A NEW CURVATURELIKE TENSOR FIELD ON TRANS-SASAKIAN MANIFOLDS

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ABSTRACT. Recently, we introduced the new curvaturelike tensor field named (CHR)-curvature tensor which is an almost contact version of M. Prvanovic [4] in almost contact metric manifolds [1]. In §1, we recall the definition of a trans-Sasakian manifold which is the generalization of a Sasakian and a Kenmotsu manifold ([2],[3]) and give many properties of this manifold which are useful in the next secton. Then, in §2, we mainly consider this tensor field in a trans-Sasakian manifold. Finally, in §3, we consider a trans-Sasakian manifold admitting the recurrent (CHR)-curvature tensor.

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1. Trans-Sasakian Manifolds

Let M be a real $(2n+1)$-dimensional almost contact metric manifold with the structure (φ, ξ, η, g), that is, they satisfy

$$\begin{cases} \varphi^2 = -I + \eta \otimes \xi, \ \ \eta o \varphi = \varphi \xi = 0, \ \ g(X, \xi) = \eta(X), \\ g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \end{cases} \tag{1.1}$$

Definition 1.1. An almost contact metric manifold M with structure (φ, ξ, η, g) is said to be a *trans-Sasakian manifodd of* (α, β)*-type* if it satisfies

$$(\nabla_X \varphi)Y = \alpha\{g(X, Y)\xi - \eta(Y)X\} + \beta\{g(\varphi X, Y)\xi - \eta(Y)\varphi X\}, \tag{1.2}$$

for certain functions α and β on M, which are called *associated functions*, where ∇ means the covariant differentiation with respect to g.

Remark 1.2. In a trans-Sasakian manifold, if $\alpha = 0$ (resp. $\beta = 0$), we say it is a β-Kenmotsu (resp. an α-Sasakian) manifold.

Remark 1.3. In [2], Oubina gave an example of a 3-dimensional trans-Sasakian manifold.

We know the following formulae in a trans-Sasakian manifold M of (α, β)-type.
About the structure vector fields ξ and η, we can easily get from (1.1) and (1.2)

$$\begin{cases} \nabla_X \xi = -\alpha \varphi X + \beta\{X - \eta(X)\xi\}, \\ (\nabla_Y \eta)(X) = -\alpha g(\varphi Y, X) + \beta\{g(Y, X) - \eta(Y)\eta(X)\}. \end{cases} \tag{1.3}$$

Next, about the Riemannian curvature tensor R, we have

$$R(X, Y, Z, \xi) = (X\alpha)g(\varphi Y, Z) - (Y\alpha)g(\varphi X, Z) \tag{1.4}$$

$$-(X\beta)\{g(Y,Z)-\eta(Y)\eta(Z)\}+(Y\beta)\{g(X,Z)-\eta(X)\eta(Z)\}$$
$$+(\alpha^2-\beta^2)A(X,Y,Z)-2\alpha\beta B(X,Y,Z),$$

where we put

$$A(X,Y,Z)=g(Z,Y)X-g(Z,X)Y,\quad B(X,Y,Z)=A(X,Y,\varphi Z). \tag{1.5}$$

From (1.4), we can easily obtain

$$\rho(X,\xi)=(X\tilde{\alpha})+(2n-1)(X\beta)+\{(\xi\beta)-2n(\alpha^2-\beta^2)\}\eta(X), \tag{1.6}$$

where ρ means the Ricci tensor with respect to g.

From (1.2), we can obtain

$$\begin{aligned} R(X,Y,W,\varphi Z)-R(X,Y,Z,\varphi W)=(X\alpha)A(Z,W,Y)\\ -(Y\alpha)A(Z,W,X)+(X\beta)B(Z,W,Y)-(Y\beta)B(Z,W,X)\\ +(\alpha^2-\beta^2)\{g(X,W)g(\varphi Y,Z)-g(X,Z)g(\varphi Y,W)-g(Y,W)g(\varphi X,Z)\\ +g(Y,Z)g(\varphi X,W)\}-2\alpha\beta\{g(X,W)g(Y,Z)-g(X,Z)g(Y,W)\\ -g(\varphi X,W)g(\varphi Y,Z)+g(\varphi X,Z)g(\varphi Y,W)\}. \end{aligned} \tag{1.7}$$

From the above equation, we can easily have

$$\begin{aligned} R(X,Y,\varphi Z,\varphi W)=R(X,Y,Z,W)+(X\alpha)B(Z,W,Y)\\ -(Y\alpha)B(Z,W,X)-(X\beta)A(Z,W,Y)+(Y\beta)A(Z.W,X)\\ -(\alpha^2-\beta^2)\{g(X,W)g(Y,Z)-g(X,Z)g(Y,W)\\ -g(\varphi X,W)g(\varphi Y,Z)+g(\varphi X,Z)g(\varphi Y,W)\}\\ -2\alpha\beta\{g(X,W)g(\varphi Y,Z)-g(X,Z)g(\varphi Y,W)\\ -g(Y,W)g(\varphi X,Z)-g(Y,Z)g(\varphi X,W)\} \end{aligned} \tag{1.8}$$

and

$$\begin{aligned} R(\varphi X,\varphi Y,\varphi Z,\varphi W)=R(X,Y,Z,W)+(Z\alpha)B(X,Y,W)\\ -(W\alpha)B(X,Y,Z)-(Z\beta)A(X,Y,W)+(W\beta)A(X,Y,Z)\\ -(\varphi X\alpha)A(Z,W,Y)+(\varphi Y\alpha)A(Z,W,X)-(\varphi X\beta)B(Z,W,Y)\\ +(\varphi Y\beta)A(Z,W,X)+(\alpha^2-\beta^2)\{A(Z,W,Y)\eta(X)-A(Z,W,Y)\eta(Y)\}\\ +2\alpha\beta[2\{g(X,W)g(\varphi Y,Z)-g(X,Z)g(\varphi Y,W)-g(Y,W)g(\varphi X,Z)\\ +g(Y,Z)g(\varphi X,W)\}+B(Z,W,Y)\eta(X)-B(Z,W,X)\eta(Y)]. \end{aligned} \tag{1.9}$$

From (1.4), we obtain

$$\begin{aligned} R(\xi,X,Y,\xi)=R(\xi,\varphi X,\varphi Y,\xi)=\{(\alpha^2-\beta^2)\\ -(\xi\beta)\}\{g(X,Y)-\eta(X)\eta(Y)\}. \end{aligned} \tag{1.10}$$

About the associated functions α and β, we have

$$(\xi\alpha)+2\alpha\beta=0 \tag{1.11}$$

$$\begin{aligned} \{(X\alpha)+(\varphi X\beta)\}B(Z,W,Y)-\{(Y\alpha)+(\varphi Y\beta)\}B(Z,W,X)\\ -\{(Z\alpha)+(\varphi Z\beta)\}B(X,Y,W)+\{(W\alpha)+(\varphi W\beta)\}B(X,Y,Z)\\ -\{(X\beta)-(\varphi X\alpha)\}A(Z,W,Y)+\{(Y\beta)-(\varphi Y\alpha)\}A(Z,W,X)\\ +\{(Z\beta)-(\varphi Z\alpha)\}A(X,Y,W)-\{(W\beta)-(\varphi W\alpha)\}A(X,Y,Z)\\ =4\alpha\beta[2\{g(X,W)g(\varphi Y,Z)-g(X,Z)g(\varphi Y,Z)-g(Y,W)g(\varphi X,Z)\\ +g(Y,Z)g(\varphi X,W)\}+B(Z,W,Y)\eta(X)-B(Z,W,X)\eta(Y)]. \end{aligned} \tag{1.12}$$

Remark 1.4. The equation (1.12) is important to obtain the (CHR)-curvature tensor in a trans-Sasakian manifold.

2. (CHR)-curvature tensor

Definition 2.1. In an almost contact metric manifold $M(\varphi,\xi,\eta,g)$, a new curvaturelike tensor field named (CHR)-*curvature tensor* is defined by

$$
\begin{aligned}
16(CHR)(X,Y,Z,W) &= 3\{R(X,Y,Z,W)+R(\varphi X,\varphi Y,Z,W)\\
&+R(X,Y,\varphi Z,\varphi W)+R(\varphi X,\varphi Y,\varphi Z,\varphi W)\}-R(X,Z,\varphi W,\varphi Y)\\
&-R(\varphi X,\varphi Z,W,Y)-R(X,W,\varphi Y,\varphi Z)-R(\varphi X\varphi W,Y,Z)\\
&+R(\varphi X,Z,\varphi W,Y)+R(X,\varphi Z,W,\varphi Y)+R(\varphi X,W,Y,\varphi Z)\\
&+R(X,\varphi W,\varphi Y,Z)+\eta(X)P(Z,W,Y)-\eta(Y)P(Z,W,X)\\
&+\eta(Z)P(X,Y,W)-\eta(W)P(X,Y,Z)+\eta(X)\eta(W)Q(Y,Z)\\
&-\eta(X)\eta(Z)Q(Y,W)+\eta(Y)\eta(Z)Q(W,X)-\eta(Y)\eta(W)Q(Z,X),
\end{aligned}
\tag{2.1}
$$

where we put

$$
\begin{aligned}
P(X,Y,Z) &= 3\{R(X,Y,Z,\xi)+R(\varphi X,\varphi Y,Z,\xi)\}\\
&+R(\varphi X,\varphi Z,Y,\xi)+R(\varphi Z,\varphi Y,X,\xi)-R(X,\varphi Z,\varphi Y,\xi)\\
&-R(\varphi Z,Y,\varphi X,\xi)
\end{aligned}
\tag{2.2}
$$

and

$$
Q(X,Y)=3R(\xi,X,Y,\xi)-R(\xi,\varphi X,\varphi Y,\xi). \tag{2.3}
$$

Remark 2.2. A (0.4) tensor field T is said to be *curvaturelike* if it satisfies

$$
\begin{cases}
T(X,Y,Z,W)=-T(Y,X,Z,W),\\
T(X,Y,Z,W)=T(Z,W,X,Y),\\
T(X,Y,Z,W)+T(X,Z,W,Y)+T(X,W,Y,Z)=0.
\end{cases}
$$

Remark 2.3. It is easy to check that the above $(CHR)(X,Y,Z,W)$ is curvaturelike.

From now on, we consider the (CHR)-curvature tensor in a trans-Sasakian manifold.

Using the formulae in §1, the (CHR)-curvature tensor in a trans-Sasakian manifold of (α,β)-type is given by

$$
\begin{aligned}
16(CHR)(X,Y,Z,W) &= 16R(X,Y,Z,W)+(X\alpha)\{7B(Z,W,Y)\\
&+4g(\varphi Z,W)\eta(Y)\}-(Y\alpha)\{7B(Z,W,X)+4g(\varphi Z,W)\eta(X)\}\\
&+(Z\alpha)\{7B(X,Y,W)+4g(\varphi X,Y)\eta(W)\}-(W\alpha)\{7B(X,Y,Z)+4g(\varphi X,Y)\eta(Z)\}\\
&-9\{(X\beta)A(Z,W,Y)-(Y\beta)A(Z,W,X)+(Z\beta)A(X,Y,W)-(W\beta)A(X,Y,Z)\}\\
&-5\{(\varphi X\alpha)A(Z,W,Y)-(\varphi Y\alpha)A(Z,W,X)+(\varphi Z\alpha)A(X,Y,W)\\
&-(\varphi W\alpha)A(X,Y,Z)\}-(\varphi X\beta)\{3B(Z,W,Y)+4g(\varphi Z,W)\eta(Y)\}\\
&+(\varphi Y\beta)\{3B(Z,W,X)+4g(\varphi Z,W)\eta(X)\}-(\varphi Z\beta)\{3B(X,Y,W)+4g(\varphi X,Y)\eta(W)\}\\
&+(\varphi W\beta)\{3B(X,YZ)+4g(\varphi X,Y)\eta(Z)\}-4(\alpha^2-\beta^2)[3\{g(X,W)g(Y,Z)\\
&-g(X,Z)g(Y,W)\}-g(\varphi X,W)g(\varphi Y,Z)+g(\varphi X,Z)g(\varphi Y,W)\\
&+2g(\varphi X,Y)g(\varphi Z,W)-A(Z,W,Y)\eta(X)+A(Z,W,X)\eta(Y)].\\
&+2(\xi\beta)\{A(Z,W,Y)\eta(X)-A(Z,W,X)\eta(Y)\}
\end{aligned}
\tag{2.4}
$$

and

$$
\begin{aligned}
16(CHR)(X,Y,Z,\xi) &= 3\{3(X\alpha)+(\varphi X\beta)\}g(\varphi Y,Z)\\
&-3\{3(Y\alpha)+(\varphi Y\beta)\}g(\varphi X,Z)+4\{(Z\alpha)-(\varphi Z\beta)\}g(\varphi X,Y).
\end{aligned}
\tag{2.5}
$$

From (2.4), the (CHR)-Ricci tensor $\rho(CHR)(X,Y)$ is given by

$$
8\rho(CHR)(X,Y)=8\rho(X,Y)+(5n+3)\{(\varphi X\alpha)\eta(Y) \tag{2.6}
$$

$$+(\varphi Y\alpha)\eta(X)\} + (9n-1)\{(X\beta)\eta(Y) + (Y\beta)\eta(X)\}$$
$$+2(\xi\beta)\{4g(X,Y) - (n+3)\eta X)\eta(Y)\} - 4(\alpha^2 - \beta^2)\{(3n-1)g(X,Y)$$
$$+(n+1)\eta(X)\eta(Y)\}.$$

Moreover, we have from (2.6), the (CHR)-scaler curvature $\tau(CHR)$ is

$$\tau(CHR) = \tau + 4n(\xi\beta) - n(3n+1)(\alpha^2 - \beta^2), \tag{2.7}$$

where τ means the scalar curvature with respect to g.
By virtue of (2.5), we obtain the following

$$-16\alpha(CHR)(X,Y,Z,\varphi W) + 16\beta(CHR)(X,Y,Z,W) = \tag{2.8}$$
$$-16\{\nabla_W(CHR)\}(X,Y,Z,\xi) + 16\beta(CHR)(X,Y,Z,\xi)\eta(W) + 3\{3(\nabla_W\alpha)(X)$$
$$+(\nabla_W\tilde{\beta})(X)g(\varphi Y,Z) - 3\{3(\nabla_W\alpha)(Y) + (\nabla_W\tilde{\beta})(Y)\}g(\varphi X,Z)$$
$$+4\{(\nabla_W\alpha)(Z) - (\nabla_W\tilde{\beta})(Z)\}g(\varphi X,Y) - 3\{3(X\alpha) + (X\tilde{\beta})\}\{A(Y,Z,W)$$
$$+B(Y,Z,W)\} + 3\{3(Y\alpha) + (Y\tilde{\beta})\}\{A(X,Z,W) + B(X,Z,W)\}$$
$$-4\{(Z\alpha) - (Z\tilde{\beta})\}\{A(X,Y,W) + B(X,Y,W)\} + \{5(\nabla_W\tilde{\alpha}(X)$$
$$-7(\nabla_W\beta)(X)\}\{g(Y,Z) - \eta(Y)\eta(Z)\} - \{5(\nabla_W\tilde{\alpha}(Y) - 7(\nabla_W\beta)(Y)\}\{g(X,Z)$$
$$-\eta(X)\eta(Z)\} - \{5(X\tilde{\alpha}) - 7(X\beta)\}[\alpha\{g(\varphi Y,W)\eta(Z) + g(\varphi Z,W)\eta(Y)\}$$
$$+\beta\{g(Y,W)\eta(Z) + g(Z,W)\eta(Y) - 2\eta(Y)\eta(Z)\eta(W)\}]$$
$$+\{5(Y\tilde{\alpha}) - 7(Y\beta)\}[\alpha\{g(\varphi X,W)\eta(Z) + g(\varphi Z,W)\eta(X)\} + \beta\{g(Y,W)\eta(Z)$$
$$+g(Z,W)\eta(Y) - 2\eta(Y)\eta(Z)\eta(W)\}].$$

Using the above equation, we obtain

$$16(\alpha^2+\beta^2)(CHR)(X,Y,X,W) = -16\beta\{\nabla_W(CHR)\}(X,Y,Z,\xi) \tag{2.9}$$
$$-16\alpha\{\nabla_{\varphi_W}(CHR)(X,Y,X,\xi)\} + 3\beta[3\{\nabla_W(X\alpha) + \nabla_W(\varphi X\beta)\}]g(\varphi Y,Z)$$
$$+3\alpha[3\{\nabla_{\varphi W}(X\alpha) + \nabla_{\varphi W}(\varphi X\beta)\}]g(\varphi Y,Z)$$
$$-3\beta[3\{\nabla_W(Y\alpha) + \nabla_W(\varphi Y\beta)\}]g(\varphi X,Z)$$
$$-3\alpha[3\{\nabla_{\varphi W}(Y\alpha) + \nabla_{\varphi W}(\varphi Y\beta)\}]g(\varphi X,Z)$$
$$+4\beta[\{\nabla_W(Z\alpha) - \nabla_W(\varphi Z\beta)\}]g(\varphi X,Y)$$
$$+4\alpha[\{\nabla_{\varphi W}(Z\alpha) - \nabla_{\varphi W}(\varphi Z\beta)\}]g(\varphi X,Y)$$
$$+\beta[5\{\nabla_W(\varphi X\alpha)\} - 7\{\nabla_W(X\beta)\}]\{g(Y,Z) - \eta(Y)\eta(Z)\}$$
$$+\alpha[5\{\nabla_{\varphi W}(\varphi X\alpha)\} - 7\{\nabla_{\varphi W}(X\beta)\}]\{g(Y,Z) - \eta(Y)\eta(Z)\}$$
$$+7\beta\{\nabla_W(\xi\beta)\}A(X,Y,Z) + 7\alpha\{\nabla_{\varphi W}(\xi\beta)\}A(X,Y,Z)$$
$$+\beta\{\nabla_W(\xi\alpha)\}\{9B(X,Y,Z) - 4g(\varphi X,Y)\eta(Z)\}$$
$$+\beta\{\nabla_{\varphi W}(\xi\alpha)\}\{9B(X,Y,Z) - 4g(\varphi X,Y)\eta(Z)\}$$
$$+(\alpha^2+\beta^2)\big[16(CHR)(X,Y,Z,\xi)\eta(W) - 4\{(Z\alpha) - (\varphi XZ)\}B(X,Y,W)$$
$$-\{5(\varphi X\alpha) - 7(X\beta)\}\{g(Y,W)\eta(Z) + g(Z,W)\eta(Y) - 2\eta(Y)\eta(Z)\eta(W)\}$$
$$+\{5(\varphi Y\alpha) - 7(Y\beta)\}\{g(X,W)\eta(Z) + g(Z,W)\eta(X) - 2\eta(X)\eta(Z)\eta(W)\}$$
$$+7(\xi\beta)\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W) - A(X,Y,Z)\eta(W)\}$$
$$-(\xi\alpha)[9\{g(X,W)g(\varphi Y,Z) - g(Y,W)g(\varphi X,,Z) + B(X,Y,W)\eta(Z)$$
$$+B(X,Y,Z)\eta(W)\} + 4B(X,Y,W)\eta(Z) + 4g(\varphi X,Y)\{g(Z,W) - \eta(Z)\eta(W)\}]\big].$$

3. (CHR)-recurrent curvature tensors

In this section, we recall the recurrent tensor field in a Riemannian manifold. Then we consider a trans-Sasakian manifold has the recurrent (CHR)-curvature tensor.

Definition 3.1. A tensor field T in a Riemannian manifold is called *reccurent* if it satisfies

$$\nabla_X T = \Pi(X)T,$$

for a certain 1-form Π which is called its *recurrent form.*

Let the (CHR) curvature tensor be recurrent with its recurrent form Π. Then, the (CHR)-curvature tensor is written by

$$\nabla_W(CHR)(X,Y,Z,W) = \Pi(W)(CHR)(X,Y,Z,W). \tag{3.1}$$

Then by virtue of (2.5) and (2.9), the (CHR)curvature tensor is written by

$$\begin{aligned}
16(\alpha^2+\beta^2)(CHR)(X,Y,Z,W) = &-\{\beta\Pi(W)+\alpha\Pi(\varphi W)\}[3\{(X\alpha)\\
&+(\varphi X\beta)\}g(\varphi Y,Z) - 3\{(Y\alpha)+(\varphi Y\beta)\}g(\varphi X,Z)\\
&+4\{(Z\alpha)-(\varphi Z\beta)\}g(\varphi X,Y) + \{5(\varphi X\alpha)-7(X\beta)\}\{g(Y,Z)-\eta(Y)\eta(Z)\}\\
&-\{5(\varphi Y\alpha)-7(Y\beta)\}\{g(X,Z)-\eta(X)\eta(Z)\} + 7(\xi\beta)A(X,Y,Z)\\
&+(\xi\alpha)\{9B(X,Y,Z)-4g(\varphi X,Y)\eta(Z)\}]\\
&+3[\beta\{3\nabla_W(Y\alpha)+\nabla_W(\varphi Y\beta)\}+\alpha\{3\nabla_{\varphi W}(Y\alpha)+\nabla_{\varphi W}(\varphi Y\beta)\}]g(\varphi X,Z)\\
&-3[\beta\{3\nabla_W(X\alpha)+\nabla_W(\varphi X\beta)\}+\alpha\{3\nabla_{\varphi W}(X\alpha)+\nabla_W(\varphi X\beta)\}]g(\varphi Y,Z)\\
&+4[\beta\{\nabla_W(Z\alpha)-\nabla_W(\varphi Z\beta)\}+\alpha\{\nabla_{\varphi W}(Z\alpha)-\nabla_{\varphi W}(\varphi Z\beta)\}]g(\varphi X,Y)\\
&+[\beta\{5\nabla_W(\varphi X\alpha)-7\nabla_W(X\beta)\}+\alpha\{5\nabla_{\varphi W}(\varphi X\alpha)-7\nabla_{\varphi W}(X\beta)\}]\{g(Y,Z)-\\
&\eta(Y)\eta(Z)\} - [\beta\{5\nabla_W(\varphi Y\alpha)-7\nabla_W(Y\beta)\}+\alpha\{5\nabla_{\varphi W}(\varphi Y\alpha)\\
&-7\nabla_{\varphi W}(Y\beta)\}]\{g(X,Z)-\eta(X)\eta(Z)\} - 7\{\beta\nabla_W(\xi\beta)+\alpha\nabla_{\varphi W}(\xi\beta)\}A(X,Y,Z)\\
&+\{\beta\nabla_W(\xi\alpha)+\alpha\nabla_{\varphi W}(\xi\alpha)\}\{9B(X,Y,Z)-4g(\varphi X,Y)\eta(Z)\}\\
&+(\alpha^2+\beta^2)\big[-\{(X\alpha)+(\varphi X\beta)\}B(Y,Z,W)+\{(Y\alpha)+(\varphi Y\beta)\}B(X,Z,W)\\
&-4\{(Z\alpha)-(\varphi Z\beta)\}B(X,Y,W) - \{5(\varphi X\alpha)-7(X\beta)\}\{g(Y,W)\eta(Z)+g(Z,W)\eta(Y)\\
&-\eta(Y)\eta(Z)\eta(W)\} + \{5(\varphi Y\alpha)-7(Y\beta)\}\{g(X,W)\eta(Z)+g(Z,W)\eta(X)\\
&-\eta(X)\eta(Z)\eta(W)\} + 7(\xi\beta)\{g(X,W)g(Y,Z)-g(X,Z)g(Y,W)-A(X,Y,Z)\eta(W)\}\\
&+(\xi\alpha)\{9\{g(X,W)g(\varphi Y,Z)-g(Y,W)g(\varphi X,Z)+B(X,Y,W)\eta(Z)\\
&+B(X,Y,Z)\eta(W)\} - 4B(X,Y,W)\eta(Z)+4g(\varphi X,Y)\{g(Z,W)-\eta(Z)\eta(W)\}\}\big].
\end{aligned} \tag{3.2}$$

From (2.4) and (3.2), the Riemannian curvature tensor R is given by

$$\begin{aligned}
16(\alpha^2+\beta^2)R(X,Y,Z,W) = &-\{\beta\Pi(W)+\alpha\Pi(\varphi W)\}[3\{(X\alpha)\\
&+(\varphi X\beta)\}g(\varphi Y,Z) - 3\{(Y\alpha)+(\varphi Y\beta)\}g(\varphi X,Z)\\
&+4\{(Z\alpha)-(\varphi Z\beta)\}g(\varphi X,Y) + \{5(\varphi X\alpha)-7(X\beta)\}\{g(Y,Z)-\eta(Y)\eta(Z)\}\\
&-\{5(\varphi Y\alpha)-7(Y\beta)\}\{g(X,Z)-\eta(X)\eta(Z)\} + 7(\xi\beta)A(X,Y,Z)\\
&+(\xi\alpha)\{9B(X,Y,Z)-4g(\varphi X,Y)\eta(Z)\}]\\
&+3[\beta\{3\nabla_W(Y\alpha)+\nabla_W(\varphi Y\beta)\}+\alpha\{\nabla_{\varphi W}(Y\alpha)+\nabla_{\varphi W}(\varphi Y\beta)\}]g(\varphi X,Z)\\
&-3[\beta\{3\nabla_W(X\alpha)+\nabla_W(\varphi X\beta)\}+\alpha\{3\nabla_{\varphi W}(X\alpha)+\nabla_W(\varphi X\beta)\}]g(\varphi Y,Z)\\
&+4[\beta\{\nabla_W(Z\alpha)-\nabla_W(\varphi Z\beta)\}+\alpha\{\nabla_{\varphi W}(Z\alpha)-\nabla_{\varphi W}(\varphi Z\beta)\}]g(\varphi X,Y)\\
&+[\beta\{5\nabla_W(\varphi X\alpha)-7\nabla_W(X\beta)\}+\alpha\{5\nabla_{\varphi W}(\varphi X\alpha)-7\nabla_{\varphi W}(X\beta)\}]\{g(Y,Z)-\\
&\eta(Y)\eta(Z)\} - [\beta\{5\nabla_W(\varphi Y\alpha)-7\nabla_W(Y\beta)\}+\alpha\{5\nabla_{\varphi W}(\varphi Y\alpha)\\
&-7\nabla_{\varphi W}(Y\beta)\}]\{g(X,Z)-\eta(X)\eta(Z)\} - 7\{\beta\nabla_W(\xi\beta)+\alpha\nabla_{\varphi W}(\xi\beta)\}A(X,Y,Z)\\
&+\{\beta\nabla_W(\xi\alpha)+\alpha\nabla_{\varphi W}(\xi\alpha)\}\{9B(X,Y,Z)-4g(\varphi X,Y)\eta(Z)\}
\end{aligned} \tag{3.3}$$

$$
\begin{aligned}
&+(\alpha^2+\beta^2)\big[-\{(X\alpha)+(\varphi X\beta)\}B(Y,Z,W)+\{(Y\alpha)+(\varphi Y\beta)\}B(X,Z,W)\\
&-4\{(Z\alpha)-(\varphi Z\beta)\}B(X,Y,W)-\{5(\varphi X\alpha)-7(X\beta)\}\{g(Y,W)\eta(Z)+g(Z,W)\eta(Y)\\
&-\eta(Y)\eta(Z)\eta(W)\}+\{5(\varphi Y\alpha)-7(Y\beta)\}\{g(X,W)\eta(Z)+g(Z,W)\eta(X)\\
&-\eta(X)\eta(Z)\eta(W)\}+7(\xi\beta)\{g(X,W)g(Y,Z)-g(X,Z)g(Y,W)-A(X,Y,Z)\eta(W)\}\\
&+(\xi\alpha)\{9\{g(X,W)g(\varphi Y,Z)-g(Y,W)g(\varphi X,Z)+B(X,Y,W)\eta(Z)+B(X,Y,Z)\eta(W)\}\\
&-4B(X,Y,W)\eta(Z)+4g(\varphi X,Y)\{g(Z,W)-\eta(Z)\eta(W)\}\}\big]\\
&-(X\alpha)\{7B(Z,W,Y)+4g(\varphi Z,W)\eta(Y)\}+(Y\alpha)\{7B(Z,W,X)+4g(\varphi Z,W)\eta(X)\}\\
&+(Z\alpha)\{7B(X,Y,W)+4g(\varphi X,Y)\eta(W)\}-(W\alpha)\{7B(X,Y,Z)+4g(\varphi X,Y)\eta(Z)\}\\
&+9\{(X\beta)A(Z,W,Y)-(Y\beta)A(Z,W,X)+(Z\beta)A(X,Y,W)-(W\beta)A(X,Y,Z)\}\\
&-5\{(\varphi X\alpha)A(Z,W,Y)-(\varphi Y)A(Z,W,X)+(\varphi Z\alpha)A(X,Y,W)-(\varphi W\alpha)A(X,Y,Z)\}\\
&+(\varphi X\beta)\{3B(Z,W,Y)+4g(\varphi Z,W)\eta(Y)\}+(\varphi Y\beta)\{3B(Z,W,X)-4g(\varphi Z,W)\eta(X)\}\\
&+(\varphi Z\beta)\{3B(X,Y,W,)+4g(\varphi X,Y)\eta(W)\}-(\varphi W\beta)\{3B(X,YZ)+4g(\varphi X,Y)\eta(Z)\}\\
&-2(\xi\beta)\{A(Z,W,Y)\eta(X)-A(Z,W,X)\eta(Y)\}+4(\alpha^2-\beta^2)[3\{g(X,W)g(Y,Z)\\
&-g(X,Z)g(Y,W)\}-g(\varphi X,W)g(\varphi Y,Z)+g(\varphi X,Z)g(\varphi Y,W)+2g(\varphi X,Y)g(\varphi Z,W)\\
&-A(Z,W,Y)\eta(X)+A(Z,W,X)\eta(Y)].
\end{aligned}
$$

Thus, we obtain obtain

Theorem 3.2. *If a real* $(2n+1)$*-dimensional trans-Sasakian manifold* M *of* (α,β)*-type with the* (φ,ξ,η,g) *has the recurrent* (CHR)*-curvature tensor of the recurrent form* Π*, then the* (CHR)*-curvature tensor and the Riemannian curvature tensor are respectively given by* (3.2) *and* (3.3).

Remark 3.3. From (3.2) (resp. (3.3)), if a trans-Sasakian manifold has the (CHR)-recurrent curvature tensor, then we can obtain the (CHR)-Ricci (resp. Ricci) tensor and the (CHR)-scalar (resp. the scalar) curvature.

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