# THE FIRST EIGENVALUE OF THE LAPLACIAN FOR A COMPACT SPACELIKE SUBMANIFOLD IN LORENTZ-MINKOWSKI SPACETIME OF ARBITRARY DIMENSION

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Abstract. According to the well-known Reilly result [10], the first eigenvalue of an $n$-dimensional compact submanifold in an $(n+p)$-dimensional Euclidean space is bounded above by $n$ times the average value of the square of the norm of the mean curvature vector field. Furthermore, if the eigenvalue achieves this bound, then the submanifold lies minimally in a hypersphere. However, through a counterexample, we will show, following [9], that Reilly's result does not hold for a compact spacelike submanifold of Lorentz-Minkowski spacetime. In the search for an alternative result, we revisit Reilly's original proof. Subsequently, we explain the new technique introduced in [9], which is based on an integral formula on a compact spacelike section of the light cone in Lorentz-Minkowski spacetime. We derive a family of upper bounds for the first eigenvalue of the Laplacian of a compact spacelike submanifold of Lorentz-Minkowski spacetime. The equality for one of these inequalities is characterized. On the way, we reprove Reilly's original result if a compact submanifold of Euclidean space is naturally seen as a compact spacelike submanifold of Lorentz-Minkowski spacetime through a spacelike hyperplane.

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## 1. Introduction

Let $(M^n, g)$ be a (connected) $n$-dimensional Riemannian manifold and $\Delta$ its Laplacian, i.e.,

$$\Delta f := \mathrm{div}(\nabla f) = \mathrm{trace}_g \mathrm{Hess}(f) = \frac{1}{\sqrt{\det(g_{kl})}} \sum_{i\,j} \frac{\partial}{\partial x_i}\Big(\sqrt{\det(g_{kl})}\, g^{ij} \frac{\partial f}{\partial x_j}\Big),$$

where $f \in C^\infty(M^n)$, div, $\nabla$ and Hess are divergence, Hessian, and gradient operators, respectively, $\mathrm{trace}_g\mathrm{Hess}(f)$ denotes the trace of the $(1,1)$ tensor field $g$-equivalent to $\mathrm{Hess}(f)$ and $(x_1, ..., x_n)$ a local coordinate system in $M^n$.

The Laplacian is a Riemannian invariant in the following sense

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**Proposition 1.1.** *If $F : (M^n, g) \to (M'^n, g')$ is an isometry between Riemannian manifolds, then*
$$(\Delta' f') \circ F = \Delta(f' \circ F),$$
*for all $f' \in C^\infty(M'^n)$, where $\Delta$, $\Delta'$ are the respective Laplacians.*

The Laplacian is an elliptic differential operator and, if $M^n$ is compact, then it is self-adjoint and positive definite for the $L^2$-inner product
$$\langle f, h\rangle = \int_{M^n} f\, h\, d\mu_g,$$
$f, h \in C^\infty(M^n)$, where $d\mu_g$ is the canonical measure on $M^n$ defined from $g$, [3].

**Definition 1.2.** *A real number $\lambda$ is an eigenvalue of $\Delta$ if there exists $f \in C^\infty(M^n)$, $f \neq 0$, such that*
$$\Delta f + \lambda\, f = 0.$$
*In this case, $f$ is called an eigenfunction of $\Delta$.*[1]

For any compact Riemannian manifold $(M^n, g)$ we have, [1], [3]

(1) 0 is an eigenvalue,
(2) Any eigenvalue $\lambda$ satisfies $\lambda \geq 0$,
(3) The eigenvalues of $\Delta$ are collected in an increasing discrete sequence $\lambda_0 := 0 < \lambda_1 < \lambda_2 ... < \lambda_k ... \nearrow \infty$
(4) From Proposition 1.1, if a two Riemannian manifolds $(M^n, g)$ and $(M'^n, g')$ are isometric, then both have the same eigenvalues (including multiplicities).
(5) The converse is not true in general: there exist two flat Riemannian tori of dimension 16 which are not (globally) isometric, and whose Laplacians have the same sequence of eigenvalues [6]. However, two flat Riemannian tori of dimension 2 whose Laplacian has the same sequence of eigenvalues are isometric [1, Prop. B.II.5].

**Example 1.3.** For the case of the $n$-dimensional round sphere with radius $r$, $\mathbb{S}^n(r)$, we have:
$$\lambda_k = \frac{k}{r^2}(n + k - 1), \quad k = 0, 1, 2, ...$$
and the corresponding eigenfunctions are the restrictions to $\mathbb{S}^n(r)$ of the homogeneous harmonic polynomials of $n+1$ variables and of degree $k$. This is carried out relating, by means of the Gauss formula of $\mathbb{S}^n(r)$ in $\mathbb{R}^{n+1}$, the functions $\Delta(f_{|\mathbb{S}^n})$ and $(\Delta^0 f)_{|\mathbb{S}^n}$, for $f \in C^\infty(\mathbb{R}^{n+1})$ where $\Delta$, $\Delta^0$ are the Laplacians of the metric of $\mathbb{S}^n$ and of the usual Riemannian metric of $\mathbb{R}^{n+1}$, respectively [1, Prop. C.I.1]. In particular, for the unit round sphere, $\mathbb{S}^n$, have
$$\lambda_1 = n,$$
with $\Delta f + n\, f = 0$, where $f = \varphi_{|\mathbb{S}^n}$ and $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}$ is linear and $\varphi \neq 0$.

In general, computing the sequence of eigenvalues for a concrete compact Riemannian manifold is not easy. Often, we only have an intrinsic or extrinsic bound of one specific eigenvalue, with interesting geometric information when equality holds. That is the case for the following relevant result

**Theorem 1.4.** [10] (Reilly).*For an $n$-dimensional compact submanifold $M^n$ (immersed) in the $(n+p)$-dimensional Euclidean space $\mathbb{R}^{n+p}$, the first non-trivial eigenvalue $\lambda_1$ of the Laplacian of the induced metric $g$ on $M^n$ satisfies*
$$\text{(Re)} \qquad \lambda_1 \leq n \frac{\displaystyle\int_{M^n} \|\mathbf{H}\|^2\, d\mu_g}{\mathrm{vol}(M^n)},$$

[1] In the usual notation of Linear Algebra, $\lambda$ and $f$ are an eigenvalue and an eigenfunction of the operator $-\Delta$, respectively.

*where* $\|\mathbf{H}\|^2$ *is the squared length of the mean curvature vector field* $\mathbf{H}$*, and* $\mathrm{vol}(M^n)$ *is the volume of* $M^n$*. Moreover, the equality holds in* (Re) *if and only if* $M^n$ *lies minimally in some hypersphere of radius* $\sqrt{n/\lambda_1}$ *in* $\mathbb{R}^{n+p}$*.*

Now consider $(n+p)$-dimensional Lorentz-Minkowski spacetime $\mathbb{L}^{n+p}$ , i.e., $\mathbb{R}^{n+p}$ endowed with the Lorentzian metric $-dx_1^2 + \sum_{i=2}^{n+p} dx_i^2$, where $(x_1, ..., x_{n+p})$ are the usual coordinates of $\mathbb{R}^{n+p}$. An immersion $\psi : M^n \to \mathbb{L}^{n+p}$ is said to be spacelike if the induced metric on $M^n$ via $\psi$ is Riemannian. In this case $M^n$ is called a spacelike submanifold of $\mathbb{L}^{n+p}$. In view of Theorem 1.4, the following question arises naturally:

> Does Reilly inequality (Re) work for any compact spacelike submanifold $M^n$ of $\mathbb{L}^{n+p}$, $p \geq 2$?

We point out that there exists no compact spacelike hypersurface in $\mathbb{L}^m$ (see [5], for instance), this is the reason to write $p \geq 2$.

Along this paper, we will focus on this question following mainly [9]. Thus, we will show a counter-example in Section 2 that says that the answer to this question is in general negative. Moreover, after revisiting the Reilly argument for the proof of Theorem 1.4 in Section 3, we will detail in Section 4 several unavoidable technical difficulties that make impossible the translation of Reilly's technique to our setting. However, there are several concrete assumptions on the compact spacelike submanifold where (Re) remains true. For instance: (*i*) Any compact spacelike submanifold $M^n$ contained in a spacelike hyperplane of $\mathbb{L}^{n+p}$ satisfies obviously (Re). (*ii*) Any compact spacelike surface in $\mathbb{L}^4$ contained in a light cone satisfies (Re) [8, Thm. 5.4]. (*iii*) Any $n$-dimensional compact submanifold in the hyperbolic space $\mathbb{H}^{m-1} \subset \mathbb{L}^m$ also satisfies (Re) [11, Thm. 1]. Moreover, (*iv*) any compact 4-dimensional spacelike submanifold $M^4$ of $\mathbb{L}^6$ contained in a light cone we have $\lambda_1^2 \leq 16 \int_{M^4} \|\mathbf{H}\|^4 d\mu_g / \mathrm{vol}(M^4)$ [7, Cor. 5.6]. In all cases equality is geometrically characterized.

Section 5 is devoted to explaining a key technical result, Lemma 5.1, that has the same role in our approach as [10, Main Lemma] in Reilly's approach, to get Theorem 1.4. Finally, in Section 6 we will explain Theorem 6.1, obtained originally in [9, Thm. 6.10], as an alternative inequality to (Re), pointing out in what sense it is a generalization of Theorem 1.4 and exposing some of its consequences.

## 2. A counter-example

We denote a point of the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ by $(t, x)$ where $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$ and a point of the $(n+2)$-dimensional Lorentz-Minkowski spacetime $\mathbb{L}^{n+2}$ by $(t, s, x)$ where $t, s \in \mathbb{R}$ and $x \in \mathbb{R}^n$. Following [PR, Section 3], we consider the map

$$\Psi : \mathbb{R}^{n+1} \to \mathbb{L}^{n+2}, \quad (t, x) \mapsto (\cosh(t), \sinh(t), x), \tag{2.1}$$

which is an embedding, and the metric induced, via $\Psi$, on $\mathbb{R}^{n+1}$ is the usual one, i.e., $\Psi$ is an isometric embedding. Note that

$$\Psi(\mathbb{R}^{n+1}) = \{(x_1, x_2, x) \in \mathbb{L}^{n+2} : x_1^2 - x_2^2 = 1, x_1 > 0, x \in \mathbb{R}^n\},$$

is a cylinder over a (branch of a) Lorentzian circle in the Lorentzian plane $x = 0$ (see [2], for instance).

Let us consider now the $n(\geq 1)$-dimensional unit round sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ endowed with its usual Riemannian metric $g$. Then, we have that

$$\psi := \Psi\,|_{\mathbb{S}^n} : \mathbb{S}^n \to \mathbb{L}^{n+2} \tag{2.2}$$

is also an isometric embedding.

In order to compute the mean curvature vector field $\mathbf{H}$ of $\psi$ we make use of the Beltrami equation

$$\Delta\psi = n\,\mathbf{H}. \tag{2.3}$$

Thus, we get

$$\mathbf{H}(t,x) = \big(f_1(t),\, f_2(t)\,, -x\big),$$

for all $(t,x) \in \mathbb{S}^n$, where

$$f_1(t) = -t\sinh t + \frac{1}{n}(1-t^2)\cosh t, \quad f_2(t) = -t\cosh t + \frac{1}{n}(1-t^2)\sinh t.$$

Therefore, we obtain

$$0 \le \|\mathbf{H}(t,x)\|^2 = 1 - \frac{1}{n^2}(1-t^2)^2 \le 1,$$

and equality to 1 holds if and only if $(t,x) = (\pm 1, 0)$. Thus, we have

$$\int_{\mathbb{S}^n} \|\mathbf{H}\|^2 \, dV \le \mathrm{vol}(\mathbb{S}^n),$$

where $dV$ is the canonical measure on $\mathbb{S}^n$. Moreover, equality does not hold. Otherwise, we will arrive to $\|\mathbf{H}\|^2 = 1$, which is untrue.

Consequently, if (Re) holds, we get

$$\lambda_1 = n \le n \,\frac{\displaystyle\int_{\mathbb{S}^n} \|\mathbf{H}\|^2 \, dV}{\mathrm{vol}(\mathbb{S}^n)} < n\,,$$

which is a contradiction.

## 3. Revisiting Reilly's proof

Consider now a compact manifold $M^n$ and an immersion $\psi : M^n \to \mathbb{R}^{n+p}$, $\psi = (\psi_1, ..., \psi_{n+p})$, and denote by $g$ the induced metric on $M^n$ via $\psi$. We will break down Reilly's argument into several steps to elucidate our approach in the Lorentzian setting.

*Step* 1. By composing $\psi$ with a suitable translation, we can assume that the gravity center of $\psi$ is located at the origin. Importantly, this 'change' does not alter the intrinsic or extrinsic geometry of $M^n$. Therefore we have

$$\int_{M^n} \psi_j \, d\mu_g = 0,$$

for all $j = 1, ..., n+p$. Observe that this is equivalent to

$$\int_{M^n} \langle \psi, v \rangle \, d\mu_g = 0, \tag{3.1}$$

for all $v \in \mathbb{R}^{n+p}$.

*Step* 2. According to the Minimum Principle for $\lambda_1$ [1, Lemme D.II.3] we have:

$$\lambda_1 \int_{M^n} \langle \psi, v \rangle^2 \, d\mu_g \le \int_{M^n} \|\nabla \langle \psi, v \rangle\|^2 \, d\mu_g, \tag{3.2}$$

for any $v \in \mathbb{R}^{n+p}$, or equivalently, for any $v \in \mathbb{S}^{n+p-1}$. Moreover, the equality holds in (3.2) for some $v$ with $\langle \psi, v \rangle \ne 0$, if and only if

$$\Delta \langle \psi, v \rangle + \lambda_1 \langle \psi, v \rangle = 0.$$

*Step* 3. Integrating on $\mathbb{S}^{n+p-1}$ both members in integral inequality (3.2), we get:

$$\lambda_1 \int_{v \in \mathbb{S}^{n+p-1}} \int_{M^n} \langle \psi, v \rangle^2 \, d\mu_g \, dV \le \int_{v \in \mathbb{S}^{n+p-1}} \int_{M^n} \|\nabla \langle \psi, v \rangle\|^2 \, d\mu_g \, dV, \tag{3.3}$$

where $dV$ is the canonical measure on $\mathbb{S}^{n+p-1}$.

Now, we can use Fubini's theorem in inequality (3.3) to get:

$$\lambda_1 \int_{x\in M^n} \left\{ \int_{v\in \mathbb{S}^{n+p-1}} \langle \psi(x), v\rangle^2 \, dV \right\} d\mu_g \leq$$

$$\leq \int_{x\in M^n} \left\{ \int_{v\in \mathbb{S}^{n+p-1}} \|\nabla \langle \psi, v\rangle\|^2(x) \, dV \right\} d\mu_g, \tag{3.4}$$

*Step* 4. To compute both members in the previous integral inequality, we need the following technical result

**Lemma 3.1.** (Averaging Principle). *If $T : \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \to \mathbb{R}$ is a symmetric bilinear form and $\Phi : \mathbb{R}^{m+1} \to \mathbb{R}$, $\Phi(v) := T(v, v)$, is the corresponding quadratic form, then,*

$$\int_{\mathbb{S}^m} \Phi \, dV = \frac{1}{m+1} \operatorname{trace}_{g^0}(T) \operatorname{vol}(\mathbb{S}^m),$$

*where $g^0$ is the usual metric on $\mathbb{R}^{m+1}$ and $\operatorname{trace}_{g^0}(T)$ denotes the trace of the operator $g^0$-equivalent to $T$.*

Now, for each (fixed) $x \in M^n$, let us first consider $T : \mathbb{R}^{n+p} \times \mathbb{R}^{n+p} \to \mathbb{R}$ given by

$$T(u, v) := \langle \psi(x), u\rangle \langle \psi(x), v\rangle, \tag{3.5}$$

that satisfies

$$\operatorname{trace}_{g^0}(T) = \|\psi(x)\|^2.$$

Then, using previous Lemma 3.1 for $T$ given by (3.5) we obtain

$$\int_{v\in \mathbb{S}^{n+p-1}} \langle \psi(x), v\rangle^2 \, dV = \frac{1}{n+p} \|\psi(x)\|^2 \operatorname{vol}(\mathbb{S}^{n+p-1}). \tag{3.6}$$

Next, consider

$$T(u, v) := \langle \nabla \langle \psi, u\rangle, \nabla \langle \psi, v\rangle\rangle(x). \tag{3.7}$$

Taking into account that

$$\nabla \langle \psi, v\rangle(x) = v^\top(x),$$

we get

$$\operatorname{trace}_{g^0}(T) = n.$$

Consequently, using again Lemma 3.1 for the symmetric bilinear form given by (3.7), we achieve

$$\int_{v\in \mathbb{S}^{n+p-1}} \|\nabla \langle \psi, v\rangle\|^2(x) \, dV = \frac{1}{n+p} n \operatorname{vol}(\mathbb{S}^{n+p-1}). \tag{3.8}$$

Therefore, the integral inequality (3.4) reduces to

$$\lambda_1 \int_{M^n} \|\psi\|^2 \, d\mu_g \leq n \operatorname{vol}(M^n), \tag{Re$^*$}$$

which is [10, Main Lemma]. Moreover, from Step 2, the equality holds in (Re$^*$) if and only if

$$\Delta \langle \psi, v\rangle + \lambda_1 \langle \psi, v\rangle = 0,$$

for all $v \in S^{n+p-1}$, i.e., if and only if

$$\Delta \psi + \lambda_1 \psi = 0 \,..$$

**Theorem 3.2.** Takahashi [12, Thm. 3] *If an immersion $\psi : M^n \to \mathbb{R}^m$ of a manifold $M^n$ in Euclidean space $\mathbb{R}^{n+p}$ satisfies*

$$\Delta\psi + \lambda\psi = 0, \tag{3.9}$$

*for some constant $\lambda \neq 0$, then $\lambda$ is necessarily positive, and $\psi$ realizes a minimal immersion in an hypersphere $\mathbb{S}^{n+p-1}(\sqrt{n/\lambda})$ of radius $\sqrt{n/\lambda}$ in $\mathbb{R}^{n+p}$. Conversely, if $\psi$ realizes a minimal immersion in a hypersphere of radius $R$ in $\mathbb{R}^{n+p}$, then $\psi$ satisfies* (3.9) *up to a parallel displacement in $\mathbb{R}^{n+p}$ and $\lambda = n/R^2$.*

Therefore, equality in (Re*) holds if and only if $\psi(M^n)$ lies minimally in a hypersphere in $\mathbb{R}^{n+p}$ of radius $\sqrt{n/\lambda_1}$. Note that the characterization of equality in (Re*) is also a part of [10, Main Lemma].

*Step* 5. (The final step). Given an immersion $\Psi : M^n \to \mathbb{R}^{n+p}$, we construct $\psi : M^n \to \mathbb{R}^{n+p}$ such that the gravity center of $\psi$ is located at the origin as above. Multiplying both members of (Re*) by $\int_{M^n} \|\mathbb{H}\|^2 \, d\mu_g$, the left-hand side can be bounded from below in the following way

$$\begin{aligned} \int_{M^n} \|\psi\|^2 \, d\mu_g \int_{M^n} \|\mathbb{H}\|^2 \, d\mu_g &\geq \left( \int_{M^n} \|\psi\| \, \|\mathbb{H}\| \, d\mu_g \right)^2 \\ &\geq \left( \int_{M^n} \langle \psi, \mathbb{H} \rangle \, d\mu_g \right)^2, \end{aligned} \tag{3.10}$$

where we have used $L^2$-Schwarz inequality and Schwarz inequality in $\mathbb{R}^{n+p}$.

Now, we use the general formula

$$\Delta\|\psi\|^2 = 2n\,(1 + \langle \psi, \mathbb{H} \rangle)$$

to write

$$\int_{M^n} \langle \psi, \mathbb{H} \rangle \, d\mu_g = -\operatorname{vol}(M^n). \tag{3.11}$$

Substituting (3.11) in the integral inequality (3.10) we arrive to

$$\int_{M^n} \|\psi\|^2 \, d\mu_g \int_{M^n} \|\mathbb{H}\|^2 \, d\mu_g \geq \operatorname{vol}(M^n)^2. \tag{3.12}$$

From (Re*), taking into account (3.11) we obtain

$$\lambda_1 \operatorname{vol}(M^n)^2 \leq n \operatorname{vol}(M^n) \int_{M^n} \|\mathbb{H}\|^2 \, d\mu_g \,,$$

which establishes inequality (Re). The equality holds in (Re) if and only if the equality holds in (Re*). Hence, if and only if $\psi(M^n)$ lies minimally in a hypersphere in $\mathbb{R}^{n+p}$ of radius $\sqrt{n/\lambda_1}$.

## 4. The approach for compact spacelike submanifolds

In view of the previous section, we assert that Reilly's technique is not applicable to compact spacelike submanifolds of $\mathbb{L}^m$ for several significant reasons, namely:

(1) The Averaging Principle, shown in Lemma 3.1 cannot be extended by substituting $\mathbb{S}^m$ with either the hypersurface of unit spacelike vectors or the hypersurface of (pointing future) unit timelike vectors in $\mathbb{L}^{m+1}$, both of which are non-compact.

(2) The normal bundle of a codimension $\geq 2$ spacelike submanifold of $\mathbb{L}^m$ has a Lorentzian signature, in particular $\|H\|^2$ may not have a definite sign.

(3) The Schwarz inequality for vectors in $\mathbb{R}^m$ does not hold for vectors in $\mathbb{L}^m$.

Considering both these difficulties and the counter-example presented in Section 2, we will introduce an alternative approach to the one used by Reilly, as described in Section 3, specifically tailored for compact submanifolds of $\mathbb{L}^m$.

To begin our technique, let us introduce a new geometric object. Denote by $\langle\ ,\ \rangle$ the Lorentzian metric of $\mathbb{L}^{m+1}$. For each unit timelike vector $a \in \mathbb{L}^{m+1}$, i.e., with $\langle a, a\rangle = -1$, we define the spherical section in $\mathbb{L}^{m+1}$ relative to $a$ as

$$\mathbb{S}_a^{m-1} := \{v \in \mathbb{L}^{m+1} : \langle v, v\rangle = 0,\ \ \langle a, v\rangle = -1\}, \tag{4.1}$$

that is, $\mathbb{S}_a^{m-1}$ is the intersection of the light cone of $\mathbb{L}^{m+1}$ with the spacelike hyperplane

$$\mathbb{R}_a^m := \{v \in \mathbb{L}^{m+1} : \langle a, v\rangle = -1\}.$$

Clearly, $\mathbb{S}_a^{m-1}$ is a hypersphere not centered at the origin in the spacelike hyperplane $\mathbb{R}_a^m$,and, therefore, a compact codimension two spacelike submanifold of $\mathbb{L}^{m+1}$. Moreover, it is isometric to the unit round sphere $\mathbb{S}^{m-1}$ in Euclidean space $\mathbb{R}^m$. In particular, we have

$$\mathrm{vol}(\mathbb{S}_a^{m-1}) = \mathrm{vol}(\mathbb{S}^{m-1}), \tag{4.2}$$

for all unit timelike vector $a$.

The essential tool we will use here is the following integral formula

**Lemma 4.1.** (Generalized Averaging Principle) [4, Lemma 3.4(b)] *If $T : \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \to \mathbb{R}$ is a symmetric bilinear form, $\Phi : \mathbb{R}^{m+1} \to \mathbb{R}$, $\Phi(v) = T(v, v)$, the corresponding quadratic form, and $a$ is a unit timelike vector in $\mathbb{L}^{m+1}$, we have,*

$$\int_{\mathbb{S}_a^{m-1}} \Phi \, dV_a = \frac{1}{m}\Big[(m+1)\, T(a, a) + \mathrm{trace}_{\langle\,,\,\rangle}(T)\Big]\mathrm{vol}(\mathbb{S}^{m-1}),$$

*where $dV_a$ is the canonical measure on $\mathbb{S}_a^{m-1}$, $\mathrm{trace}_{\langle\,,\,\rangle}(T) := \mathrm{trace}(A_T)$ and $A_T$ is the linear operator of $\mathbb{L}^{m+1}$ defined by $\langle A_T(u), v\rangle = T(u, v)$, for all $u, v \in \mathbb{L}^{m+1}$.*

## 5. A key technical result

**Lemma 5.1.** [9, Lemma 6.4] *Let $M^n$ be an $n$-dimensional compact manifold and $\psi : M^n \to \mathbb{L}^{n+p}$, $p \geq 2$, be a spacelike immersion with gravity center located at the origin.*

*For every unit timelike vector $a \in \mathbb{L}^{n+p}$, the first eigenvalue $\lambda_1$ of the Laplacian of the induced metric $g$ on $M^n$ satisfies*

$$\lambda_1 \int_{M^n} \|\psi_a\|^2 \, d\mu_g \leq n\, \mathrm{vol}(M^n) + \int_{M^n} \|a^\top\|^2 \, d\mu_g, \tag{PR$^*$}$$

*where $\psi_a := \psi + \langle\psi, a\rangle a$ is, at each point $x \in M^n$, the orthogonal projection of $\psi(x)$ on the spacelike hyperplane $a^\perp$ of $\mathbb{L}^{n+p}$. Moreover, the equality holds in* (PR$^*$) *for some unit timelike vector $a$ if and only if*

$$\Delta\psi_a + \lambda_1\psi_a = 0\,.$$

**Remark 5.2.** (a) Note that the previous characterization of the equality in (PR$^*$) turns to be equivalent to $\Delta\psi + \lambda_1\psi = f_a\, a$, for some $f_a \in C^\infty(M^n)$. (b) Particularly, if $\psi = \psi_a$ for some $a$, i.e., if $\psi(M^n) \subset a^\perp$, then $a^\top = 0$.Therefore, in this case, the result is just inequality (Re$^*$); that is to say, [10, Main Lemma].

**Remark 5.3.** Before delving into the proof of Lemma 5.1, it would be clarifying to provide a few comments on the map $\psi_a$, introduced in that result. Note that $\psi_a$ may be considered as a map

$$\psi_a : M^n \to a^\perp,$$

and that satisfies $d\psi_a(v) = d\psi(v) + \langle d\psi(v), a\rangle\, a$. Therefore,

$$d\psi_a(v) = 0 \quad \Longleftrightarrow \quad \underbrace{d\psi(v)}_{spacelike} = - \underbrace{\langle d\psi(v), a\rangle\, a}_{\text{timelike or zero}}\ .$$

Thus, $d\psi_a(v) = 0$ gives $v = 0$, showing that $\psi_a$ is an immersion. Furthermore, the Riemannian metric $g^a$ on $M^n$ induced from the metric on $a^\perp$, via $\psi_a$, satisfies

$$g^a(u, v) = g(u, v) + \langle d\psi(u), a\rangle\langle d\psi(v), a\rangle,$$

for any $u, v \in T_xM^n$, $x \in M^n$. Consequently, $g^a = g$ if and only if $a \perp d\psi_x(T_xM^n)$ for any $x \in M^n$. Thus, [10, Main Lemma] cannot be applied to $\psi_a : M^n \to a^\perp$ to prove Lemma 5.1.

*Proof.* By hypothesis, we known that the center of gravity of $\psi$ is located at the origin. Therefore, from (3.1) we have

$$\int_{M^n} \langle \psi_a, v\rangle\, d\mu_g = 0, \tag{5.1}$$

for all $v \in \mathbb{L}^{n+p}$. Using now the Minimum Principle for the function $\langle \psi_a, v\rangle$, we get

$$\lambda_1 \int_{M^n} \langle \psi_a, v\rangle^2\, d\mu_g \leq \int_{M^n} \|\nabla\langle \psi_a, v\rangle\|^2\, d\mu_g, \tag{5.2}$$

and the equality holds for some $v$, with $\langle \psi_a, v\rangle \neq 0$, if and only if

$$\Delta\langle \psi_a, v\rangle + \lambda_1\langle \psi_a, v\rangle = 0.$$

Integrating both members of inequality (5.2) and using, as Section 3, Fubiny's theorem, we get

$$\lambda_1 \int_{x\in M^n} \left\{\int_{v\in\mathbb{S}_a^{n+p-2}} \langle \psi_a(x), v\rangle^2\, dV_a\right\} d\mu_g \leq$$

$$\leq \int_{x\in M^n} \left\{\int_{v\in\mathbb{S}_a^{n+p-2}} \|\nabla\langle \psi_a, v\rangle\|^2(x)\, dV_a\right\} d\mu_g, \tag{5.3}$$

where $dV_a$ is the canonical measure on $\mathbb{S}_a^{n+p-2}$.

For each (fixed) $x \in M^n$ consider

$$T(u, v) = \langle \psi_a(x), u\rangle\langle \psi_a(x), v\rangle, \tag{5.4}$$

that satisfies

$$\mathrm{trace}_{\langle\ ,\ \rangle}(T) = \|\psi_a(x)\|^2 \quad \text{and} \quad T(a, a) = 0.$$

Then, using Lemma 4.1 for the symmetric bilinear form given by (5.4), we get

$$\int_{v\in\mathbb{S}_a^{n+p-2}} \langle \psi_a(x), v\rangle^2\, dV_a = \frac{1}{n+p-1}\, \|\psi_a(x)\|^2\, \mathrm{vol}(\mathbb{S}^{n+p-2}). \tag{5.5}$$

On the other hand, consider now

$$T(u, v) = \langle \nabla\langle \psi_a, u\rangle, \nabla\langle \psi_a, v\rangle\rangle(x). \tag{5.6}$$

Clearly, $T(a, a) = 0$ and $\nabla\langle \psi_a, v\rangle(x) = v^\top(x) + \langle a, v\rangle a^\top(x)$.

If we take an orthonormal basis of $\mathbb{L}^{n+p}$ so that

$$(\underbrace{e_1, ..., e_n}_{\text{in } T_xM^n}, \underbrace{e_{n+1}, ..., e_{n+p}}_{\text{in } T_x^\perp M^n}),$$

then, we compute

$$\mathrm{trace}_{\langle\ ,\ \rangle}(T) = \sum_{i=1}^{n} \big\langle e_i + \langle a, e_i\rangle a^\top(x), e_i + \langle a, e_i\rangle a^\top(x)\big\rangle +$$

$$+\sum_{j=n+1}^{n+p}\langle a,e_j\rangle^2\|a^\top(x)\|^2,$$

Thus,

$$\begin{aligned}\mathrm{trace}_{\langle\ ,\ \rangle}(T)&=n+\|a^\top(x)\|^4+2\|a^\top(x)\|^2+\|a^N(x)\|^2\|a^\top(x)\|^2\\&=n+\|a^\top(x)\|^2,\end{aligned}$$

where, for the normal component $a^N$ of $a$, we have used $\|a^N\|^2=-1-\|a^\top\|^2$, everywhere on $M^n$.

Consequently, using again Lemma 4.1 for the symmetric bilinear form given by (5.6), we achieve

$$\int_{v\in\mathbb{S}_a^{n+p-2}}\|\nabla\langle\psi_a,v\rangle\|^2(x)\,dV_a=\frac{n+\|a^\top(x)\|^2}{n+p-1}\,\mathrm{vol}(\mathbb{S}^{n+p-2}),\tag{5.7}$$

which ends the proof of Lemma 5.1. □

## 6. Upper bounds for $\lambda_1$

**Theorem 6.1.** [9, Thm 6.10]. *Let $M^n$ be a compact $n$-dimensional manifold and $\psi:M^n\to\mathbb{L}^{n+p}$, $p\geq 2$, be a spacelike immersion. For each unit timelike vector $a\in\mathbb{L}^{n+p}$, the first non-trivial eigenvalue $\lambda_1$ of the Laplacian of the induced metric $g$ on $M^n$ via $\psi$ satisfies*

$$\lambda_1\leq n\,\frac{\displaystyle\int_{M^n}\|\mathbf{H}_a\|^2\,d\mu_g}{\mathrm{vol}(M^n)+\dfrac{1}{n}\displaystyle\int_{M^n}\|a^T\|^2\,d\mu_g},\tag{PR}$$

*where $\mathbf{H}_a:=\mathbf{H}+\langle\mathbf{H},a\rangle a$ is, at any point $x\in M^n$, the orthogonal projection of the vector $\mathbf{H}(x)$ on the spacelike hyperplane $a^\perp$. Moreover, the equality in* (PR) *holds if and only the equality holds in Lemma 5.1, i.e., if and only if there exists $f(=f_a)\in C^\infty(M^n)$ such that*

$$\Delta\widetilde{\psi}+\lambda_1\widetilde{\psi}=f\,a,$$

*where $\widetilde{\psi}:=\psi-(1/\mathrm{vol}(M^n))\,c$, with $c_j=\int_{M^n}\psi_j\,d\mu_g$, $j=1,...,n+p$.*

*Proof.* To facilitate a comparison with the proof of Reilly's theorem in Section 3, we will divide our argument into several steps

*Step* 1. Our starting point is the inequality in Lemma 5.1 for $\widetilde{\psi}_a$, namely

$$\lambda_1\int_{M^n}\|\widetilde{\psi}_a\|^2\,d\mu_g\leq n\,\mathrm{vol}(M^n)+\int_{M^n}\|a^\top\|^2\,d\mu_g.\tag{6.1}$$

Multiplying both members of inequality (6.1) by the non-negative quantity $\int_{M^n}\|\mathbb{H}_a\|^2\,d\mu_g$, we obtain

$$\begin{aligned}\big(n\,\mathrm{vol}(M^n)+\int_{M^n}\|a^\top\|^2\,d\mu_g\big)\int_{M^n}\|\mathbb{H}_a\|^2\,d\mu_g\geq&\\ \geq\lambda_1\int_{M^n}\|\widetilde{\psi}_a\|^2\,d\mu_g\int_{M^n}\|\mathbb{H}_a\|^2\,d\mu_g&\\ \geq\lambda_1\Big(\int_{M^n}\|\widetilde{\psi}_a\|\,\|\mathbb{H}_a\|\,d\mu_g\Big)^2.&\end{aligned}$$

Using now $L^2$-Schwarz inequality and Schwarz inequality in $a^\perp$, we conclude

$$\big(n\,\mathrm{vol}(M^n)+\int_{M^n}\|a^\top\|^2\,d\mu_g\big)\int_{M^n}\|\mathbb{H}_a\|^2\,d\mu_g\geq\lambda_1\Big(\int_{M^n}\langle\widetilde{\psi}_a,\mathbb{H}_a\rangle\,d\mu_g\Big)^2.\tag{6.2}$$

*Step* 2. In order to compute the right hand side of inequality (6.2), observe that

$$\langle \widetilde{\psi}_a, \mathbb{H}_a \rangle = \langle \widetilde{\psi}, \mathbb{H} \rangle + \langle \widetilde{\psi}, a \rangle \langle \mathbb{H}, a \rangle$$

and recall the general formula

$$\Delta \|\widetilde{\psi}\|^2 = 2n\left(1 + \langle \widetilde{\psi}, \mathbb{H} \rangle\right).$$

Therefore, we can write

$$\int_{M^n} \langle \widetilde{\psi}_a, \mathbb{H}_a \rangle \, d\mu_g = -\mathrm{vol}(M^n) + \int_{M^n} \langle \widetilde{\psi}, a \rangle \langle \mathbb{H}, a \rangle \, d\mu_g. \tag{6.3}$$

Finally, using

$$\frac{1}{2} \Delta \langle \widetilde{\psi}, a \rangle^2 = n \langle \widetilde{\psi}, a \rangle \langle \mathbb{H}, a \rangle + \|a^\top\|^2,$$

we can rewrite (6.3) as follows

$$\int_{M^n} \langle \widetilde{\psi}_a, \mathbb{H}_a \rangle \, d\mu_g = -\mathrm{vol}(M^n) - \frac{1}{n} \int_{M^n} \|a^\top\|^2 \, d\mu_g \, . \tag{6.4}$$

Consequently, making use of (6.4), the integral inequality in (6.2) gives

$$\left(n \,\mathrm{vol}(M^n) + \int_{M^n} \|a^\top\|^2 \, d\mu_g\right) \int_{M^n} \|\mathbb{H}_a\|^2 \, d\mu_g \geq$$

$$\geq \frac{\lambda_1}{n^2} \left(n \,\mathrm{vol}(M^n) + \int_{M^n} \|a^\top\|^2 \, d\mu_g\right)^2, \tag{6.5}$$

that yields

$$\int_{M^n} \|\mathbb{H}_a\|^2 \, d\mu_g \geq \frac{\lambda_1}{n^2} \left(n \,\mathrm{vol}(M^n) + \int_{M^n} \|a^\top\|^2 \, d\mu_g\right).$$

We conclude the proof by noting that equality holds in the previous inequality if and only if equality holds in the integral inequality stated in Lemma 5.1. ☐

**Corollary 6.2.** *For each unit timelike vector $a \in \mathbb{L}^{n+p}$, the first eigenvalue $\lambda_1$ of the Laplacian for a compact $n$-dimensional spacelike submanifold $M^n$ in Lorentz-Minkowski spacetime $\mathbb{L}^{n+p}$, $p \geq 2$, satisfies*

$$\lambda_1 \leq n \, \frac{\displaystyle\int_{M^n} \|\mathbf{H}_a\|^2 \, d\mu_g}{\mathrm{vol}(M^n)}.$$

*The equality holds if and only if $M^n$ factors through a spacelike affine hyperplane $\pi_a$, and $M^n$ is minimal in some hypersphere in $\pi_a$ with radius $\sqrt{n/\lambda_1}$.*

**Remark 6.3.** In particular, if $\psi(M^n)$ is contained in a spacelike hyperplane $\pi_a$ with unit timelike normal vector $a$, then $a^\top = 0$ and $\langle \mathbf{H}, a \rangle = 0$ hold, reproving Reilly's inequality (Re) and the corresponding characterization of the equality.

Finally, we observe that the family of inequalities (PR) is parametrized by the $(n+p-1)$-dimensional unit hyperbolic space $\mathbb{H}^{n+p-1}$ in $\mathbb{L}^{n+p}$. Consequently, we can collect all these inequalities to derive a vector-independent upper bound, namely: for any compact $n$-dimensional spacelike submanifold $M^n$ in Lorentz-Minkowski spacetime $\mathbb{L}^{n+p}$, the first eigenvalue $\lambda_1$ of the Laplacian of $M^n$ satisfies

$$\lambda_1 \leq \inf_{a \in \mathbb{H}^{n+p-1}} n \, \frac{\displaystyle\int_{M^n} \|\mathbf{H}_a\|^2 \, d\mu_g}{\mathrm{vol}(M^n) + \dfrac{1}{n} \displaystyle\int_{M^n} \|a^\top\|^2 \, d\mu_g} \, .$$

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## References


[1] M. Berger, P. Gauduchon and E. Mazet, *Le spectre d'une variété Riemannienne*, Lect. Notes Math. **194**, Springer Verlag, Berlin, New York, 1974,

[2] G.S. Birman and K. Nomizu, *Trigonometry in Lorentzian geometry*, Amer. Math. Mon. **9** (1984), 543–549.

[3] I. Chavel, *Eigenvalues in Riemannian Geometry*, Pure and Applied Math **115**, Academic Press, 1984.

[4] M. Gutiérrez, F.J. Palomo and A. Romero, *A Berger-Green type inequality for compact Lorentzian manifolds*, Trans. Amer. Math. Soc. **354** (2002), 4505–4523.

[5] S.G. Harris, *Closed and complete spacelike hypersurfaces in Minkowski space*, Class. Quantum Grav. **5** (1988), 111–119.

[6] J. Milnor, *Eigenvalues of the Laplace operator on certain manifolds*, Proc. Nat. Acad. Sci., U.S.A., **51** (1964), 542.

[7] O. Palmas, F.J. Palomo and A. Romero, *On the total mean curvature of a compact space-like submanifold in Lorentz–Minkowski spacetime*, P. Roy. Soc. Edinb. A Mat. **148A** (2018), 199–210.

[8] F.J. Palomo and A. Romero, *On spacelike surfaces in* 4*-dimensional Lorentz-Minkowski spacetime through a lightcone*, P. Roy. Soc. Edinb. A Mat. **143A** (2013), 881–892.

[9] F.J. Palomo and A. Romero, *On the first eigenvalue of the Laplace operator for compact spacelike submanifolds in Lorentz–Minkowski spacetime*, Proc. Roy. Soc. Edinburgh Sect. A Mathematics **152** (2022), 311–330.

[10] R.C. Reilly, *On the first eigenvalue of the Laplace operator for compact submanifolds of Euclidean space*, Comment. Mat. Helvetici **52** (1977), 525–533.

[11] A. El Soufi, and S. Ilias, *Une inégalité du type "Reilly" pour les sousvariétés de l'espace hyperbolique*, Comment. Mat. Helvetici **67** (1992), 167–181.

[12] T. Takahashi, *Minimal immersions of Riemannian manifolds*, J. Math. Soc. Japan **18** (1966), 380–385.



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