

# KOROVKIN THEORY FOR BANACH SPACE VALUED FUNCTIONS

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ABSTRACT. Here we study quantitatively the rate of convergence of sequences of linear operators acting on Banach space valued continuous functions to the unit operator. These operators are bounded by real positive companion linear operators. The Banach spaces considered here are general and no positivity assumption is made on the initial linear operators whose we study their approximation properties. We derive pointwise and uniform estimates which imply the approximation of these operators to the unit. Our inequalities are of Shisha-Mond type and they imply an elegant Korovkin type theorem.

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## 1. MOTIVATION

Let  $(X, \|\cdot\|)$  be a Banach space,  $n \in \mathbb{N}$ . Consider  $g \in C([0, 1])$  and the classic Bernstein polynomials

$$(1) \quad (\tilde{B}_n g)(t) = \sum_{k=0}^n g\left(\frac{k}{n}\right) \binom{n}{k} t^k (1-t)^{n-k}, \quad \forall t \in [0, 1].$$

Let also  $f \in C([0, 1], X)$  and define the vector valued in  $X$  Bernstein linear operators

$$(2) \quad (B_n f)(t) = \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} t^k (1-t)^{n-k}, \quad \forall t \in [0, 1].$$

That is  $(B_n f)(t) \in X$ .

Clearly here  $\|f\| \in C([0, 1])$ .

We notice that

$$(3) \quad \|(B_n f)(t)\| \leq \sum_{k=0}^n \left\| f\left(\frac{k}{n}\right) \right\| \binom{n}{k} t^k (1-t)^{n-k} = (\tilde{B}_n(\|f\|))(t),$$

$\forall t \in [0, 1]$ .

The property

$$(4) \quad \|(B_n f)(t)\| \leq (\tilde{B}_n(\|f\|))(t), \quad \forall t \in [0, 1],$$

is shared by almost all summation/integration similar operators and motivates our work here.

If  $f(x) = c \in X$  the constant function, then

$$(5) \quad (B_n c) = c.$$

If  $g \in C([0, 1])$  and  $c \in X$ , then  $cg \in C([0, 1], X)$  and

$$(6) \quad (B_n(cg)) = c\tilde{B}_n(g).$$

Again (5), (6) are fulfilled by many summation/integration operators.

In fact here (6) implies (5), when  $g \equiv 1$ .

The above can be generalized from  $[0, 1]$  to any interval  $[a, b] \subset \mathbb{R}$ . All this discussion motivates us to consider the following situation.

Let  $L_n : C([a, b], X) \hookrightarrow C([a, b], X)$ ,  $(X, \|\cdot\|)$  a Banach space,  $L_n$  is a linear operator,  $\forall n \in \mathbb{N}$ ,  $x_0 \in [a, b]$ . Let also  $\tilde{L}_n : C([a, b]) \hookrightarrow C([a, b])$ , a sequence of positive linear operators,  $\forall n \in \mathbb{N}$ .

We assume that

$$(7) \quad \|(L_n(f))(x_0)\| \leq (\tilde{L}_n(\|f\|))(x_0),$$

$\forall n \in \mathbb{N}$ ,  $x_0 \in X$ ,  $f \in C([a, b], X)$ .

When  $g \in C([a, b])$ ,  $c \in X$ , we assume that

$$(8) \quad (L_n(cg)) = c\tilde{L}_n(g).$$

The special case of

$$(9) \quad \tilde{L}_n(1) = 1,$$

implies

$$(10) \quad L_n(c) = c, \quad \forall c \in X.$$

We call  $\tilde{L}_n$  the companion operator of  $L_n$ .

Based on the above fundamental properties we study the approximation properties of the sequence of linear operators  $\{L_n\}_{n \in \mathbb{N}}$ , i.e. their convergence to the unit operator. No kind of positivity property of  $\{L_n\}_{n \in \mathbb{N}}$  is assumed. See also [1], [2].

## 2. MAIN RESULTS

We present the following pointwise convergence

**Theorem 1.** *Let  $L_n : C([a, b], X) \rightarrow C([a, b], X)$ , where  $(X, \|\cdot\|)$  is a Banach space and  $L_n$  is a linear operator,  $\forall n \in \mathbb{N}$ ,  $x_0 \in [a, b]$ . Let the positive linear operators  $\tilde{L}_n : C([a, b]) \hookrightarrow C([a, b])$ , such that*

$$(11) \quad \|(L_n(f))(x_0)\| \leq (\tilde{L}_n(\|f\|))(x_0), \quad \forall n \in \mathbb{N},$$

where  $f \in C([a, b], X)$ .

Furthermore assume that

$$(12) \quad \tilde{L}_n(1) = 1, \quad L_n(c) = c, \quad \forall c \in X.$$

Then

$$(13) \quad \|(L_n(f))(x_0) - f(x_0)\| \leq 2\omega_1\left(f, \left(\tilde{L}_n(|\cdot - x_0|)\right)(x_0)\right),$$

where

$$(14) \quad \omega_1(f, \delta) := \sup_{\substack{x, y \in [a, b]: \\ |x - y| \leq \delta}} \|f(x) - f(y)\|, \quad 0 < \delta \leq b - a,$$

is the first modulus of continuity.

*Proof.* We notice that

$$\begin{aligned} & \|(L_n(f))(x_0) - f(x_0)\| = \|(L_n(f))(x_0) - (L_n(f(x_0)))(x_0)\| = \\ (15) \quad & \|(L_n(f - f(x_0)))(x_0)\| \leq \left(\tilde{L}_n(\|f - f(x_0)\|)\right)(x_0) \end{aligned}$$

(let  $h > 0$ , and by Lemma 7.1.1, p. 208 of [1])

$$\begin{aligned} & \leq \left(\tilde{L}_n\left(\omega_1(f, h)\left(1 + \frac{|\cdot - x_0|}{h}\right)\right)\right)(x_0) = \\ (16) \quad & \omega_1(f, h) \left[1 + \frac{1}{h} \left(\tilde{L}_n(|\cdot - x_0|)\right)(x_0)\right] = 2\omega_1\left(f, \left(\tilde{L}_n(|\cdot - x_0|)\right)(x_0)\right), \end{aligned}$$

by choosing

$$(17) \quad h := \left(\tilde{L}_n(|\cdot - x_0|)\right)(x_0),$$

if  $\left(\tilde{L}_n(|\cdot - x_0|)\right)(x_0) > 0$ .

Next we consider the case of  $\left(\tilde{L}_n(|\cdot - x_0|)\right)(x_0) = 0$ .

By Riesz representation theorem there exists a probability measure  $\mu_{x_0}$  such that

$$(18) \quad \left(\tilde{L}_n(g)\right)(x_0) = \int_{[a,b]} g(t) d\mu_{x_0}(t), \quad \forall g \in C([a, b]).$$

That is

$$(19) \quad \int_{[a,b]} |t - x_0| d\mu_{x_0}(t) = 0,$$

which implies  $|t - x_0| = 0$ , a.e, hence  $t - x_0 = 0$ , a.e, and  $t = x_0$ , a.e.

Consequently  $\mu_{x_0}(\{t \in [a, b] : t \neq x_0\}) = 0$ . That is  $\mu_{x_0} = \delta_{x_0}$ , the Dirac measure with support only  $\{x_0\}$ . Hence in that case  $\left(\tilde{L}_n(g)\right)(x_0) = g(x_0)$ .

Consequently it holds  $\omega_1\left(f, \left(\tilde{L}_n(|\cdot - x_0|)\right)(x_0)\right) = \omega_1(f, 0) = 0$ , and

$$\left(\tilde{L}_n(\|f - f(x_0)\|)\right)(x_0) = \|f(x_0) - f(x_0)\| = 0,$$

and by (11), (15)  $\|(L_n(f))(x_0) - f(x_0)\| = 0$ , imply  $(L_n(f))(x_0) = f(x_0)$ . That is proving inequality (13) is always true.  $\square$

**Remark 2.** (related to the proof of Theorem 1) By Schwartz's inequality we get

$$(20) \quad \int_{[a,b]} |t - x_0| d\mu_{x_0}(t) \leq \left(\int_{[a,b]} (t - x_0)^2 d\mu_{x_0}(t)\right)^{\frac{1}{2}},$$

that is

$$(21) \quad \left(\tilde{L}_n(|\cdot - x_0|)\right)(x_0) \leq \left(\left(\tilde{L}_n\left((\cdot - x_0)^2\right)\right)(x_0)\right)^{\frac{1}{2}}.$$

**Corollary 3.** (to Theorem 1) It holds

$$(22) \quad \|(L_n(f))(x_0) - f(x_0)\| \leq 2\omega_1\left(f, \left(\left(\tilde{L}_n\left((\cdot - x_0)^2\right)\right)(x_0)\right)^{\frac{1}{2}}\right).$$

*Proof.* By (13) and (21).  $\square$

We further obtain

**Corollary 4.** (to Corollary 3) It holds

$$(23) \quad \|(B_n f)(t) - f(t)\| \leq 2\omega_1 \left( f, \left( \frac{t(1-t)}{n} \right)^{\frac{1}{2}} \right) \leq 2\omega_1 \left( f, \frac{1}{2\sqrt{n}} \right),$$

$\forall t \in [0, 1], \forall f \in C([0, 1], X)$ ,  $(X, \|\cdot\|)$  is a Banach space,  $\forall n \in \mathbb{N}$ , where  $B_n$  are the vectorial Bernstein polynomials.

*Proof.* Notice that  $\left( \tilde{B}_n \left( (\cdot - t)^2 \right) \right) (t) = \frac{t(1-t)}{n}, \forall t \in [0, 1]$ . □

**Corollary 5.** We have that

$$(24) \quad \|\|B_n f - f\|\|_{\infty, [0, 1]} \leq 2\omega_1 \left( f, \frac{1}{2\sqrt{n}} \right), \quad \forall f \in C([0, 1], X).$$

**Conclusion 6.** (from (24)) Clearly as  $n \rightarrow \infty, \omega_1 \left( f, \frac{1}{2\sqrt{n}} \right) \rightarrow 0$  and

$$\|\|B_n f - f\|\|_{\infty, [0, 1]} \rightarrow 0.$$

The last implies  $\|B_n f - f\| \rightarrow 0$ , uniformly in  $t \in [0, 1]$ , as  $n \rightarrow \infty$ , equivalently, it holds  $\lim_{n \rightarrow \infty} B_n f = f$ , uniformly in  $t \in [0, 1]$ .

We say that  $B_n \rightarrow I$ , uniformly as  $n \rightarrow \infty$ , where  $I$  is the unit operator i.e.  $I(f) = f$ .

A related comment follows

**Conclusion 7.** By (13) and assuming  $\left( \tilde{L}_n (|\cdot - x_0|) \right) (x_0) \rightarrow 0$ , implies

$(L_n(f))(x_0) \rightarrow f(x_0)$ , as  $n \rightarrow \infty$ . By (22) and assuming  $\left( \tilde{L}_n \left( (\cdot - x_0)^2 \right) \right) (x_0) \rightarrow 0$ , we get again that  $(L_n(f))(x_0) \rightarrow f(x_0)$ , as  $n \rightarrow \infty$ .

We present the more general theorem of pointwise convergence.

**Theorem 8.** Let  $L_n : C([a, b], X) \rightarrow C([a, b], X)$ , where  $(X, \|\cdot\|)$  is a Banach space and  $L_n$  is a linear operator,  $\forall n \in \mathbb{N}, x_0 \in [a, b]$ . Let the positive linear operators  $\tilde{L}_n : C([a, b]) \hookrightarrow C([a, b])$ , such that

$$(25) \quad \|(L_n(f))(x_0)\| \leq \left( \tilde{L}_n (\|f\|) \right) (x_0), \quad \forall n \in \mathbb{N},$$

where  $f \in C([a, b], X)$ .

Furthermore assume that

$$(26) \quad L_n(cg) = c\tilde{L}_n(g), \quad \forall g \in C([a, b]), \forall c \in X.$$

Then

$$(27) \quad \|(L_n(f))(x_0) - f(x_0)\| \leq \|f(x_0)\| \left| \left( \tilde{L}_n(1) \right) (x_0) - 1 \right| + \left[ \left( \tilde{L}_n(1) \right) (x_0) + 1 \right] \omega_1 \left( f, \left( \tilde{L}_n (|\cdot - x_0|) \right) (x_0) \right).$$

(Notice if  $\left( \tilde{L}_n(1) \right) (x_0) = 1$ , then (27) collapses to (13). So Theorem 8 generalizes Theorem 1.)

By (27), as  $\left( \tilde{L}_n(1) \right) (x_0) \rightarrow 1$  and  $\left( \tilde{L}_n (|\cdot - x_0|) \right) (x_0) \rightarrow 0$ , then  $(L_n(f))(x_0) \rightarrow f(x_0)$ , as  $n \rightarrow \infty$ , and as here  $\left( \tilde{L}_n(1) \right) (x_0)$  is bounded.

*Proof.* We observe that

$$(28) \quad \|(L_n(f))(x_0) - f(x_0)\| = \|(L_n(f))(x_0) - (L_n(f(x_0)))(x_0) + (L_n(f(x_0)))(x_0) - f(x_0)\| \leq$$

$$\begin{aligned}
& \| (L_n(f))(x_0) - (L_n(f(x_0)))(x_0) \| + \| (L_n(f(x_0)))(x_0) - f(x_0) \| = \\
& \| (L_n(f - f(x_0)))(x_0) \| + \| f(x_0) \left( \tilde{L}_n(1) \right) (x_0) - f(x_0) \| = \\
(29) \quad & \| (L_n(f - f(x_0)))(x_0) \| + \| f(x_0) \| \left| \left( \tilde{L}_n(1) \right) (x_0) - 1 \right| \leq \\
& \left( \tilde{L}_n(\|f - f(x_0)\|) \right) (x_0) + \| f(x_0) \| \left| \left( \tilde{L}_n(1) \right) (x_0) - 1 \right| \leq
\end{aligned}$$

(let  $h > 0$ , and by Lemma 7.1.1, p. 208 of [1])

$$\begin{aligned}
(30) \quad & \left( \tilde{L}_n \left( \omega_1(f, h) \left( 1 + \frac{|\cdot - x_0|}{h} \right) \right) (x_0) \right) + \| f(x_0) \| \left| \left( \tilde{L}_n(1) \right) (x_0) - 1 \right| = \\
& \omega_1(f, h) \left[ \left( \tilde{L}_n(1) \right) (x_0) + \frac{1}{h} \left( \tilde{L}_n(|\cdot - x_0|) \right) (x_0) \right] + \| f(x_0) \| \left| \left( \tilde{L}_n(1) \right) (x_0) - 1 \right| =
\end{aligned}$$

$$(31) \quad \omega_1 \left( f, \left( \tilde{L}_n(|\cdot - x_0|) \right) (x_0) \right) \left[ \left( \tilde{L}_n(1) \right) (x_0) + 1 \right] + \| f(x_0) \| \left| \left( \tilde{L}_n(1) \right) (x_0) - 1 \right|,$$

by choosing

$$(32) \quad h := \left( \tilde{L}_n(|\cdot - x_0|) \right) (x_0),$$

if  $\left( \tilde{L}_n(|\cdot - x_0|) \right) (x_0) > 0$ .

Next we consider the case of

$$(33) \quad \left( \tilde{L}_n(|\cdot - x_0|) \right) (x_0) = 0.$$

By Riesz representation theorem there exists a positive finite measure  $\mu_{x_0}$  such that

$$(34) \quad \left( \tilde{L}_n(g) \right) (x_0) = \int_{[a,b]} g(t) d\mu_{x_0}(t), \quad \forall g \in C([a, b]).$$

That is

$$(35) \quad \int_{[a,b]} |t - x_0| d\mu_{x_0}(t) = 0,$$

which implies  $|t - x_0| = 0$ , a.e, hence  $t - x_0 = 0$ , a.e, and  $t = x_0$ , a.e. on  $[a, b]$ . Consequently  $\mu_{x_0}(\{t \in [a, b] : t \neq x_0\}) = 0$ . That is  $\mu_{x_0} = \delta_{x_0}M$  (where  $0 < M := \mu_{x_0}([a, b]) = \left( \tilde{L}_n(1) \right) (x_0)$ ).

Hence, in that case  $\left( \tilde{L}_n(g) \right) (x_0) = g(x_0)M$ .

Consequently it holds  $\omega_1 \left( f, \left( \tilde{L}_n(|\cdot - x_0|) \right) (x_0) \right) = 0$ , and the right hand side of (27) equals  $\|f(x_0)\| |M - 1|$ .

Also, it is  $\left( \tilde{L}_n(\|f - f(x_0)\|) \right) (x_0) = 0$ , implying (see (29))  $\| (L_n(f - f(x_0)))(x_0) \| = 0$ . Hence,  $(L_n(f - f(x_0)))(x_0) = 0$ , and

$$(36) \quad (L_n(f))(x_0) = f(x_0) \left( \tilde{L}_n(1) \right) (x_0) = Mf(x_0).$$

Consequently the left hand side of (27) becomes

$$(37) \quad \| (L_n(f))(x_0) - f(x_0) \| = \| Mf(x_0) - f(x_0) \| = \| f(x_0) \| |M - 1|.$$

So that (27) becomes an equality, both sides equal  $\|f(x_0)\| |M - 1|$

in the extreme case of  $\left( \tilde{L}_n(|\cdot - x_0|) \right) (x_0) = 0$ . Thus inequality (27) is proved completely in all cases.  $\square$

**Remark 9.** (on Theorem 8) By Schwartz's inequality we get

$$(38) \quad \left( \tilde{L}_n(|\cdot - x_0|) \right) (x_0) \leq \left( \left( \tilde{L}_n \left( (\cdot - x_0)^2 \right) \right) (x_0) \right)^{\frac{1}{2}} \left( \left( \tilde{L}_n(1) \right) (x_0) \right)^{\frac{1}{2}}.$$

Another pointwise convergence result follows

**Corollary 10.** (to Theorem 8) It holds

$$(39) \quad \begin{aligned} & \| (L_n(f))(x_0) - f(x_0) \| \leq \| f(x_0) \| \left| \left( \tilde{L}_n(1) \right) (x_0) - 1 \right| + \\ & \left[ \left( \tilde{L}_n(1) \right) (x_0) + 1 \right] \omega_1 \left( f, \left( \tilde{L}_n(1) \right) (x_0) \right)^{\frac{1}{2}} \left( \left( \tilde{L}_n \left( (\cdot - x_0)^2 \right) \right) (x_0) \right)^{\frac{1}{2}}. \end{aligned}$$

*Proof.* By (27) and (38). □

**Remark 11.** (to Corollary 10) Denote

$$(40) \quad \mu_n := \left\| \left( \tilde{L}_n \left( (\cdot - x)^2 \right) \right) (x) \right\|_{\infty, [a, b]}^{\frac{1}{2}}.$$

By [4], we get that

$$(41) \quad \mu_n \leq \sqrt{\left\| \tilde{L}_n(t^2; x) - x^2 \right\|_{\infty, [a, b]} + 2c_1 \left\| \tilde{L}_n(t; x) - x \right\|_{\infty, [a, b]} + c_1^2 \left\| \tilde{L}_n(1; x) - 1 \right\|_{\infty, [a, b]}},$$

where  $c_1 := \max(|a|, |b|)$ .

We give the following theorem related to uniform convergence, which gives a Shisha-Mond ([4]) type inequality.

**Theorem 12.** Let  $L_n : C([a, b], X) \rightarrow C([a, b], X)$ , where  $(X, \|\cdot\|)$  is a Banach space and  $L_n$  is a linear operator,  $\forall n \in \mathbb{N}$ . Let the positive linear operators  $\tilde{L}_n : C([a, b]) \hookrightarrow C([a, b])$ , such that

$$(42) \quad \| (L_n(f))(x) \| \leq \left( \tilde{L}_n(\|f\|) \right) (x), \quad \forall n \in \mathbb{N}, \quad \forall x \in [a, b],$$

where  $f \in C([a, b], X)$ .

Furthermore assume that

$$(43) \quad L_n(cg) = c\tilde{L}_n(g), \quad \forall g \in C([a, b]), \quad \forall c \in X.$$

Then

$$(44) \quad \begin{aligned} & \| \| L_n(f) - f \| \|_{\infty, [a, b]} \leq \| \| f \| \|_{\infty, [a, b]} \left\| \tilde{L}_n(1) - 1 \right\|_{\infty, [a, b]} + \\ & \left\| \tilde{L}_n(1) + 1 \right\|_{\infty, [a, b]} \omega_1 \left( f, \left\| \tilde{L}_n(1) \right\|_{\infty, [a, b]}^{\frac{1}{2}} \right. \\ & \left. \sqrt{\left\| \tilde{L}_n(t^2; x) - x^2 \right\|_{\infty, [a, b]} + 2c_1 \left\| \tilde{L}_n(t; x) - x \right\|_{\infty, [a, b]} + c_1^2 \left\| \tilde{L}_n(1; x) - 1 \right\|_{\infty, [a, b]}} \right), \end{aligned}$$

where  $c_1 := \max(|a|, |b|)$ .

*Proof.* Using Corollary 10 and Remark 11; see (39), (40), (41). □

It follows a Korokvin type theorem ([3]) for Banach space valued functions.

**Theorem 13.** All assumptions as in Theorem 12. Additionally assume that  $\tilde{L}_n(1) \xrightarrow{u} 1$ ,  $\tilde{L}_n(id) \xrightarrow{u} id$ ,  $\tilde{L}_n(id^2) \xrightarrow{u} id^2$ , uniformly, where  $id = \text{identity map}$ , as  $n \rightarrow \infty$ .

Then  $L_n(f) \xrightarrow{u} f$ , uniformly in  $t \in [a, b]$ , i.e.  $L_n \rightarrow I$ , uniformly, as  $n \rightarrow \infty$ , where  $I$  is the unit operator.

*Proof.* We use (44). Since  $\tilde{L}_n(1) \xrightarrow{u} 1$ , uniformly, we get that  $\|\tilde{L}_n(1)\|_{\infty, [a, b]}$  is bounded. Thus  $\|\tilde{L}_n(1) + 1\|_{\infty, [a, b]}$  is also bounded. Clearly under our assumptions  $\omega_1$  trends to zero. The rest of the right hand side of (44) goes to zero too, proving the claim.  $\square$

Next we present another general theorem of pointwise convergence but proved differently.

**Theorem 14.** *Let  $L_n : C([a, b], X) \rightarrow C([a, b], X)$ , where  $(X, \|\cdot\|)$  is a Banach space and  $L_n$  is a linear operator,  $\forall n \in \mathbb{N}$ ,  $x_0 \in [a, b]$ . Let the positive linear operators  $\tilde{L}_n : C([a, b]) \hookrightarrow C([a, b])$ , such that*

$$(45) \quad \|(L_n(f))(x_0)\| \leq \left(\tilde{L}_n(\|f\|)\right)(x_0), \quad \forall n \in \mathbb{N},$$

where  $f \in C([a, b], X)$ .

Furthermore assume that

$$(46) \quad L_n(cg) = c\tilde{L}_n(g), \quad \forall g \in C([a, b]), \forall c \in X.$$

Then

$$(47) \quad \|(L_n(f))(x_0) - f(x_0)\| \leq \|f(x_0)\| \left| \left(\tilde{L}_n(1)\right)(x_0) - 1 \right| + \left[ \left(\tilde{L}_n(1)\right)(x_0) + 1 \right] \omega_1 \left( f, \left( \left(\tilde{L}_n((\cdot - x_0)^2)\right)(x_0) \right)^{\frac{1}{2}} \right).$$

As  $\left(\tilde{L}_n(1)\right)(x_0) \rightarrow 1$  and  $\left(\tilde{L}_n((\cdot - x_0)^2)\right)(x_0) \rightarrow 0$ , we get  $(L_n(f))(x_0) \rightarrow f(x_0)$ , as  $n \rightarrow \infty$ . Clearly here  $\left(\tilde{L}_n(1)\right)(x_0)$  is bounded.

*Proof.* Let  $x_0 \in [a, b]$  and  $\delta > 0$ . Let  $t \in [a, b]$ . If  $|t - x_0| > \delta$ , then

$$(48) \quad \|f(t) - f(x_0)\| \leq \omega_1(f, |t - x_0|) = \omega_1(f, |t - x_0| \delta^{-1} \delta) \leq \left(1 + \frac{|t - x_0|}{\delta}\right) \omega_1(f, \delta) \leq \left(1 + \frac{(t - x_0)^2}{\delta^2}\right) \omega_1(f, \delta).$$

The estimate

$$(49) \quad \|f(t) - f(x_0)\| \leq \left(1 + \frac{(t - x_0)^2}{\delta^2}\right) \omega_1(f, \delta)$$

also holds trivially when  $|t - x_0| \leq \delta$ .

So (49) is true always,  $\forall t \in [a, b]$ , for any  $x_0 \in [a, b]$ .

We can rewrite

$$(50) \quad \|f(\cdot) - f(x_0)\| \leq \left(1 + \frac{(\cdot - x_0)^2}{\delta^2}\right) \omega_1(f, \delta).$$

Hence it holds

$$(51) \quad \left(\tilde{L}_n(\|f - f(x_0)\|)\right)(x_0) \leq \left[ \left(\tilde{L}_n(1)\right)(x_0) + \frac{1}{\delta^2} \left(\tilde{L}_n((\cdot - x_0)^2)\right)(x_0) \right] \omega_1(f, \delta).$$

As in the proof of Theorem 8 we have

$$\|(L_n(f))(x_0) - f(x_0)\| \leq \dots \leq$$

$$(52) \quad \begin{aligned} & \|(L_n(f - f(x_0)))(x_0)\| + \|f(x_0)\| \left| (\tilde{L}_n(1))(x_0) - 1 \right| \leq \\ & \left( \tilde{L}_n(\|f - f(x_0)\|) \right)(x_0) + \|f(x_0)\| \left| (\tilde{L}_n(1))(x_0) - 1 \right| \stackrel{(51)}{\leq} \\ & \left[ (\tilde{L}_n(1))(x_0) + \frac{1}{\delta^2} \left( \tilde{L}_n((\cdot - x_0)^2) \right)(x_0) \right] \omega_1(f, \delta) \end{aligned}$$

$$(53) \quad \begin{aligned} & + \|f(x_0)\| \left| (\tilde{L}_n(1))(x_0) - 1 \right| = \\ & \left[ (\tilde{L}_n(1))(x_0) + 1 \right] \omega_1 \left( f, \left( \tilde{L}_n((\cdot - x_0)^2) \right)(x_0) \right)^{\frac{1}{2}} \\ & + \|f(x_0)\| \left| (\tilde{L}_n(1))(x_0) - 1 \right|, \end{aligned}$$

by choosing

$$(54) \quad \delta := \left( \left( \tilde{L}_n((\cdot - x_0)^2) \right)(x_0) \right)^{\frac{1}{2}},$$

if  $\left( \tilde{L}_n((\cdot - x_0)^2) \right)(x_0) > 0$ .

Next we consider the case

$$(55) \quad \left( \tilde{L}_n((\cdot - x_0)^2) \right)(x_0) = 0.$$

By Riesz representation theorem there exists a positive finite measure  $\mu_{x_0}$  such that

$$(56) \quad \left( \tilde{L}_n(g) \right)(x_0) = \int_{[a,b]} g(t) d\mu_{x_0}(t), \quad \forall g \in C([a, b]).$$

That is

$$(57) \quad \int_{[a,b]} (t - x_0)^2 d\mu_{x_0}(t) = 0,$$

which implies  $(t - x_0)^2 = 0$ , a.e, hence  $t - x_0 = 0$ , a.e, and  $t = x_0$ , a.e. on  $[a, b]$ .

Consequently  $\mu_{x_0}(\{t \in [a, b] : t \neq x_0\}) = 0$ . That is  $\mu_{x_0} = \delta_{x_0} M$  (where  $0 < M := \mu_{x_0}([a, b]) = \left( \tilde{L}_n(1) \right)(x_0)$ ). Hence, we get here that

$$(58) \quad \left( \tilde{L}_n(g) \right)(x_0) = g(x_0) M.$$

Since  $\omega_1 \left( f, \left( \tilde{L}_n((\cdot - x_0)^2) \right)(x_0) \right)^{\frac{1}{2}} = 0$ , the right hand side of (47) equals  $\|f(x_0)\| |M - 1|$ .

Also, it holds  $\left( \tilde{L}_n(\|f - f(x_0)\|) \right)(x_0) = 0$ , implying (see (52))  $\|(L_n(f - f(x_0)))(x_0)\| = 0$ . Therefore,  $(L_n(f - f(x_0)))(x_0) = 0$ , and

$$(59) \quad (L_n(f))(x_0) = f(x_0) \left( \tilde{L}_n(1) \right)(x_0) = M f(x_0).$$

Consequently the left hand side of (47) becomes

$$(60) \quad \|(L_n(f))(x_0) - f(x_0)\| = \|f(x_0)\| |M - 1|.$$

Thus (47) becomes an equality, both sides are equal  $\|f(x_0)\| |M - 1|$ , in the extreme case of

$$\left( \tilde{L}_n((\cdot - x_0)^2) \right)(x_0) = 0.$$

Inequality (47) is proved in all cases. □

A combined pointwise result follows



**Corollary 15.** *All as in Theorem 14. It holds*

$$(61) \quad \begin{aligned} & \| (L_n(f))(x_0) - f(x_0) \| \leq \| f(x_0) \| \left| \left( \tilde{L}_n(1) \right) (x_0) - 1 \right| + \\ & \left[ \left( \tilde{L}_n(1) \right) (x_0) + 1 \right] \min \left\{ \omega_1 \left( f, \left( \left( \tilde{L}_n \left( (\cdot - x_0)^2 \right) \right) (x_0) \right)^{\frac{1}{2}} \right), \right. \\ & \left. \omega_1 \left( f, \left( \left( \tilde{L}_n(1) \right) (x_0) \right)^{\frac{1}{2}} \left( \left( \tilde{L}_n \left( (\cdot - x_0)^2 \right) \right) (x_0) \right)^{\frac{1}{2}} \right) \right\}. \end{aligned}$$

*Proof.* By (47) and (39). □

So (39) is better than (47) only if  $\left( \tilde{L}_n(1) \right) (x_0) < 1$ .

A sharpened Shisha-Mond type inequality follows

**Corollary 16.** *All as in Theorem 12. Then*

$$(62) \quad \begin{aligned} & \| \| L_n(f) - f \| \|_{\infty, [a, b]} \leq \| \| f \| \|_{\infty, [a, b]} \left\| \tilde{L}_n(1) - 1 \right\|_{\infty, [a, b]} + \\ & \left\| \tilde{L}_n(1) + 1 \right\|_{\infty, [a, b]} \min \left\{ \omega_1 \left( f, \right. \right. \\ & \left. \sqrt{\left\| \tilde{L}_n(t^2; x) - x^2 \right\|_{\infty, [a, b]} + 2c_1 \left\| \tilde{L}_n(t; x) - x \right\|_{\infty, [a, b]} + c_1^2 \left\| \tilde{L}_n(1; x) - 1 \right\|_{\infty, [a, b]}} \right), \\ & \left. \omega_1 \left( f, \left\| \tilde{L}_n(1) \right\|_{\infty, [a, b]}^{\frac{1}{2}} \right) \right\} \\ & \left. \sqrt{\left\| \tilde{L}_n(t^2; x) - x^2 \right\|_{\infty, [a, b]} + 2c_1 \left\| \tilde{L}_n(t; x) - x \right\|_{\infty, [a, b]} + c_1^2 \left\| \tilde{L}_n(1; x) - 1 \right\|_{\infty, [a, b]}} \right) \right\}, \end{aligned}$$

where  $c_1 := \max(|a|, |b|)$ .

*Proof.* Using Theorem 14 and Theorem 12, see also (41). □

Clearly, one can also use (62) to prove the Korovkin type Theorem 13.

Under convexity we have the following sharp general pointwise convergence theorem.

**Theorem 17.** *All as in Theorem 8. Additionally, assume that  $x_0 \in (a, b)$ ,*

$$(63) \quad 0 \leq \left( \left( \tilde{L}_n(|\cdot - x_0|) \right) (x_0) \right) \leq \min(x_0 - a, b - x_0),$$

and  $\|f(t) - f(x_0)\|$  is convex in  $t \in [a, b]$ .

Then

$$(64) \quad \begin{aligned} & \| (L_n(f))(x_0) - f(x_0) \| \leq \| f(x_0) \| \left| \left( \tilde{L}_n(1) \right) (x_0) - 1 \right| + \\ & \omega_1 \left( f, \left( \tilde{L}_n(|\cdot - x_0|) \right) (x_0) \right). \end{aligned}$$

*Proof.* Let  $x_0 \in (a, b)$ ,  $0 < h \leq \min(x_0 - a, b - x_0)$ . Here  $g(t) := \|f(t) - f(x_0)\|$  is assumed to be convex in  $t \in [a, b]$ , and obviously  $g(x_0) = 0$ . Then by Lemma 8.1.1, p. 243 of [1], we obtain

$$(65) \quad g(t) \leq \frac{\omega_1(g, h)}{h} |t - x_0|, \quad \forall t \in [a, b].$$

We notice the following

$$(66) \quad \begin{aligned} & \|f(t_1) - f(x_0)\| = \|f(t_1) - f(t_2) + f(t_2) - f(x_0)\| \leq \\ & \|f(t_1) - f(t_2)\| + \|f(t_2) - f(x_0)\|, \end{aligned}$$

hence

$$(67) \quad \|f(t_1) - f(x_0)\| - \|f(t_2) - f(x_0)\| \leq \|f(t_1) - f(t_2)\|.$$

Similarly, it holds

$$(68) \quad \|f(t_2) - f(x_0)\| - \|f(t_1) - f(x_0)\| \leq \|f(t_1) - f(t_2)\|.$$

Therefore for any  $t_1, t_2 \in [a, b] : |t_1 - t_2| \leq h$  we get:

$$(69) \quad \| \|f(t_1) - f(x_0)\| - \|f(t_2) - f(x_0)\| \| \leq \|f(t_1) - f(t_2)\| \leq \omega_1(f, h).$$

That is

$$(70) \quad \omega_1(g, h) \leq \omega_1(f, h).$$

The last implies

$$(71) \quad \|f(t) - f(x_0)\| \leq \frac{\omega_1(f, h)}{h} |t - x_0|, \quad \forall t \in [a, b].$$

As in the proof of Theorem 8 we have

$$(72) \quad \begin{aligned} & \| (L_n(f))(x_0) - f(x_0) \| \leq \dots \leq \\ & \| (L_n(f - f(x_0)))(x_0) \| + \|f(x_0)\| \left| \left( \tilde{L}_n(1) \right) (x_0) - 1 \right| \leq \\ & \left( \tilde{L}_n(\|f - f(x_0)\|) \right) (x_0) + \|f(x_0)\| \left| \left( \tilde{L}_n(1) \right) (x_0) - 1 \right| \stackrel{(71)}{\leq} \\ (73) \quad & \frac{\omega_1(f, h)}{h} \left( \left( \tilde{L}_n(|\cdot - x_0|) \right) (x_0) \right) + \|f(x_0)\| \left| \left( \tilde{L}_n(1) \right) (x_0) - 1 \right| = \\ & \omega_1 \left( f, \left( \tilde{L}_n(|\cdot - x_0|) \right) (x_0) \right) + \|f(x_0)\| \left| \left( \tilde{L}_n(1) \right) (x_0) - 1 \right|, \end{aligned}$$

by choosing

$$(74) \quad h := \left( \left( \tilde{L}_n(|\cdot - x_0|) \right) (x_0) \right) > 0,$$

if the last is positive. The case of  $\left( \left( \tilde{L}_n(|\cdot - x_0|) \right) (x_0) \right) = 0$  is treated the same way as in the proof of Theorem 8. The theorem is proved.  $\square$

**Theorem 18.** *All as in Theorem 17. Inequality (64) is sharp, infact it is attained by  $f(t) = \vec{i} |t - x_0|$ ,  $\vec{i}$  is a unit vector of  $(X, \|\cdot\|)$ ,  $t \in [a, b]$ .*

*Proof.* Indeed,  $f$  here fulfills all the assumptions of the theorem.

We further notice that  $f(x_0) = 0$ , and  $\|f(t) - f(x_0)\| = |t - x_0|$  is convex in  $t \in [a, b]$ .

The left hand side of (64) is

$$(75) \quad \begin{aligned} & \| (L_n(f))(x_0) - f(x_0) \| = \left\| \left( L_n \left( \vec{i} |\cdot - x_0| \right) \right) (x_0) \right\| \\ & \stackrel{(26)}{=} \left\| \vec{i} \left( \tilde{L}_n(|\cdot - x_0|) \right) (x_0) \right\| = \left( \left( \tilde{L}_n(|\cdot - x_0|) \right) (x_0) \right). \end{aligned}$$

The right hand side of (64) is

$$\begin{aligned} & \omega_1 \left( f, \left( \left( \tilde{L}_n(|\cdot - x_0|) \right) (x_0) \right) \right) = \\ & \omega_1 \left( \vec{i} |\cdot - x_0|, \left( \left( \tilde{L}_n(|\cdot - x_0|) \right) (x_0) \right) \right) = \end{aligned}$$

$$\begin{aligned}
& \sup_{\substack{t_1, t_2 \in [a, b]: \\ |t_1 - t_2| \leq ((\tilde{L}_n(|\cdot - x_0|))(x_0))}} \left\| \vec{i} |t_1 - x_0| - \vec{i} |t_2 - x_0| \right\| = \\
(76) \quad & \sup_{\substack{t_1, t_2 \in [a, b]: \\ |t_1 - t_2| \leq ((\tilde{L}_n(|\cdot - x_0|))(x_0))}} \| |t_1 - x_0| - |t_2 - x_0| \| \leq \\
& \sup_{\substack{t_1, t_2 \in [a, b]: \\ |t_1 - t_2| \leq ((\tilde{L}_n(|\cdot - x_0|))(x_0))}} |t_1 - t_2| = \left( (\tilde{L}_n(|\cdot - x_0|))(x_0) \right).
\end{aligned}$$

Hence we have found that

$$(77) \quad \omega_1 \left( f, \left( (\tilde{L}_n(|\cdot - x_0|))(x_0) \right) \right) \leq \left( (\tilde{L}_n(|\cdot - x_0|))(x_0) \right).$$

Clearly (64) is attained. The theorem is proved.  $\square$

#### REFERENCES

- [1] G.A. Anastassiou, *Moments in Probability and Approximation Theory*, Pitman Research Notes in Math., Vol. 287, Longman Sci. & Tech., Harlow, U.K., 1993.
- [2] G.A. Anastassiou, *Lattice homomorphism - Korovkin type inequalities for vector valued functions*, Hokkaido Mathematical Journal, Vol. 26 (1997), 337-364.
- [3] P.P. Korovkin, *Linear Operators and Approximation Theory*, Hindustan Publ. Corp. Delhi, India, 1960.
- [4] O. Shisha and B. Mond, *The degree of convergence of sequences of linear positive operators*, Nat. Acad. of Sci. U.S., 60, (1968), 1196-1200.

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