ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2013, VOLUME 3, ISSUE 1, p.41-47

Fourier transform of Dini-Lipschitz functions in the space $L^2(\mathbb{R}^n)$

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Abstract: Using a spherical mean operator, we obtain an analog and a generalization of Younis's Theorem 5.2 in [5] for the Fourier transform in the space $L^2(\mathbb{R}^n)$.

Keywords: Fourier transform, Spherical mean operator.

Mathematics Subject Classification: 42A38, 42B10 .

1 Introduction and preliminaries

The Fourier transform, as well as Fourier series, is widely used in various fields of calculus, mathematical physics...

In [5], Younis proved an estimate for the Fourier transform in the space $L^2(\mathbb{R})$. In this paper, we prove an analog and a generalization of this estimate in the space $L^2(\mathbb{R}^n)$.

Assume that $L^2(\mathbb{R}^n)$ is the Hilbert space of 2-power integrable function $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ with the norm

$$\|f\|_2 = \left(\int_{\mathbb{R}^n} |f(x)|^2 dx\right)^{1/2}.$$

Let $f(x) \in L^2(\mathbb{R}^n)$. The Fourier transform $\widehat{f}$ of f is defined by

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-i\xi\cdot x} dx.$$

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The inverse formula of Fourier transform is defined by

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{i\xi \cdot x} d\xi.$$

We have from [3] the Parseval's equality

$$\|\widehat{f}\|_2 = \|f\|_2. \tag{1}$$

Consider in $\mathrm{L}^2(\mathbb{R}^n)$ the spherical mean operator (see [2])

$$\mathrm{M}_h f(x) = \frac{1}{w_{n-1}} \int_{\mathbb{S}^{n-1}} f(x + hw) dw,$$

where $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$, w_{n-1} its total surface measure with respect to the usual induced measure dw.

For $\alpha \geq -\frac{1}{2}$, we introduce the normalized spherical Bessel function j_α defined by

$$j_\alpha(z) = \Gamma(\alpha + 1) \sum_{j=0}^{\infty} \frac{(-1)^j (z/2)^{2j}}{j! \Gamma(j + \alpha + 1)}, \ z \in \mathbb{C}.$$

Lemma 1.1 *For $x \in \mathbb{R}$ the following inequalities are fulfilled*

1. $|j_\alpha(x)| \leq 1$,
2. $|1 - j_\alpha(x)| \leq |x|$,
3. $|1 - j_\alpha(x)| \geq c$ *with* $|x| \geq 1$, *where* $c > 0$ *is a certain constant which depends only on* α.

Proof. Similarly as the proof of Lemma 2.9 in [1] ■

Lemma 1.2 *Let $f \in \mathrm{L}^2(\mathbb{R}^n)$, then*

$$\widehat{(M_h f)}(\xi) = j_{\frac{n-2}{2}}(h|\xi|) \widehat{f}(\xi).$$

Proof. The statement follows easily from representation of Fourier transform of radial functions (see [4], Chapter IV). ■

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2 Main Result

In this section we give the main result of this paper.

Theorem 2.1 *Let $f(x)$ belong to $\mathrm{L}^2(\mathbb{R}^n)$, and let*

$$\|\mathrm{M}_h f(x) - f(x)\|_2 = O\left(\frac{h^\alpha}{(\log \frac{1}{h})^\gamma}\right), \ \alpha > 0, \gamma > 0$$

as $h \longrightarrow 0$. Then

$$\int_{|\xi|\geq r} |\widehat{f}(\xi)|^2 d\xi = O\left(r^{-2\alpha}(\log r)^{-2\gamma}\right) \ \text{as } r \longrightarrow +\infty.$$

Proof. We have

$$\|\mathrm{M}_h f(x) - f(x)\|_2 = O\left(\frac{h^\alpha}{(\log \frac{1}{h})^\gamma}\right), \ \text{as } h \longrightarrow 0.$$

If $|\xi| \in [\frac{1}{h}, \frac{2}{h}]$, then $h|\xi| \geq 1$, and (3) of Lemma 1.1 implies that

$$1 \leq \frac{1}{c^2}|1 - j_{\frac{n-2}{2}}(h|\xi|)|^2. \tag{2}$$

Lemma 1.2 and Parseval's equality (1) give

$$\|\mathrm{M}_h f(x) - f(x)\|_2^2 = \int_{\mathbb{R}^n} |1 - j_{\frac{n-2}{2}}(h|\xi|)|^2 |\widehat{f}(\xi)|^2 d\xi. \tag{3}$$

Hence, by (1) and Lemma 1.2, it follows that

$$\begin{aligned}
\int_{1/h\leq|\xi|\leq 2/h} |\widehat{f}(\xi)|^2 d\xi &\leq \frac{1}{c^2}\int_{1/h\leq|\xi|\leq 2/h} |1 - j_{\frac{n-2}{2}}(h|\xi|)|^2 |\widehat{f}(\xi)|^2 d\xi \\
&\leq \frac{1}{c^2}\int_{\mathbb{R}^n} |1 - j_{\frac{n-2}{2}}(h|\xi|)|^2 |\widehat{f}(\xi)|^2 d\xi \\
&= \frac{1}{c^2}\|\mathrm{M}_h f(x) - f(x)\|_2^2 \\
&= O\left(\frac{h^{2\alpha}}{(\log \frac{1}{h})^{2\gamma}}\right),
\end{aligned}$$

or, equivalently,

$$\int_{r\leq|\xi|\leq 2r} |\widehat{f}(\xi)|^2 d\xi = O\left(\frac{r^{-2\alpha}}{(\log r)^{2\gamma}}\right) \ \text{as } r \longrightarrow +\infty.$$

Thus there exists then a positive constant C such that

$$\int_{r\le|\xi|\le 2r} |\widehat{f}(\xi)|^2 d\xi \le C\frac{r^{-2\alpha}}{(\log r)^{2\gamma}}.$$

Hence

$$\begin{aligned}
\int_{|\xi|\ge r} |\widehat{f}(\xi)|^2 d\xi &= \left[\int_{r\le|\xi|\le 2r} + \int_{2r\le|\xi|\le 4r} + \int_{4r\le|\xi|\le 8r} +....\right] |\widehat{f}(\xi)|^2 d\xi \\
&\le C\frac{r^{-2\alpha}}{(\log r)^{2\gamma}} + C\frac{(2r)^{-2\alpha}}{(\log 2r)^{2\gamma}} + C\frac{(4r)^{-2\alpha}}{(\log 4r)^{2\gamma}} + ... \\
&\le C\frac{r^{-2\alpha}}{(\log r)^{2\gamma}} + C\frac{(2r)^{-2\alpha}}{(\log r)^{2\gamma}} + C\frac{(4r)^{-2\alpha}}{(\log r)^{2\gamma}} + ... \\
&\le C\frac{r^{-2\alpha}}{(\log r)^{2\gamma}}(1 + 2^{-2\alpha} + (2^{-2\alpha})^2 + (2^{-2\alpha})^3 + ... \\
&\le CK\frac{r^{-2\alpha}}{(\log r)^{2\gamma}},
\end{aligned}$$

where $K = (1-2^{-2\alpha})^{-1}$.

This prove that

$$\int_{|\xi|\ge r} |\widehat{f}(\xi)|^2 d\xi = O\left(r^{-2\alpha}(\log r)^{-2\gamma}\right) \ \ as \ r \longrightarrow +\infty$$

and this ends the proof. ■

Definition 2.2 *A function $f \in L^2(\mathbb{R}^n)$ is said to be in the ψ-Dini Lipschitz class, denote by $Lip(2,\psi)$, if*

$$\|M_h f(x) - f(x)\|_2 = O\left(\frac{\psi(h)}{(\log\frac{1}{h})^{\gamma}}\right) \ \gamma > 0, \ as \ h \longrightarrow 0,$$

where

1. *$\psi(t)$ is a continuous increasing function on $[0,\infty)$,*
2. *$\psi(0) = 0$,*
3. *$\psi(ts) = \psi(t)\psi(s)$ for all $s, t \in [0,\infty)$,*
4. *$\int_0^{1/h} x\frac{\psi(x^{-2})}{(\log x)^{2\gamma}}dx = O\left(\frac{1}{h^2}\frac{\psi(h^2)}{(\log\frac{1}{h})^{2\gamma}}\right)$.*

Theorem 2.3 *Let $f \in L^2(\mathbb{R}^n)$ and let ψ be a fixed function satisfying the conditions of Definition 2.2. Then the following statements are equivalent*

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1. $f \in Lip(2,\psi)$,

2. $\int_{|\xi|\geq r} |\widehat{f}(\xi)|^2 d\xi = O(\psi(r^{-2})(\log r)^{-2\gamma})$ *as* $r \longrightarrow +\infty$.

Proof. 1) $\Longrightarrow$ 2) Assume that $f \in Lip(2,\psi)$. Then we have

$$\|\mathrm{M}_h f(x) - f(x)\|_2 = O\left(\frac{\psi(h)}{(\log\frac{1}{h})^{\gamma}}\right) \text{ as } h \longrightarrow 0.$$

If $|\xi| \in [\frac{1}{h}, \frac{2}{h}]$, then $h|\xi| \geq 1$, and similarly as in the proof of Theorem 2.1 we obtain

$$\begin{aligned}\int_{1/h\leq|\xi|\leq 2/h} |\widehat{f}(\xi)|^2 d\xi &\leq \frac{1}{c^2}\|\mathrm{M}_h f(x) - f(x)\|_2^2 \\ &= O\left(\frac{\psi(h^2)}{(\log\frac{1}{h})^{2\gamma}}\right).\end{aligned}$$

Thus there exists then a positive constant C_1 such that

$$\int_{r\leq|\xi|\leq 2r} |\widehat{f}(\xi)|^2 d\xi \leq C_1 \frac{\psi(r^{-2})}{(\log r)^{2\gamma}}.$$

Hence

$$\begin{aligned}\int_{|\xi|\geq r} |\widehat{f}(\xi)|^2 d\xi &= \left[\int_{r\leq|\xi|\leq 2r} + \int_{2r\leq|\xi|\leq 4r} + \int_{4r\leq|\xi|\leq 8r} + ...\right] |\widehat{f}(\xi)|^2 d\xi \\ &\leq C_1\frac{\psi(r^{-2})}{(\log r)^{2\gamma}} + C_1\frac{\psi((2r)^{-2})}{(\log 2r)^{2\gamma}} + C_1\frac{\psi((4r)^{-2})}{(\log 4r)^{2\gamma}} + ... \\ &\leq C_1\frac{\psi(r^{-2})}{(\log r)^{2\gamma}} + C_1\frac{\psi((2r)^{-2})}{(\log r)^{2\gamma}} + C\frac{\psi((4r)^{-2})}{(\log r)^{2\gamma}} + ... \\ &\leq C_1\frac{\psi(r^{-2})}{(\log r)^{2\gamma}}(1 + \psi(2^{-2}) + (\psi(2^{-2}))^2 + (\psi(2^{-2}))^3 + ... \\ &\leq C_1 K_1 \frac{\psi(r^{-2})}{(\log r)^{2\gamma}},\end{aligned}$$

where $K_1 = (1-\psi(2^{-2}))^{-1}$, since by (1) and (3) from Definition 2.2 it follows that $\psi(2^{-2}) < 1$.

This proves that

$$\int_{|\xi|\geq r} |\widehat{f}(\xi)|^2 d\xi = O\left(\psi(r^{-2})(\log r)^{-2\gamma}\right) \text{ as } r \longrightarrow +\infty.$$

2) $\Longrightarrow$ 1) Suppose now that

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$$\int_{|\xi|\geq r} |\widehat{f}(\xi)|^2 d\xi = O\left(\psi(r^{-2})(\log r)^{-2\gamma}\right) \;\; as \; r \longrightarrow +\infty.$$

By (3) it follows that we have to show that

$$\int_0^\infty x^{n-1}|1 - j_{\frac{n-2}{2}}(hx)|^2\varphi(x)dx = O\left(\frac{\psi(h^2)}{(\log\frac{1}{h})^{2\gamma}}\right),$$

where

$$\varphi(x) = \int_{\mathbb{S}^{n-1}} |\widehat{f}(xy)|^2 dy.$$

We write

$$\int_0^{+\infty} x^{n-1}|1 - j_{\frac{n-2}{2}}(hx)|^2\varphi(x)dx = \mathrm{I}_1 + \mathrm{I}_2,$$

where

$$\mathrm{I}_1 = \int_0^{1/h} x^{n-1}|1 - j_{\frac{n-2}{2}}(hx)|^2\varphi(x)dx.$$

and

$$\mathrm{I}_2 = \int_{1/h}^{+\infty} x^{n-1}|1 - j_{\frac{n-2}{2}}(hx)|^2\varphi(x)dx.$$

Firstly, from (1) of Lemma 1.1 we see that

$$\begin{aligned} \mathrm{I}_2 &= \int_{1/h}^{+\infty} x^{n-1}|1 - j_{\frac{n-2}{2}}(hx)|^2\varphi(x)dx \\ &\leq 4\int_{1/h}^{+\infty} x^{n-1}\varphi(x)dx \\ &= O\left(\frac{\psi(h^2)}{(\log\frac{1}{h})^{2\gamma}}\right) \end{aligned}$$

Set

$$g(x) = \int_x^\infty s^{n-1}\varphi(s)ds.$$

From (2) of Lemma 1.2, an integration by parts yields

$$
\begin{aligned}
\mathrm{I}_1 &= \int_0^{1/h} x^{n-1}|1-j_{\frac{n-2}{2}}(hx)|^2\varphi(x)dx \\
&\leq -h^2\int_0^{1/h} x^2 g'(x)dx \\
&\leq -g(\frac{1}{h}) + 2h^2\int_0^{1/h} xg(x)dx \\
&\leq C_2h^2\int_0^{\infty} x\psi(x^{-2})(\log x)^{-2\gamma}dx \\
&\leq C_2\frac{\psi(h^2)}{(\log\frac{1}{h})^{2\gamma}},
\end{aligned}
$$

where C_2 is a positive constant, and this ends the proof. ■

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