TWO TENSORS OF TYPE (1, 2) ASSOCIATED TO THE SHAPE OPERATOR OF A REAL HYPERSURFACE IN THE COMPLEX PROJECTIVE SPACE

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Abstract. Following S. Tachibana we study purity and hybridness of two tensors of type (1, 2) associated to the shape operator of a real hypersurface in complex projective space with respect to either the structure operator or the shape operator.

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1. Introduction

We will denote by $\mathbb{C}P^m$ the complex projective space with complex dimension $m \geq 2$ equipped with the Kählerian structure (J, g), where g is the Fubini-Study metric with constant holomorphic sectional curvature 4. Consider a connected real hypersurface M in $\mathbb{C}P^m$ with local unit normal vector field N and define the structure (or Reeb) vector field on M by $\xi = -JN$. If for any vector field X tangent to M we write $JX = \phi X + \eta(X)N$, where ϕX denotes the tangential component of JX, ϕ is a tensor of type (1, 1) on M called the structure operator of M and the 1-form η is given by $\eta(X) = g(X, \xi)$, for any X tangent to M. We continue denoting by g the restriction of the metric on $\mathbb{C}P^m$ to M. Then (ϕ, ξ, η, g) is an almost contact metric structure on M. Therefore $\phi\xi = 0$, $\eta(\xi) = 1$, $\phi^2 X = -X + \eta(X)\xi$ and $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$, for any X, Y tangent to M, [1].

Denote by ∇ the Levi-Civita connection on M and by A the shape operator on M associated to N. As J is parallel with respect to the Levi-Civita connection $\bar{\nabla}$ on $\mathbb{C}P^m$, we get

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \phi AX,$$

for any X, Y tangent to M. The Codazzi equation is given by

$$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi,$$

for any X, Y tangent to M.

M is called Hopf if ξ is an eigenvector of A. That is, $A\xi = \alpha\xi$ for a certain function α on M, called the Reeb curvature of M. The maximal holomorphic distribution $\mathbb{D}$ on M is given by $\mathbb{D}(p) = \{X \in T_pM / g(X, \xi) = 0\}$, for any $p \in M$. Takagi, [14], [15], classified homogeneous real hypersurfaces of $\mathbb{C}P^m$ in 6 types. Kimura, [5], proved that such types are the unique Hopf real hypersurfaces in $\mathbb{C}P^m$ with constant principal curvatures. We mention the following types in Takagi's list:

- **(A_1):** Geodesic hyperspheres of radius r, $0 < r < \frac{\pi}{2}$. They are the unique real hypersurfaces in $\mathbb{C}P^m$ with 2 distinct principal curvatures, [2].

(A_2): Tubes of radius r, $0 < r < \frac{\pi}{2}$, around totally geodesic complex projective spaces $\mathbb{C}P^n$, $0 < n < m-1$. They have 3 distinct constant principal curvatures.

Okumura, [9], proved that both types, that we will call type (A) real hypersurfaces, are the unique ones in $\mathbb{C}P^m$ satisfying $A\phi = \phi A$.

As examples of non Hopf real hypersurfaces in $\mathbb{C}P^m$ we can mention ruled real hypersurfaces, introduced by Kimura, [6], as real hypersurfaces such that $\mathbb{D}$ is integrable and have $\mathbb{C}P^{m-1}$ as integral manifolds. Equivalently $g(A\mathbb{D}, \mathbb{D}) = 0$. For examples, see [6] and [7].

The Tanaka-Webster connection is the canonical affine connection defined on a non-degenerate, pseudo-Hermitian CR-manifold independently by Tanaka, [16], and Webster, [18]. Tanno [17] generalized such a connection for contact metric manifolds and from this generalization, for any nonnull real number k, Cho, [3], [4], defined the k-th generalized Tanaka-Webster connection on a real hypersurface M of $\mathbb{C}P^m$ by

$$\hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y,$$

for any X, Y tangent to M. This is a metric connection, $\hat{\nabla}^{(k)} g = 0$, and also $\hat{\nabla}^{(k)}\phi = 0$, $\hat{\nabla}^{(k)}\xi = 0$ and $\hat{\nabla}^{(k)}\eta = 0$. In the particular case of $\phi A + A\phi = 2k\phi$, M is a contact manifold and $\hat{\nabla}^{(k)}$ coincides with the Tanaka-Webster connection.

The k-th Cho operator associated to X, tangent to M, is defined by $F^{(k)}_X Y = g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y$, for any Y tangent to M. Then the torsion of $\hat{\nabla}^{(k)}$ is $T^{(k)}(X, Y) = F^{(k)}_X Y - F^{(k)}_Y X$. Notice that if $X \in \mathbb{D}$, $F^{(k)}_X$ does not depend on k and we will denote it simply by F_X.

We will also call the k-th torsion operator associated to the vector field X tangent to M to $T^{(k)}_X Y = T^{(k)}(X, Y)$, for any Y tangent to M.

Let $\mathcal{L}$ be the Lie derivative on M. We know that for any X, Y tangent to M, $\mathcal{L}_X Y = \nabla_X Y - \nabla_Y X$. This expression allows us to define a differential operator of first order on M, that we will call the derivative of Lie type associated to $\hat{\nabla}^{(k)}$ and is given by $\mathcal{L}^{(k)}_X Y = \hat{\nabla}^{(k)}_X Y - \hat{\nabla}^{(k)}_Y X = \mathcal{L}_X Y + T^{(k)}_X Y$ for any X, Y tangent to M.

Let B be a symmetric operator on M. We will consider the tensor field of type (1, 2) on M given by $B^{(k)}_F(X, Y) = ((\hat{\nabla}^{(k)}_X - \nabla_X)B)Y = [F^{(k)}_X, B]Y = F^{(k)}_X BY - BF^{(k)}_X Y$ for any X, Y tangent to M.

We also can consider a second tensor field of type (1, 2) on M given by $B^{(k)}_T(X, Y) = ((\mathcal{L}^{(k)}_X - \mathcal{L}_X)B)Y = [T^{(k)}_X, B]Y = T^{(k)}_X BY - BT^{(k)}_X Y$, for any X, Y tangent to M.

Let Θ be another operator on M and Q a tensor of type (1, 2) on M. Tachibana, [13], introduced the notion of Q being pure with respect to Θ if $Q(\Theta X, Y) = Q(X, \Theta Y)$ for any $X, Y \in TM$. In the case of the same equality for any $X, Y \in \mathbb{D}$, we will say that Q is η-pure with respect to Θ. Tachibana also gave the following definition: Q is hybrid with respect to Θ if $Q(\Theta X, Y) = -Q(X, \Theta Y)$ for any $X, Y \in TM$. If this equality is satisfied for any $X, Y \in \mathbb{D}$ we will say that Q is η-hybrid with respect to Θ.

In [10] we presented some generalizations of the conditions $A^{(k)}_F \equiv 0$, $A^{(k)}_T \equiv 0$. Now we will study purity (η-purity) and hybridness (η-hybridness) of such tensors with respect to either ϕ or A.

2. Purity and hybridness of $A_F^{(k)}$ and $A_T^{(k)}$ with respect to ϕ

Suppose that $m \geq 2$ and $A_F^{(k)}$ is η-pure with respect to ϕ. Then we have

$$(2.1)\quad g(\phi A\phi X, AY) - \eta(AY)\phi A\phi X - g(\phi A\phi X, Y)A\xi = g(\phi AX, A\phi Y)\xi - \eta(A\phi Y)\phi AX - g(AX, Y)A\xi,$$

for any $X, Y \in \mathbb{D}$.

If we suppose that M is Hopf with Reeb curvature α, we know, [8], that α is constant and if $X \in \mathbb{D}$ satisfies $AX = \lambda X$, then $2\lambda - \alpha \neq 0$ and $A\phi X = \mu\phi X$, with $\mu = \frac{\alpha\lambda+2}{2\lambda-\alpha}$. Then the scalar product of (2.1) and ξ gives $A\phi A\phi X - \alpha\phi A\phi X = -\phi A\phi AX - \alpha\phi AX$, for any $X \in \mathbb{D}$. If $X \in \mathbb{D}$ satisfies $AX = \lambda X$, we get $2\lambda\mu = \alpha(\mu + \lambda)$. From the value of μ this yields $\alpha\lambda^2 + 2\lambda = \alpha\lambda^2 + \alpha$. Then $\lambda = \frac{\alpha}{2}$. This implies $\alpha\lambda = \frac{\alpha^2}{2} = 0$ and therefore, $\alpha = \lambda = 0$, a contradiction, as $2\lambda - \alpha \neq 0$.
If M is non Hopf we can write $A\xi = \alpha\xi + \beta U$, where U is a unit vector field in $\mathbb{D}$ and α, β are functions with $\beta \neq 0$, at least on a neighbourhood of a point $p \in M$. Call $\mathbb{D}_U = \{X \in \mathbb{D}/g(X, U) = g(X, \phi U) = 0\}$. Taking the scalar product of (2.1) and either ϕU or U or $Z \in \mathbb{D}_U$ we obtain either $AU = \beta\xi$, $A\phi U = 0$, $AZ = 0$, for any $Z \in \mathbb{D}_U$. Therefore we get

Theorem 2.1 ([11])**.** *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 2$. Then $A_F^{(k)}$ is η-pure with respect to ϕ if and only if M is locally congruent to a ruled real hypersurface.*

A similar proof for the case of $A_F^{(k)}$ being η-hybrid yields

Theorem 2.2 ([11])**.** *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 2$, and k a nonnull real number. Then $A_F^{(k)}$ is η-hybrid with respect to ϕ if and only if M is locally congruent to one of the following real hypersurfaces:*

- *a tube of radius $\frac{\pi}{4}$ around a complex submanifold of $\mathbb{C}P^m$,*
- *a real hypersurface of type (A),*
- *a ruled real hypersurface.*

Remark 2.3. First case in Theorem 2.2 corresponds to Hopf real hypersurfaces with Reeb curvature equal to 0, see [2].

Remark 2.4. If we suppose that $A_F^{(k)}$ is pure with respect to ϕ we should have

$$(2.2)\quad g(\phi A\phi X, AY)\xi - \eta(AY)\phi A\phi X - g(\phi A\phi X, Y)A\xi + \eta(Y)A\phi A\phi X = g(\phi AX, A\phi Y)\xi - \eta(A\phi Y)\phi AX - k\eta(X)\phi A\phi Y - g(\phi AX, \phi Y)A\xi + k\eta(X)A\phi^2 Y,$$

for any X, Y tangent to M. From Theorem 2.1 we have that, in particular, M must be locally congruent to either a real hypersurface of type (A) or to a ruled one. In the first case, bearing in mind that $A\phi = \phi A$ and taking $X \in \mathbb{D}$, $Y = \xi$ in $\mathbb{D}$ we arrive to a contradiction. Moreover, if M is ruled and we take $X = \xi$, $Y = U$ in (2.2) we should have $-\beta A\xi - kAU = 0$. Its scalar product with U yields $\beta^2 = 0$, which is impossible. Therefore,

Corollary 2.5. *There does not exist any real hypersurface in $\mathbb{C}P^m$, $m \geq 2$, such that $A_F^{(k)}$ is pure with respect to ϕ, for any nonnull real number k.*

Also from Theorem 2.2 we have a similar non-existence result for real hypersurfaces in $\mathbb{C}P^m$, $m \geq 2$, for which $A_F^{(k)}$ is hybrid with respect to ϕ, for any nonnull real number k.

For $A_T^{(k)}$ we obtain the following result

Theorem 2.6 ([11]). *There does not exist any real hypersurface in $\mathbb{C}P^m$, $m \geq 3$, such that $A_T^{(k)}$ is η-pure with respect to ϕ, for any nonnull real number k.*

If we suppose now that $A_T^{(k)}$ is η-hybrid with respect to ϕ, we have

$$(2.3)\quad g(\phi AX, AY)\xi - \eta(AY)\phi A\phi X - g(A^2Y, X)\xi - k\eta(AY)X + g(\phi AX, A\phi Y)\xi \\ - \eta(A\phi Y)\phi AX - g(\phi A^2\phi Y, X)\xi - k\eta(A\phi Y)\phi X = 0,$$

for any $X, Y \in \mathbb{D}$. If M is Hopf with Reeb curvature α the scalar product of (2.3) and ξ yields $A\phi A\phi X - A^2X - \phi A\phi AX - \phi A^2\phi X = 0$, for any $X \in \mathbb{D}$. Taking $X \in \mathbb{D}$ such that $AX = \lambda X$, we obtain $\lambda^2 = \mu^2$. If $\lambda + \mu = 0$ we get $2\lambda^2 + 2 = 0$, which is impossible. Therefore $\lambda = \mu$ and $\phi A = A\phi$.

If M is non Hopf we write again $A\xi = \alpha\xi + \beta U$, with the same conditions as in the previous proof. Then the scalar product of (2.3) and ϕU, taking $Y = \phi U$ yields $AU = \beta\xi + kU$. And the corresponding scalar product of (2.3) with U, gives $A\phi U = k\phi U$. Both expressions allow us to assume that $\mathbb{D}_U$ is A-invariant.

Then the scalar product of (2.3) and $Z \in \mathbb{D}_U$ implies that for any $Z \in \mathbb{D}_U$, $AZ = kZ$. If we apply Codazzi equation to $Z \in \mathbb{D}_U$ and ϕZ we obtain $k\beta = 0$, which is impossible and proves

Theorem 2.7. *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$, and k a nonnull real number. Then $A_T^{(k)}$ is η-hybrid with respect to ϕ if and only if M is locally congruent to a real hypersurface of type (A).*

As in the previous section from Theorem 2.7 it is easy to prove non-existence of real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$, such that $A_T^{(k)}$ is hybrid with respect to ϕ, for any nonnull real number k.

3. Purity and hybridness of $A_F^{(k)}$ and $A_T^{(k)}$ with respect to A

The following results appear in [12].

Theorem 3.1. *There does not exist any real hypersurface M in $\mathbb{C}P^m$, $m \geq 3$, such that $A_F^{(k)}$ is pure with respect to A, for any nonnull real number k.*

If we suppose that $A_F^{(k)}$ is η-hybrid with respect to A we have

$$(3.1)\quad g(\phi A^2X, AY)\xi - \eta(AY)\phi A^2X - k\eta(AX)\phi AY - g(\phi A^2X, Y)A\xi + k\eta(AX)A\phi Y \\ + g(\phi AX, A^2Y)\xi - \eta(A^2Y)\phi AX - g(\phi AX, AY)A\xi + \eta(AY)A\phi AX = 0,$$

for any $X, Y \in \mathbb{D}$. If M is Hopf with Reeb curvature α (3.1) yields $A\phi A^2X - \alpha\phi A^2X + A^2\phi AX - \alpha A\phi AX = 0$, for any $X \in \mathbb{D}$, and if we suppose that $X \in \mathbb{D}$ satisfies $AX = \lambda X$ we obtain $(\mu - \alpha)\lambda(\lambda + \mu) = 0$. As before $\lambda + \mu$ does not vanish. Then, either $\lambda = 0$ and in this case $\mu = -\frac{2}{\alpha}$ (recall that $2\lambda - \alpha \neq 0$) or $\mu = \alpha$ and $\lambda = \frac{\alpha^2+2}{\alpha}$. Then M has at most 3 distinct principal curvatures. Looking at Takagi's list, both cases are impossible and M must be non Hopf.

Write again $A\xi = \alpha\xi + \beta U$. The scalar product of (3.1) and ϕU, for several choices of X and Y in $\mathbb{D}_U$, implies $AU = \beta\xi + \gamma U$, for a certain function γ and $A\phi U = \delta\phi U$, for a function δ. The scalar product of (3.1) and U gives either $\delta = 0$ or $\delta = -\left(\frac{\alpha+\gamma}{2}\right)$. We also know that $\mathbb{D}_U$ is A-invariant. From (3.1), if we suppose that $X \in \mathbb{D}_U$ satisfies $AX = \lambda X$ we obtain that either $\lambda = 0$ or $A\phi X = -\lambda\phi X$. But the scalar product of (3.1) and $Z \in \mathbb{D}_U$ yields $AZ = 0$, for any $Z \in \mathbb{D}_U$. Then the Codazzi equation applied to Z and ϕZ, $Z \in \mathbb{D}_U$ implies $\gamma = 0$. Now, if we suppose that $\delta = -\frac{\alpha}{2} \neq 0$, the scalar product of (3.1) and ξ gives $\alpha^2 + 4\beta^2 = 0$, which is impossible. Thus we have $\delta = 0$ and prove the

Theorem 3.2. *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$, and k a nonnull real number. Then $A_F^{(k)}$ is η-hybrid with respect to A if and only if M is locally congruent to a ruled real hypersurface.*

Now, from Theorem 3.2 we can obtain the

Corollary 3.3. *There does not exist any real hypersurface M in $\mathbb{C}P^m$, $m \geq 3$, such that $A_F^{(k)}$ is hybrid with respect to A, for any nonnull real number k.*

In the case of $A_T^{(k)}$ we can prove, in a similar but much more complicated way the following

Theorem 3.4. *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$, and k a nonnull real number. Then $A_T^{(k)}$ is pure with respect to A if and only if M is locally congruent to a geodesic hypersphere of radius r, $0 < r < \frac{\pi}{2}$, such that $\cot(2r) = \frac{k^2-1}{2k}$.*

If now we suppose that $A_T^{(k)}$ is η-hybrid with respect to A we have

$$
\begin{aligned}
(3.2) \quad & g(\phi A^2X, AY)\xi - \eta(AY)\phi A^2X - k\eta(AX)\phi AY - g(\phi A^2Y, AX)\xi + \eta(AX)\phi A^2Y \\
& + k\eta(AY)\phi AX - g(\phi A^2X, Y)A\xi + k\eta(AX)A\phi Y + g(\phi AY, AX)A\xi - \eta(AX)A\phi AY \\
& + g(\phi AX, A^2Y)\xi - \eta(A^2Y)\phi AX - g(\phi A^3Y, X)\xi + k\eta(A^2Y)\phi X - g(\phi AX, AY)A\xi \\
& + \eta(AY)A\phi AX + g(\phi A^2Y, X)A\xi - k\eta(AY)A\phi X = 0,
\end{aligned}
$$

for any $X, Y \in \mathbb{D}$. Let us suppose that M is Hopf with Reeb curvature α. From (3.2) we get $A\phi A^2X + 2A^2\phi AX - \alpha\phi A^2X - 2\alpha A\phi AX + A^3\phi X - \alpha A^2\phi X = 0$, for any $X \in \mathbb{D}$. If we take $X \in \mathbb{D}$ such that $AX = \lambda X$, $(\lambda + \mu)^2(\mu - \alpha) = 0$. As above $\mu = \alpha$, $\lambda = \frac{\alpha^2+2}{\alpha}$, M has two distinct constant principal curvatures and should be locally congruent to a geodesic hypersphere, [2]. In such a case either $2\cot(2r) = \cot(r)$ or $2\cot(2r) = -\tan(r)$, for $0 < r < \frac{\pi}{2}$, which is impossible. Thus M must be non Hopf and we continue writing $A\xi = \alpha\xi + \beta U$. Taking $X = Y = \phi U$ in (3.2) and its scalar product with ϕU we get $g(AU, \phi U) = 0$ and a similar argument with $X = Y \in \mathbb{D}_U$ yields $g(AU, X) = 0$, for any $X \in \mathbb{D}_U$. Thus, for a certain function γ, $AU = \beta\xi + \gamma U$. An analogous argument for $X = Y = U$ gives $(k - \gamma)(\alpha + \gamma) = 0$.

Taking the scalar product of (3.2) and U and several choices for X and Y we obtain

$$
(3.3) \qquad
\begin{aligned}
(\alpha + \gamma)g(A\phi U, X) &= 0, \\
2g(A\phi U, \phi AX) + g(A\phi U, A\phi X) &= 0, \\
2kg(A\phi U, X) + g(A\phi U, AX) &= 0, \\
(k - \gamma)g(A\phi U, X) - 2g(A\phi U, AX) &= 0,
\end{aligned}
$$

for any $X \in \mathbb{D}_U$. If $\gamma = k$, this equations yield $g(A\phi U, X) = 0$ for any $X \in \mathbb{D}_U$, showing that in this case $A\phi U = \delta\phi U$, for a certain function δ.

But for $X = U$, $Y = \phi U$, the scalar product of (3.2) and U implies $2\delta^2 + 2\gamma^2 + \beta^2 = 0$, which is impossible. Thus $\alpha + \gamma = 0$ and a similar reasoning gives $2\gamma + \alpha - 5k = 0$. This yields $\gamma = 5k$, $\alpha = -5k$, and there exists $Z \in \mathbb{D}_U$ such that $g(A\phi U, Z) = 0$. Taking $X = \phi U$ in (3.2) and its scalar product with $Z \in \mathbb{D}_U$ we have $2kg(A\phi U, X) - 3g(A\phi U, AX) = 0$ for any $X \in \mathbb{D}_U$. This and (3.3) yields $g(A\phi U, X) = 0$ for any $X \in \mathbb{D}_U$, a contradiction that proves

Theorem 3.5. *There does not exists any real hypersurface M in $\mathbb{C}P^m$, $m \geq 3$, such that $A_T^{(k)}$ is η-hybrid with respect to A, for any nonnull real number k.*

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