ON RICCI AND HYPERBOLIC RICCI SOLITON SUBMANIFOLDS OF ALMOST CONTACT METRIC MANIFOLDS

ADARA M. BLAGA AND CIHAN ÖZGÜR

Abstract. We determine certain properties of isometrically immersed Ricci and hyperbolic Ricci solitons into Kenmotsu and Sasakian manifolds having as potential vector field the tangential component $\xi^\top$ of the Reeb vector field ξ. We prove that a $\xi^\perp$-umbilical submanifold of a Kenmotsu manifold is a Ricci or a hyperbolic Ricci soliton if and only if it is a quasi-Einstein manifold, and a $\xi^\perp$-umbilical submanifold of a Sasakian manifold is a Ricci or a hyperbolic Ricci soliton if and only if it is an Einstein manifold. We also characterize the Ricci soliton submanifolds of a Kenmotsu and of a Sasakian manifold with Codazzi Ricci operator in terms of the second fundamental form. Finally, we deduce that all the results obtained for the Sasakian case hold true for the cosymplectic case, too.

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1. Preliminaries

Ricci solitons as submanifolds of a Riemannian manifold have been extensively treated by B.-Y. Chen in [9], completing the results from [11] for the case when the potential vector field is the tangential component of a concircular vector field on the ambient manifold. If the vector field is torse-forming, η-Ricci and η-Yamabe soliton submanifolds have been considered in [3], η-Ricci–Bourguignon soliton submanifolds in [5], Riemann soliton submanifolds in [4], hyperbolic Ricci soliton submanifolds in [6] and hyperbolic Yamabe soliton submanifolds with potential vector fields arising from a concurrent vector field, in [7]. In almost contact metric geometry, some results on Ricci solitons on invariant and anti-invariant submanifolds of a Kenmotsu manifold with respect to a quarter symmetric metric connection and quarter symmetric non-metric ϕ-connection can be found in [14].

In the present paper, we study some properties of Ricci and hyperbolic Ricci soliton submanifolds of a Kenmotsu and of a Sasakian manifold, whose potential vector field is the tangential component $\xi^\top$ of the Reeb vector field ξ. We show that a $\xi^\perp$-umbilical submanifold of a Kenmotsu manifold is a Ricci or a hyperbolic Ricci soliton if and only if it is a quasi-Einstein manifold, and a $\xi^\perp$-umbilical submanifold of a Sasakian manifold is a Ricci or a hyperbolic Ricci soliton if and only if it is an Einstein manifold. We also describe the Ricci soliton submanifolds of a Kenmotsu or of a Sasakian manifold with Codazzi Ricci operator in terms of the second fundamental form. It is known [15] that the necessary and sufficient condition for a manifold to have Codazzi Ricci operator is that the Riemann curvature tensor to be divergence-free, i.e., that the manifold to be R-harmonic [16], hence, we obtain, in this case, a characterization of R-harmonic Ricci soliton submanifolds. Finally, we deduce that all the results obtained for the Sasakian case hold true for the cosymplectic case, too.

1.1. **Quasi-Einstein manifolds and Ricci solitons.** We recall that a non-flat Riemannian manifold (M, g) is said to be a *quasi-Einstein manifold* [8] if the Ricci tensor field Ric is not identically zero and it satisfies

$$\mathrm{Ric} = ag + b\eta \otimes \eta,$$

for a and b nonzero smooth functions and η a nonzero 1-form on M. The functions a and b are called the *associated functions*. In particular, if a is a constant and $b = 0$, then the manifold is an *Einstein manifold* [1].

If (M, g) is a Riemannian manifold and V is a vector field, then (g, V) define a *Ricci soliton* [13] if there exists a real number λ such that

$$\frac{1}{2}\pounds_V g + \mathrm{Ric} = \lambda g,$$

where $\pounds_V g$ stands for the Lie derivative of g in the direction of V. On the other hand, (g, V) define a *hyperbolic Ricci soliton* [12] if there exist two real numbers λ and μ such that

$$\pounds_V \pounds_V g + \lambda \pounds_V g + \mathrm{Ric} = \mu g.$$

1.2. **Basic properties of submanifolds.** Let $(\bar{M}, \bar{g})$ be a Riemannian manifold, and let M be an isometrically immersed submanifold. We denote by g the induced Riemannian metric on M, and by $\bar{\nabla}$ and ∇ the Levi-Civita connections of $\bar{g}$ and g. Then we have the orthogonal decomposition

$$\bar{T}M = TM \oplus T^{\perp}M,$$

and any tangent vector to $\bar{M}$ decomposes into a tangential component $X^{\top}$ and a normal component $X^{\perp}$.

The Gauss and Weingarten equations are:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \bar{\nabla}_X U = -B_U(X) + \nabla^{\perp}_X U,$$

where X and Y are vector fields tangent to M, U is a vector field normal to M, h is the second fundamental form, B is the shape operator and $\nabla^{\perp}$ is the normal connection. Also, we have $\bar{g}(h(X, Y), U) = g(B_U(X), Y)$ for any vector fields X, Y tangent to M and any vector field U normal to M.

A submanifold M is said to be *U-umbilical* [9] (with respect to a normal vector field U) if $B_U = fI$, where f is a smooth function on M and I is the identity map, *totally umbilical* [10] if it is umbilical with respect to every unit normal vector field, *totally geodesic* if $B = 0$, and, *minimal* if trace$(B) = 0$ (see [10]).

1.3. **Kenmotsu, Sasakian, and cosymplectic manifolds revisited.** In 1976, Blair introduced the notion of almost contact metric structure.

Definition 1.1. [2] An odd dimensional Riemannian manifold $(\bar{M}, \bar{g})$ with a $(1, 1)$-tensor field ϕ and a vector field ξ (called the Reeb vector field) is said to be an *almost contact metric manifold* if it satisfies:
(i) $\phi^2 = -I + \eta \otimes \xi$;
(ii) $\eta(\xi) = 1$;
(iii) $\bar{g}(\phi X, \phi Y) = \bar{g}(X, Y) - \eta(X)\eta(Y)$ for any vector fields X, Y tangent to $\bar{M}$,
where η is the dual 1-form of ξ, and I is the identity map.

It immediately follows, from the definition, that

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad \bar{g}(\phi X, Y) = -\bar{g}(X, \phi Y)$$

for any vector fields X, Y tangent to $\bar{M}$.

If, for any vector fields X, Y tangent to $\bar{M}$, the Levi-Civita connection $\bar{\nabla}$ of $\bar{g}$ satisfies

$$(\bar{\nabla}_X \phi)Y = \bar{g}(\phi X, Y)\xi - \eta(Y)\phi X,$$

then $\bar{M}$ is a *Kenmotsu manifold*, if it satisfies

$$(\bar{\nabla}_X \phi)Y = \bar{g}(X, Y)\xi - \eta(Y)X,$$

then $\bar{M}$ is a *Sasakian manifold*, and if it satisfies

$$\bar{\nabla}\phi = 0,$$

then $\bar{M}$ is a *cosymplectic manifold*.

For (M, g) an isometrically immersed submanifold of $(\bar{M}, \bar{g})$, we shall further denote by TX the tangential component of ϕX and by NX the normal component of ϕX for any vector field X tangent to M. It follows that $g(TX, Y) = -g(X, TY)$ for any vector fields X, Y tangent to M.

2. Ricci and hyperbolic Ricci soliton submanifolds of a Kenmotsu manifold

Now we assume that $(\bar{M}, \bar{g}, \phi, \xi, \eta)$ is a Kenmotsu manifold. Then the Levi-Civita connection $\bar{\nabla}$ of $\bar{g}$ satisfies

$$\bar{\nabla}\xi = I - \eta \otimes \xi. \tag{2.1}$$

From the Gauss and Weingarten formulas, we have

$$\bar{\nabla}_X \xi^\top = \nabla_X \xi^\top + h(X, \xi^\top), \tag{2.2}$$

$$\bar{\nabla}_X \xi^\perp = -B_{\xi^\perp}(X) + \nabla^\perp_X \xi^\perp, \tag{2.3}$$

hence, by means of (2.1), (2.2), and (2.3), for any vector field X tangent to M, by identifying the tangential components, we get

$$\nabla_X \xi^\top = B_{\xi^\perp}(X) + X - g(X, \xi^\top)\xi^\top$$

and

$$\begin{aligned}(\pounds_{\xi^\top} g)(X, Y) &= g(\nabla_X \xi^\top, Y) + g(\nabla_Y \xi^\top, X) \\ &= 2g(B_{\xi^\perp}(X), Y) + 2g(X, Y) - 2g(X, \xi^\top)g(Y, \xi^\top).\end{aligned} \tag{2.4}$$

Then we deduce

Theorem 2.1. *The submanifold $(M, g, \xi^\top)$ is a Ricci soliton if and only if there exists a real number λ such that*

$$\mathrm{Ric}_M(X, Y) = (\lambda - 1)g(X, Y) + \eta(X)\eta(Y) - g(B_{\xi^\perp}(X), Y) \tag{2.5}$$

for any vector fields X, Y tangent to M.

Proof. The necessary and sufficient condition for $(M, g, \xi^\top)$ to be a Ricci soliton is to exist $\lambda \in \mathbb{R}$ such that

$$\frac{1}{2}\pounds_{\xi^\top} g + \mathrm{Ric}_M = \lambda g,$$

which, by means of (2.4), is equivalent to (2.5). □

From Theorem 2.1, we obtain

Corollary 2.2. *A $\xi^\perp$-umbilical Ricci soliton submanifold $(M, g, \xi^\top, \lambda)$ with $B_{\xi^\perp} = fI$ for f a smooth function on M is a quasi-Einstein manifold with associated functions $(\lambda - f - 1)$ and 1.*

Also, from Theorem 2.1, we get the expression for the Ricci operator Q_M of M:

$$Q_M(X) = (\lambda - 1)X + \eta(X)\xi^\top - B_{\xi^\perp}(X) \tag{2.6}$$

for any vector field X tangent to M, and we get

Proposition 2.3. *If the submanifold $(M, g, \xi^\top)$ is a Ricci soliton, then, for any vector fields X, Y tangent to M, we have*

$$\begin{aligned}(\nabla_X Q_M)Y = &-(\nabla_X B_{\xi^\perp})Y + \eta(Y)[B_{\xi^\perp}(X) + X] \\ &+ [g(X, Y) - 2\eta(X)\eta(Y) + g(B_{\xi^\perp}(X), Y)]\xi^\top.\end{aligned} \tag{2.7}$$

Proof. Indeed,

$$\begin{aligned}(\nabla_X Q_M)Y &= \nabla_X(Q_M(Y)) - Q_M(\nabla_X Y)\\ &= (\lambda-1)\nabla_X Y + X(g(Y,\xi^\top))\xi^\top + g(Y,\xi^\top)\nabla_X\xi^\top - \nabla_X(B_{\xi^\perp}(Y))\\ &\quad - (\lambda-1)\nabla_X Y - g(\nabla_X Y,\xi^\top)\xi^\top + B_{\xi^\perp}(\nabla_X Y)\\ &= g(Y,\nabla_X\xi^\top)\xi^\top + g(Y,\xi^\top)\nabla_X\xi^\top - (\nabla_X B_{\xi^\perp})Y\\ &= g\left(Y, B_{\xi^\perp}(X) + X - g(X,\xi^\top)\xi^\top\right)\xi^\top\\ &\quad + g(Y,\xi^\top)\left(B_{\xi^\perp}(X) + X - g(X,\xi^\top)\xi^\top\right) - (\nabla_X B_{\xi^\perp})Y\\ &= g(Y,B_{\xi^\perp}(X))\xi^\top + g(X,Y)\xi^\top - g(X,\xi^\top)g(Y,\xi^\top)\xi^\top\\ &\quad + g(Y,\xi^\top)B_{\xi^\perp}(X) + g(Y,\xi^\top)X - g(X,\xi^\top)g(Y,\xi^\top)\xi^\top - (\nabla_X B_{\xi^\perp})Y\end{aligned}$$

for any vector fields X, Y tangent to M, and we get (2.7). □

As consequences, we deduce

Corollary 2.4. *A Ricci soliton submanifold $(M, g, \xi^\top)$ has Codazzi Ricci operator if and only if*

$$(\nabla_X B_{\xi^\perp})Y - (\nabla_Y B_{\xi^\perp})X = \eta(Y)(B_{\xi^\perp} + I)(X) - \eta(X)(B_{\xi^\perp} + I)(Y) \tag{2.8}$$

for any vector fields X, Y tangent to M.

Proof. From (2.7), we have

$$\begin{aligned}(\nabla_X Q_M)Y - (\nabla_Y Q_M)X = &-(\nabla_X B_{\xi^\perp})Y + \eta(Y)[B_{\xi^\perp}(X) + X]\\ &+ (\nabla_Y B_{\xi^\perp})X - \eta(X)[B_{\xi^\perp}(Y) + Y]\end{aligned}$$

for any vector fields X, Y tangent to M, and we get (2.8). □

Corollary 2.5. *A $\xi^\perp$-umbilical Ricci soliton submanifold $(M, g, \xi^\top)$ with $B_{\xi^\perp} = fI$ has Codazzi Ricci operator if and only if*

$$\nabla f = -(f+1)\xi^\top.$$

Proof. In this case, for any vector fields X, Y tangent to M, we have

$$(\nabla_X B_{\xi^\perp})Y = \nabla_X(fY) - f(\nabla_X Y) = X(f)Y,$$

and the equation (2.8) becomes

$$X(f)Y - Y(f)X = (f+1)[\eta(Y)X - \eta(X)Y],$$

which is equivalent to

$$[df(X) + (f+1)\eta(X)]Y = [df(Y) + (f+1)\eta(Y)]X$$

and to

$$g(\nabla f + (f+1)\xi^\top, X)Y = g(\nabla f + (f+1)\xi^\top, Y)X,$$

and we get the assertion. □

Proposition 2.6. *If the submanifold $(M, g, \xi^\top)$ is a Ricci soliton, then we have*

$$T\circ Q_M - Q_M\circ T = -(T\circ B_{\xi^\perp} - B_{\xi^\perp}\circ T) + \eta\otimes T\xi^\top - (\eta\circ T)\otimes\xi^\top. \tag{2.9}$$

Proof. For any vector field X tangent to M, from (2.6), we have

$$T(Q_M(X)) = (\lambda-1)TX + \eta(X)T\xi^\top - T(B_{\xi^\perp}(X)),$$

and

$$Q_M(TX) = (\lambda-1)TX + \eta(TX)\xi^\top - B_{\xi^\perp}(TX),$$

and we get (2.9). □

As consequences, we deduce

Corollary 2.7. *If the submanifold $(M, g, \xi^\top)$ is a Ricci soliton, then, the Ricci operator Q_M commutes with the operator T if and only if*

$$T \circ B_{\xi^\perp} - B_{\xi^\perp} \circ T = \eta \otimes T\xi^\top - (\eta \circ T) \otimes \xi^\top.$$

Corollary 2.8. *A $\xi^\perp$-umbilical Ricci soliton submanifold $(M, g, \xi^\top, \lambda)$ with $B_{\xi^\perp} = fI$ such that ∇f is not pointwise collinear with $\xi^\top$ is Ricci symmetric (i.e., $\nabla Q_M = 0$) if and only if $B_{\xi^\perp} = -I$.*

Proof. In this case, for any vector fields X, Y tangent to M, from (2.7), we have

$$X(f)Y + (f+1)\eta(Y)X = -[(f+1)g(X,Y) - 2\eta(X)\eta(Y)]\xi^\top.$$

Since in the righthand side of the above equality we have a symmetric $(0, 2)$-tensor field, then the lefthand side must be symmetric, too, hence

$$X(f)Y + (f+1)\eta(Y)X = Y(f)X + (f+1)\eta(X)Y$$

for any vector fields X, Y tangent to M, which is equivalent to

$$[df(X) - (f+1)\eta(X)]Y = [df(Y) - (f+1)\eta(Y)]X,$$

and we get $\nabla f = -(f+1)\xi^\top$, hence $f = -1$. □

From Corollary 2.7, we deduce

Corollary 2.9. *If the submanifold $(M, g, \xi^\top)$ is a Ricci soliton such that the Ricci operator Q_M commutes with T, then, the shape operator in the direction of $\xi^\perp$ commutes with T if and only if $\xi^\top = 0$. In this case, the soliton is trivial and M is an Einstein manifold.*

Remark 2.10. The same conclusion, namely $\xi^\top = 0$, is reached if the submanifold $(M, g, \xi^\top)$ is a $\xi^\perp$-umbilical Ricci soliton such that the Ricci operator Q_M commutes with T.

Now, if M is a $\xi^\perp$-umbilical submanifold with $B_{\xi^\perp} = fI$, then, for any vector fields X, Y tangent to M, we have

$$\begin{aligned}
\nabla_X \xi^\top &= (f+1)X - \eta(X)\xi^\top, \\
(\pounds_{\xi^\top} g)(X,Y) &= 2(f+1)g(X,Y) - 2\eta(X)\eta(Y), \\
(\pounds_{\xi^\top}\pounds_{\xi^\top} g)(X,Y) &= 2[\xi^\top(f) + 2(f+1)^2]g(X,Y) + 4[2\|\xi^\top\|^2 - 3(f+1)]\eta(X)\eta(Y),
\end{aligned}$$

and we obtain

Theorem 2.11. *A $\xi^\perp$-umbilical submanifold $(M, g, \xi^\top)$ is a hyperbolic Ricci soliton if and only if there exist two real numbers λ and μ such that*

$$\begin{aligned}
\mathrm{Ric}_M(X,Y) = {} & [\mu - 2\xi^\top(f) - 4(f+1)^2 - 2(f+1)\lambda]g(X,Y) \\
& + [2\lambda + 12(f+1) - 8\|\xi^\top\|^2]\eta(X)\eta(Y)
\end{aligned} \tag{2.10}$$

for any vector fields X, Y tangent to M.

Proof. The necessary and sufficient condition for $(M, g, \xi^\top)$ to be a hyperbolic Ricci soliton is to exist $\lambda, \mu \in \mathbb{R}$ such that

$$\pounds_{\xi^\top}\pounds_{\xi^\top} g + \lambda \pounds_{\xi^\top} g + \mathrm{Ric}_M = \mu g,$$

which is equivalent to (2.10). □

And we deduce

Corollary 2.12. *A $\xi^\perp$-umbilical submanifold $(M, g, \xi^\top)$ is a Ricci soliton if and only if it is a quasi-Einstein manifold with associated functions*

$$\mu - 2\xi^\top(f) - 4(f+1)^2 - 2(f+1)\lambda \quad \text{and} \quad 2\lambda + 12(f+1) - 8\|\xi^\top\|^2.$$

Remark 2.13. We notice that, if the submanifold is $\xi^{\perp}$-umbilical, then $\xi^{\top}$ can not be a Killing vector field, i.e., $\pounds_{\xi^{\top}} g$ can not identically vanish.

3. Ricci and hyperbolic Ricci soliton submanifolds of a Sasakian manifold

Now we assume that $(\bar{M}, \bar{g}, \phi, \xi, \eta)$ is a Sasakian manifold. Then the Levi-Civita connection $\bar{\nabla}$ of $\bar{g}$ satisfies

$$\bar{\nabla}\xi = -\phi. \tag{3.1}$$

From the Gauss and Weingarten formulas, we have

$$\bar{\nabla}_X \xi^{\top} = \nabla_X \xi^{\top} + h(X, \xi^{\top}), \tag{3.2}$$

$$\bar{\nabla}_X \xi^{\perp} = -B_{\xi^{\perp}}(X) + \nabla^{\perp}_X \xi^{\perp}, \tag{3.3}$$

hence, by means of (3.1), (3.2), and (3.3), for any vector field X tangent to M, by identifying the tangential components, we get

$$\nabla_X \xi^{\top} = B_{\xi^{\perp}}(X) - TX$$

and

$$\begin{aligned}(\pounds_{\xi^{\top}} g)(X, Y) &= g(\nabla_X \xi^{\top}, Y) + g(\nabla_Y \xi^{\top}, X) \\ &= 2g(B_{\xi^{\perp}}(X), Y).\end{aligned} \tag{3.4}$$

Then we deduce

Theorem 3.1. *The submanifold $(M, g, \xi^{\top})$ is a Ricci soliton if and only if there exists a real number λ such that*

$$\mathrm{Ric}_M(X, Y) = \lambda g(X, Y) - g(B_{\xi^{\perp}}(X), Y) \tag{3.5}$$

for any vector fields X, Y tangent to M.

Proof. The necessary and sufficient condition for $(M, g, \xi^{\top})$ to be a Ricci soliton is to exist $\lambda \in \mathbb{R}$ such that

$$\frac{1}{2}\pounds_{\xi^{\top}} g + \mathrm{Ric}_M = \lambda g,$$

which, by means of (3.4), is equivalent to (3.5). □

From Theorem 3.1, we obtain

Corollary 3.2. *A $\xi^{\perp}$-umbilical Ricci soliton submanifold $(M, g, \xi^{\top}, \lambda)$ with $B_{\xi^{\perp}} = fI$ for f a smooth function on M is an Einstein manifold provided that $m = \dim(M) > 2$, of scalar curvature*

$$\mathrm{scal}_M = m(\lambda - f).$$

Also, from Theorem 3.1, we get the expression for the Ricci operator Q_M of M:

$$Q_M(X) = \lambda X - B_{\xi^{\perp}}(X) \tag{3.6}$$

for any vector field X tangent to M, and we get

Proposition 3.3. *If the submanifold $(M, g, \xi^{\top})$ is a Ricci soliton, then, for any vector fields X, Y tangent to M, we have*

$$(\nabla_X Q_M)Y = -(\nabla_X B_{\xi^{\perp}})Y. \tag{3.7}$$

Proof. Indeed,

$$\begin{aligned}(\nabla_X Q_M)Y &= \nabla_X(Q_M(Y)) - Q_M(\nabla_X Y)\\ &= \lambda\nabla_X Y - \nabla_X(B_{\xi^\perp}(Y)) - \lambda\nabla_X Y + B_{\xi^\perp}(\nabla_X Y)\\ &= -(\nabla_X B_{\xi^\perp})Y\end{aligned}$$

for any vector fields X, Y tangent to M, and we get (3.7). □

As consequences, we deduce

Corollary 3.4. *A Ricci soliton submanifold $(M, g, \xi^\top)$ has Codazzi (in particular, parallel) Ricci operator if and only if it has Codazzi (in particular, parallel) shape operator in the direction of $\xi^\perp$.*

Proof. From (3.7), we have

$$(\nabla_X Q_M)Y - (\nabla_Y Q_M)X = (\nabla_Y B_{\xi^\perp})X - (\nabla_X B_{\xi^\perp})Y$$

for any vector fields X, Y tangent to M, and we get the conclusion. □

Corollary 3.5. *A $\xi^\perp$-umbilical Ricci soliton submanifold $(M, g, \xi^\top)$ with $B_{\xi^\perp} = fI$ has Codazzi Ricci operator if and only if f is a constant.*

Proof. In this case, for any vector fields X, Y tangent to M, we have

$$(\nabla_X B_{\xi^\perp})Y = \nabla_X(fY) - f(\nabla_X Y) = X(f)Y,$$

and we get

$$(\nabla_X Q_M)Y - (\nabla_Y Q_M)X = Y(f)X - X(f)Y,$$

hence the conclusion. □

Proposition 3.6. *If the submanifold $(M, g, \xi^\top)$ is a Ricci soliton, then we have*

$$T \circ Q_M - Q_M \circ T = -(T \circ B_{\xi^\perp} - B_{\xi^\perp} \circ T). \tag{3.8}$$

Proof. For any vector field X tangent to M, from (3.6), we have

$$T(Q_M(X)) = \lambda TX - T(B_{\xi^\perp}(X)),$$

and

$$Q_M(TX) = \lambda TX - B_{\xi^\perp}(TX),$$

and we get (3.8). □

As consequences, we deduce

Proposition 3.7. *If the submanifold $(M, g, \xi^\top)$ is a Ricci soliton, then, the Ricci operator Q_M commutes with the operator T if and only if the shape operator in the direction of $\xi^\perp$ commutes with T.*

Proof. It follows from (3.8). □

Corollary 3.8. *For any $\xi^\perp$-umbilical Ricci soliton submanifold $(M, g, \xi^\top)$, the Ricci operator Q_M commutes with T.*

Now, if M is a $\xi^\perp$-umbilical submanifold with $B_{\xi^\perp} = fI$, then, for any vector fields X, Y tangent to M, we have

$$\begin{aligned}\nabla_X \xi^\top &= fX - TX,\\ (\pounds_{\xi^\top} g)(X, Y) &= 2fg(X, Y),\\ (\pounds_{\xi^\top}\pounds_{\xi^\top} g)(X, Y) &= 2[\xi^\top(f) + 2f^2]g(X, Y),\end{aligned}$$

and we obtain

Theorem 3.9. *A $\xi^\perp$-umbilical submanifold $(M, g, \xi^\top)$ is a hyperbolic Ricci soliton if and only if there exist two real numbers λ and μ such that*

$$\mathrm{Ric}_M(X,Y) = [\mu - 2\xi^\top(f) - 4f^2 - 2f\lambda]g(X,Y) \tag{3.9}$$

for any vector fields X, Y tangent to M.

Proof. The necessary and sufficient condition for $(M, g, \xi^\top)$ to be a hyperbolic Ricci soliton is to exist $\lambda, \mu \in \mathbb{R}$ such that

$$\pounds_{\xi^\top}\pounds_{\xi^\top} g + \lambda \pounds_{\xi^\top} g + \mathrm{Ric}_M = \mu g,$$

which is equivalent to (3.9). □

And we deduce

Corollary 3.10. *A $\xi^\perp$-umbilical submanifold $(M, g, \xi^\top)$ is a hyperbolic Ricci soliton if and only if it is an Einstein manifold provided that $m = \dim(M) > 2$, of scalar curvature*

$$\mathrm{scal}_M = m[\mu - 2\xi^\top(f) - 4f^2 - 2f\lambda].$$

Remark 3.11. We notice that, if the submanifold is $\xi^\perp$-umbilical, then $\xi^\top$ is a Killing vector field if and only if $f = 0$. In particular, if a hypersurface is $\xi^\perp$-umbilical, then $\xi^\top$ is a Killing vector field if and only if it is a totally geodesic hypersurface.

4. Ricci and hyperbolic Ricci soliton submanifolds of a cosymplectic manifold

Now we assume that $(\bar{M}, \bar{g}, \phi, \xi, \eta)$ is a cosymplectic manifold. Then the Levi-Civita connection $\bar{\nabla}$ of $\bar{g}$ satisfies

$$\bar{\nabla}\xi = 0. \tag{4.1}$$

From the Gauss and Weingarten formulas, we have

$$\bar{\nabla}_X \xi^\top = \nabla_X \xi^\top + h(X, \xi^\top), \tag{4.2}$$

$$\bar{\nabla}_X \xi^\perp = -B_{\xi^\perp}(X) + \nabla^\perp_X \xi^\perp, \tag{4.3}$$

hence, by means of (4.1), (4.2), and (4.3), for any vector field X tangent to M, by identifying the tangential components, we get

$$\nabla_X \xi^\top = B_{\xi^\perp}(X)$$

and

$$\begin{aligned}(\pounds_{\xi^\top} g)(X,Y) &= g(\nabla_X \xi^\top, Y) + g(\nabla_Y \xi^\top, X)\\ &= 2g(B_{\xi^\perp}(X), Y).\end{aligned}$$

If M is a $\xi^\perp$-umbilical submanifold with $B_{\xi^\perp} = fI$, then, for any vector fields X, Y tangent to M, we have

$$\begin{aligned}\nabla_X \xi^\top &= fX,\\ (\pounds_{\xi^\top} g)(X,Y) &= 2fg(X,Y),\\ (\pounds_{\xi^\top}\pounds_{\xi^\top} g)(X,Y) &= 2[\xi^\top(f) + 2f^2]g(X,Y),\end{aligned}$$

and we deduce that all the results obtained for the Sasakian case hold true for the cosymplectic case, too.

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Department of Mathematics, West University of Timişoara, Timişoara, Romania
Email address: `adarablaga@yahoo.com`

Department of Mathematics, Izmir Democracy University, Izmir, Türkiye
Email address: `cihan.ozgur@idu.edu.tr`