi \\
&- \{\alpha g(X,\phi\nabla_Y Z) + \beta g(X,\nabla_Y Z) + \alpha g(Y,\nabla_X\phi Z)\}\nabla_W\xi \\
&- \alpha\{g([X,Y],\phi Z) + \beta g([X,Y],Z)\}\nabla_W\xi \\
&+ \{\alpha g(Y,\phi Z) + \beta g(Y,Z)\}(-\alpha(\nabla_W\phi)X + \beta\nabla_W X) \\
&- \{\alpha g(X,\phi Z) + \beta g(X,Z)\}(-\alpha(\nabla_W\phi)Y + \beta\nabla_W Y) \\
&+ \{\alpha g(X,(\nabla_W\phi)\nabla_Y Z) + \beta g(X,\nabla_W\nabla_Y Z) + \alpha g(Y,\nabla_W\nabla_X\phi Z)\}\xi \\
&- \{\alpha g(X,(\nabla_W\phi)\nabla_Y Z) + \beta g(X,\nabla_W\nabla_Y Z) + \alpha g(Y,\nabla_W\nabla_X\phi Z)\}\xi \\
&- \alpha\{g([X,Y],(\nabla_W\phi)Z) + \beta g(\nabla_W[X,Y],Z)\}\xi \\
&+ \alpha g(Y,(\nabla_W\phi)Z)(-\alpha\phi X + \beta X) \\
&- \alpha g(X,(\nabla_W\phi)Z)(-\alpha\phi Y + \beta Y) \\
&+ \{\alpha g(W,\phi\tilde{R}(X,Y)Z) + \beta g(W,\tilde{R}(X,Y)Z)\}\xi.
\end{aligned} \tag{3.6}$$

We consider X, Y, Z and W orthogonal to ξ and using relation (2.6), (2.7), (2.8) in equation (3.6) we obtain

$$\begin{aligned}
(\tilde{\nabla}_W \tilde{R})(X,Y)Z &= df_1(W)\{g(Y,Z)X - g(X,Z)Y\} \\
&+ df_2(W)\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\
&+ f_2\{g(X,\phi Z)(\nabla_W\phi)Y + g(X,(\nabla_W\phi)Z)\phi Y \\
&- g(Y,\phi Z)(\nabla_W\phi)X - g(Y,(\nabla_W\phi)Z)\phi X \\
&+ 2g(X,\phi Y)(\nabla_W\phi)Z + 2g(X,(\nabla_W\phi)Y)\phi Z\} \\
&+ \{\alpha g(X,\phi\nabla_Y Z) + \beta g(X,\nabla_Y Z) + \alpha g(Y,\nabla_X\phi Z)\}(-\alpha\phi W + \beta W) \\
&- \{\alpha g(X,\phi\nabla_Y Z) + \beta g(X,\nabla_Y Z) + \alpha g(Y,\nabla_X\phi Z)\}(-\alpha\phi W + \beta W) \\
&- \alpha\{g([X,Y],\phi Z) + \beta g([X,Y],Z)\}(-\alpha\phi W + \beta W) \\
&+ \{\alpha g(Y,\phi Z) + \beta g(Y,Z)\}(-\alpha(\nabla_W\phi)X + \beta\nabla_W X) \\
&- \{\alpha g(X,\phi Z) + \beta g(X,Z)\}(-\alpha(\nabla_W\phi)Y + \beta\nabla_W Y) \\
&+ \{\alpha g(X,(\nabla_W\phi)\nabla_Y Z) + \beta g(X,\nabla_W\nabla_Y Z) + \alpha g(Y,\nabla_W\nabla_X\phi Z)\}\xi \\
&- \{\alpha g(X,(\nabla_W\phi)\nabla_Y Z) + \beta g(X,\nabla_W\nabla_Y Z) + \alpha g(Y,\nabla_W\nabla_X\phi Z)\}\xi \\
&- \alpha\{g([X,Y],\nabla_W\phi Z) + \beta g(\nabla_W[X,Y],Z)\}\xi \\
&+ \alpha g(Y,(\nabla_W\phi)Z)(-\alpha\phi X + \beta X) \\
&- \alpha g(X,(\nabla_W\phi)Z)(-\alpha\phi Y + \beta Y) \\
&+ \{\alpha g(W,\phi\tilde{R}(X,Y)Z) + \beta g(W,\tilde{R}(X,Y)Z)\}\xi.
\end{aligned} \tag{3.7}$$

After using relation (2.6) in (3.7) we apply ϕ^2 on both side and then we get

$$\begin{aligned}\phi^2((\tilde{\nabla}_W\tilde{R})(X,Y)Z) &= -df_1(W)\{g(Y,Z)X-g(X,Z)Y\}\\ &- df_2(W)\{g(X,\phi Z)\phi Y-g(Y,\phi Z)\phi X+2g(X,\phi Y)\phi Z\}\\ &- \{\alpha g(X,\phi\nabla_Y Z)+\beta g(X,\nabla_Y Z)+\alpha g(Y,\nabla_X\phi Z)\}(-\alpha\phi W+\beta W)\\ &+ \{\alpha g(X,\phi\nabla_Y Z)+\beta g(X,\nabla_Y Z)+\alpha g(Y,\nabla_X\phi Z)\}(-\alpha\phi W+\beta W)\\ &+ \alpha\{g([X,Y],\phi Z)+\beta g([X,Y],Z)\}(-\alpha\phi W+\beta W)\\ &+ \{\alpha g(Y,\phi Z)+\beta g(Y,Z)\}\{\beta\phi^2(\nabla_W X)\}\\ &- \{\alpha g(X,\phi Z)+\beta g(X,Z)\}\{\beta\phi^2(\nabla_W Y)\}.\end{aligned} \tag{3.8}$$

Suppose α is a constant, then by (2.11) we get $\beta=0$. In such a case from equation (3.8) we get

$$\begin{aligned}\phi^2((\tilde{\nabla}_W\tilde{R})(X,Y)Z) &= -df_1(W)\{g(Y,Z)X-g(X,Z)Y\}\\ &- df_2(W)\{g(X,\phi Z)\phi Y-g(Y,\phi Z)\phi X+2g(X,\phi Y)\phi Z\}\\ &- \{\alpha g(X,\phi\nabla_Y Z)+\beta g(X,\nabla_Y Z)+\alpha g(Y,\nabla_X\phi Z)\}(-\alpha\phi W)\\ &+ \{\alpha g(X,\phi\nabla_Y Z)+\beta g(X,\nabla_Y Z)+\alpha g(Y,\nabla_X\phi Z)\}(-\alpha\phi W)\\ &+ \alpha\{g([X,Y],\phi Z)+\beta g([X,Y],Z)\}(-\alpha\phi W).\end{aligned} \tag{3.9}$$

Taking inner product on both side of (3.9) with respect to W we get

$$\begin{aligned}g(\phi^2((\tilde{\nabla}_W\tilde{R})(X,Y)Z),W) &= -df_1(W)\{g(Y,Z)g(X,W)-g(X,Z)g(Y,W)\}\\ &- df_2(W)\{g(X,\phi Z)g(\phi Y,W)-g(Y,\phi Z)g(\phi X,W)\\ &+ 2g(X,\phi Y)g(\phi Z,W)\}.\end{aligned} \tag{3.10}$$

Since the above relation is true for any vector field W, so we can write from (3.10)

$$\begin{aligned}\phi^2((\tilde{\nabla}_W\tilde{R})(X,Y)Z) &= -df_1(W)\{g(Y,Z)X-g(X,Z)Y\}\\ &- df_2(W)\{g(X,\phi Z)\phi Y-g(Y,\phi Z)\phi X+2g(X,\phi Y)\phi Z\}.\end{aligned} \tag{3.11}$$

Thus we are in a position to state the following result:

Theorem 3.2. *A Generalized Sasakian-space-form admitting trans-Sasakian structure with respect to a generalized Tanaka Webster Okumura connection is locally ϕ-symmetric if and only if f_1 and f_2 are constants provided α is constant.*

4. η- recurrent generalized Sasakian-space-forms admitting trans-Sasakian structure with respect to a generalized Tanaka Webster Okumura connection

Definition 4.1. A (2n+1)-dimensional generalized Sasakian-space-forms admitting trans-Sasakian structure with respect to a generalized Tanaka Webster Okumura connection $\tilde{\nabla}$ is said to be η- recurrent Ricci tensor if there exist a non-zero 1-form A such that

$$(\tilde{\nabla}_X\tilde{S})(\phi Y,\phi Z)=A(X)\tilde{S}(Y,Z). \tag{4.1}$$

If a 1-form A vanishes on M then the space-form is said to have η-parallel Ricci tensor. The notion of η-parallel Ricci tensor was introduced by Kon in the context of Sasakian geometry (see [14]).

Taking inner product on both side of (2.26) with respect to a horizontal vector field W and contracting between Y and Z we get

$$\tilde{S}(X,W)=(2nf_1+3f_2-f_3+\alpha^2)g(X,W). \tag{4.2}$$

$$\tilde{Q}X=(2nf_1+3f_2-f_3+\alpha^2)X \tag{4.3}$$

$$\tilde{r}=(2n+1)(2nf_1+3f_2-f_3+\alpha^2) \tag{4.4}$$

where $\tilde{S}$, $\tilde{Q}$, $\tilde{r}$ are respectively the Ricci tensor, the Ricci operator and the scalar curvature of generalized Sasakian-space-forms admitting trans-Sasakian structure with respect to a generalized Tanaka Webster

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Okumura connection and α is constant.

Differentiating both side of (4.2) covariantly by W with respect to the connection $\tilde{\nabla}$ we obtain

$$(\tilde{\nabla}_W \tilde{S})(\phi Y, \phi Z) = d(2nf_1 + 3f_2 - f_3 + \alpha^2)(W)g(Y, Z). \tag{4.5}$$

Suppose that the space forms M is η- recurrent. Then in view of (4.5) we obtain from (4.1)

$$d(2nf_1 + 3f_2 - f_3 + \alpha^2)(W)g(Y, Z) = A(X)S(\tilde{Y}, Z). \tag{4.6}$$

In view of (4.2) we get from above

$$d(2nf_1 + 3f_2 - f_3 + \alpha^2)(W) = A(X)(2nf_1 + 3f_2 - f_3 + \alpha^2). \tag{4.7}$$

Let $(2nf_1 + 3f_2 - f_3 + \alpha^2) = f$. Then (4.7) reduces to

$$fA(W) = df(W). \tag{4.8}$$

From (4.8) we get

$$df(Y)A(W) + (\tilde{\nabla}_Y A)(W)f = d^2 f(W, Y). \tag{4.9}$$

Interchanging Y and W in above we get

$$df(W)A(Y) + (\tilde{\nabla}_W A)(Y)f = d^2 f(Y, W). \tag{4.10}$$

Subtracting (4.10) from (4.9) we get

$$(\tilde{\nabla}_W A)(Y) - (\tilde{\nabla}_Y A)(W) = 0. \tag{4.11}$$

Hence the 1-form A is closed.

Thus we have the following result:

Theorem 4.2. *In an η- recurrent generalized Sasakian-space-form admitting trans-Sasakian structure with respect to a generalized Tanaka Webster Okumura connection the 1-form A is closed.*

Since $A(W)$ is non-zero, the equation (4.7) leads us to the following:

Theorem 4.3. *If a (2n+1)- dimensional generalized Sasakian-space-form admitting trans-Sasakian structure with respect to a generalized Tanaka Webster Okumura connection has η- recurrent Ricci tensor, then $(2nf_1 + 3f_2 - f_3 + \alpha^2)$ can never be a non-zero constant.*

In view of (4.5) we also have the following result:

Theorem 4.4. *A (2n+1)- dimensional generalized Sasakian-space-form admitting trans-Sasakian structure with respect to a generalized Tanaka Webster Okumura connection has η- parallel Ricci tensor if and only if $(2nf_1 + 3f_2 - f_3 + \alpha^2)$ is constant.*

Acknowledgement: The authors are thankful to the referee for his valuable suggestions for the improvement of the paper.

References

[1] P. Alerge, D. E. Blair and A. Carriazo, *Generalized Sasakian-space-forms*, Israel J. Math., 14(2004), 157-182).

[2] P. Alegre and A. Carriazo, *Structures on generalized Sasakian-space-forms*, Differential Geometry and its Applications, 26(6)(2008),656-666.

[3] C. S. Bagewadi and E. Girish Kumar, *Note on trans-Sasakian manifolds*, Tensor(N.S.), 65(2004),80-88.

[4] D. E. Blair, *Contact manifolds in Riemannian geometry*, Lecture Notes in Math. No. 509. Springer 1976.

[5] D. E. Blair, and J. A. Oubina, *Conformal and related changes of metric on the product of two almost contact metric manifolds*, Publ. Matematiques 34(1990), 199-207.

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[6] U. C. De and M. M. Tripathi, *Ricci tensor in three dimension trans-Sasakian manifolds*, Kyungpook Math. J. 43(2) (2003), 247-255.
[7] U. C. De and A. A. Shaikh, *Complex manifolds and contact manifolds*, Narosa Publication House Pvt. Ltd., 2009.
[8] U. C. De and A. Sarkar, *On the Projective Curvature Tensor of Generalized Sasakian-space-forms*, Quaestiones Mathematicae, 33(2010), 245-252.
[9] U. C. De and A. Sarkar, *Some results on Generalized Sasakian-space-forms*, Thai. J. Math. 8, 1(2010), 1-10.
[10] A. Gray and L. M. Hervella, *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. Mat. pura Appl. (4) 123 (1980), 35-58.
[11] J. Inoguchi and J. E. Lee, *Affine biharmonic curves in 3-dimensional homogeneous geometries*, Mediterr. J. Math. 10(2013), 571-592.
[12] D. Janssens and L. Vanhecke, *Almost contact structures and curvature tensors*, Kodai Math. J. 4(1981), 1-27.
[13] U. K. Kim, *Conformally flat generalized Sasakian-space-forms and locally symmetric generalized Sasakian-space-forms*, Note di matemetica, 26(2006), 55-67.
[14] M. Kon, *Invariant submanifolds in Sasakian manifolds*, Math. Ann., 219(1976), 277-290.
[15] J. C. Marrero, *The local structure of trans-Sasakian manifolds*, Ann. Mat. Pura Appl. 162(4)(1992), 77-86.
[16] H. G. Nagaraja, *ϕ recurrent trans-Sasakian manifolds*, Mate. Bechnk 63(2011), 79-86.
[17] J. A. Oubina, *New class of almost contact metric manifolds*, Publ. Math. Debrecen 32(1985), 187-193.
[18] T. Takahashi, *Sasakian ϕ-symmetric spaces*, Tohoku Math. J., 29(1977), 91-113.

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