SOME REMARKS ON RECTIFYING MATE CURVES

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Abstract. We classify mate curves which are rectifying and also study rectifying Bertrand curves.

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1. Preliminaries

Let $\rho : I \longrightarrow \mathbf{E^3}$ (where $I \subseteq \mathbf{R}$ is an interval and $\mathbf{E^3}$ is the 3-dimensional Euclidean space, endowed with the Euclidean scalar product, $< (x_1, x_2, x_3),\ (y_1, y_2, y_3) > = x_1y_1 + x_2y_2 + x_3y_3$) be a Frenet curve and denote by s its canonical parameter, i.e. $||\dot{\rho}(s)|| = 1$; then ρ is a unit speed curve.

At any point $\rho(s)$, there exists a Frenet basis $\{t(s), n(s), b(s)\}$ such that the following Frenet formulae hold:

$$\begin{cases} \dot{t}(s) = k(s)n(s), \\ \\ \dot{n}(s) = -k(s)t(s) + \tau(s)b(s), \\ \\ \dot{b}(s) = -\tau(s)n(s), \end{cases}$$

where $t(s) = \dot{\rho}(s)$ is the unit tangent vector field, $n(s)$ is the unit principal normal vector field, $b(s)$ is the unit binormal vector field, $b(s) = t(s) \times n(s)$, $k(s)$ is the curvature of $\rho(s)$, $\tau(s)$ is the torsion of $\rho(s)$ and dot denotes the first derivative.

Remark that for a Frenet curve $k(s) > 0$, $\forall s \in I$. We suppose $\tau(s) \neq 0$, i.e. the curve is not a plane curve.

A space curve $\rho : I \longrightarrow \mathbf{E^3}$ whose position vector always lies in its rectifying plane, i.e.

$$\rho(s) = \gamma(s)t(s) + \mu(s)b(s),$$

for some functions γ and μ, is called a *rectifying curve* (see [1]). Such curves were recently characterized by their involutes and evolutes by some of the present authors ([2]).

Recall that two curves ρ and ρ^* are called *Bertrand* curves if they have common principal normal lines in corresponding points M on ρ and M^* on ρ^*; then $n(s) = \pm n^*(s^*)$.

Motivated by the definition of Bertrand curves, in this paper we will consider mate space curves, $\rho(s)$ and $\rho^*(s^*)$, where s is the canonical parameter for ρ and, respectively, s^* is the canonical parameter of ρ^*, in the following situations (cases):

Case 1) $n(s) = \pm n^*(s^*)$

Case 2) $n(s) = \pm b^*(s^*)$

Case 3) $n(s) = \pm t^*(s^*)$

Case 4) $b(s) = \pm b^*(s^*)$

Case 5) $b(s) = \pm n^*(s^*)$

Case 6) $b(s) = \pm t^*(s^*)$

Case 7) $t(s) = \pm t^*(s^*)$

Case 8) $t(s) = \pm n^*(s^*)$

Case 9) $t(s) = \pm b^*(s^*)$,

where $\{t^*(s^*), b^*(s^*), n^*(s^*)\}$ is the Frenet basis of ρ^*.

Remark 1.1. we consider in all cases 1)-9) common lines.

Obviously, the above case **1)** is exactly the case of Bertrand curve mates.

From geometrical point of view, the Bertrand mates have the following two important properties:

Corollary 1.2. *The distance between corresponding points M on ρ and M^* on ρ^* is constant.*

Corollary 1.3. *The angle between corresponding tangent lines t and t^*, in M, respectively M^*, is constant.*

2. Rectifying mate curves

First we investigate the existence of such mate curves.

Theorem 2.1. *The cases* 3), 4), 6), 7) *and* 9) *are not possible.*

Proof. **Case 3)** $n(s) = \pm t^*(s^*)$
We can write

$$\rho^*(s^*) = \rho(s) + \alpha(s)n(s).$$

Then

$$\frac{d\rho^*(s^*)}{ds^*}\frac{ds^*}{ds} = (1 - \alpha(s)k(s))t(s) + \dot{\alpha}(s)n(s) + \alpha(s)\tau(s)b(s).$$

But $n(s) \perp t(s) \implies 1 - \alpha(s)k(s) = 0$ and $n(s) \perp b(s) \implies \alpha(s)\tau(s) = 0$. Because ρ is not a plane curve, in other words, $\tau \neq 0$, we get $\alpha(s) = 0$. Then we get $1 = 0$, contradiction.
Therefore there do not exist ρ and ρ^* satisfying the case 3).

Case 4) $b(s) = \pm b^*(s^*)$
We write

$$\rho^*(s^*) = \rho(s) + \beta(s)b(s).$$

Then

$$\frac{d\rho^*(s^*)}{ds^*}\frac{ds^*}{ds} = \dot{\rho}(s) + \dot{\beta}(s)b(s) + \beta(s)\dot{b}(s) =$$
$$= t(s) + \dot{\beta}(s)b(s) - \beta(s)\tau(s)n(s).$$

But $\frac{d\rho^*(s^*)}{ds^*} = t^*(s^*)$, so $t^*(s^*) \perp b^*(s^*) \implies t^*(s^*) \perp b(s)$.
Then $\dot{\beta}(s) = 0 \implies \beta(s) = \beta = constant$, i.e., $\rho^*(s^*) = \rho(s) + \beta b(s)$
It follows that

$$\frac{d\rho^*(s^*)}{ds^*}\frac{ds^*}{ds} = t(s) - \beta\tau(s)n(s).$$

Calculating the scalar product with $t(s)$ one gets $0 = 1$, contradiction.

Case 6) $b(s) = \pm t^*(s^*)$

We have

$$\rho^*(s^*) = \rho(s) + \beta(s)b(s).$$

Then

$$\frac{d\rho^*(s^*)}{ds^*}\frac{ds^*}{ds} = \dot{\rho}(s) + \dot{\beta}(s)b(s) + \beta(s)\dot{b}(s),$$

which implies

$$\frac{d\rho^*(s^*)}{ds^*}\frac{ds^*}{ds} = t(s) + \dot{\beta}(s)b(s) + \beta(s)(-\tau(s)n(s)).$$

Calculating the scalar product with $t(s)$, we get $0 = 1$, contradiction.

Case 7) $t(s) = \pm t^*(s^*)$
We have

$$\rho^*(s^*) = \rho(s) + \gamma(s)t(s).$$

Then

$$\frac{d\rho^*(s^*)}{ds^*}\frac{ds^*}{ds} = (1 + \dot{\gamma}(s))t(s) + \gamma(s))\dot{t}(s),$$

or equivalently

$$t^*(s^*)\frac{ds^*}{ds} = (1 + \dot{\gamma}(s))t(s) + \gamma(s))k(s)n(s).$$

It follows that $\gamma(s))k(s) = 0$. Since $k(s) \neq 0$, it follows that $\gamma(s) = 0$, i.e. $\rho^* = \rho$.

Case 9) $t(s) = \pm b^*(s^*)$
Then

$$\rho^*(s^*) = \rho(s) + \gamma(s)t(s) = (s + c + \gamma(s))t(s) + \mu b(s)$$

and

$$\frac{d\rho^*(s^*)}{ds^*}\frac{ds^*}{ds} = (1 + \dot{\gamma}s))t(s) + \gamma(s)k(s)n(s) = 0.$$

The same argument as in the case 7) implies $\rho^* = \rho$.

□

For the remaining cases, we ask the following question:

If ρ is a rectifying curve, when its mate, ρ^*, is a rectifying curve too? In case of a positive answer, under which conditions is the curve ρ^* rectifying?

Because ρ is rectifying, $\rho(s) = \lambda(s)t(s) + \mu(s)b(s)$.
For cases 1), 2), ρ^* can be expressed by

$$\rho^*(s^*) = \rho(s) + \alpha(s)n(s).$$

Then

$$\rho^*(s^*) = \lambda(s)t(s) + \mu(s)b(s) + \alpha(s)n(s).$$

Similarly, for case 5), ρ^* can be expressed by

$$\rho^*(s^*) = \lambda(s)t(s) + \mu(s)b(s) + \beta(s)b(s) = \lambda(s)t(s) + [\mu(s) + \beta(s)]b(s).$$

For case8), ρ^* can be expressed by

$$\rho^* s^*) = \lambda(s)t(s) + \mu(s)b(s) + \gamma(s)t(s) = [\lambda(s) + \gamma(s)]\, t(s) + \mu(s)b(s).$$

Remark 2.2. From **[1]**, one has $\lambda(s) = s + c$, where c is a constant and $\mu(s) = \mu$ =constant, i.e. ρ will be written as

$$\rho(s) = (s + c)t(s) + \mu b(s).$$

Case 1) $n(s) = \pm n^*(s^*)$ (Bertrand curves)
One can write $\rho^*(s^*) = \rho(s) + \alpha(s)n(s) = (s + c)t(s) + \mu b(s) + \alpha(s)n(s)$.

By differentiation, we obtain

$$\frac{d\rho^*(s^*)}{ds^*}\frac{ds^*}{ds} = \dot{\rho}(s) + \dot{\alpha}(s)n(s) + \alpha(s)\dot{n}(s) =$$

.

$$= t(s) + \dot{\alpha}(s)n(s) + \alpha(s)\left[-k(s)t(s) + \tau(s)b(s)\right] =$$
$$= (1 - \alpha(s)k(s))\, t(s) + \dot{\alpha}(s)n(s) + \alpha(s)\tau(s)b(s).$$

But $\frac{d\rho^*(s^*)}{ds^*} = t^*(s^*)$ which is orthogonal to $n^*(s^*)$, i.e. $t^*(s^*)$ is orthogonal to $n(s)$. Then $\dot{\alpha}(s) = 0 \implies \alpha(s) = \alpha = \mathit{constant} \neq 0$ ($\alpha = 0 \implies \rho^*(s^*) = \rho(s)$, α is the distance between corresponding points M and M^*).

We obtain $\rho^*(s^*) = (s + c)t(s) + \mu b(s) + \alpha n(s)$.

Then $< \rho^*(s^*), n^*(s^*) > = \pm < \rho^*(s^*), n(s) > = \alpha \neq 0$.

It follows that in Case 1), ρ^* is not a rectifying curve.

Remark 2.3. $\alpha(s) = \alpha = \mathit{constant}$ implies the distance between the corresponding points is constant (see Corollary 1.2).

Remark 2.4. For Bertrand curves, $\angle(t, t^*) = \mathit{constant}$ (see [3] and Corollary 1.3).

Case 2) $n(s) = \pm b^*(s^*)$

$$\rho^*(s^*) = \rho(s) + \alpha(s)n(s) \implies$$
$$\frac{d\rho^*(s^*)}{ds^*}\frac{ds^*}{ds} = (1 - \alpha(s)k(s))t(s) + \dot{\alpha}(s)n(s) + \alpha(s)\tau(s)b(s).$$

But $\frac{d\rho^*(s^*)}{ds^*} = t^*(s^*)$ orthogonal to $b^*(s^*)$, i.e. orthogonal to $n(s) \implies \dot{\alpha}(s) = 0 \implies \alpha(s) = \alpha = \mathit{constant} \implies \rho^*(s^*) = (s + c)t(s) + \mu b(s) + \alpha n(s) \implies < \rho^*(s^*), n^*(s^*) > =< (s + c)t(s) + \mu b(s) + \alpha n(s), n^*(s^*) > = (s + c) < t(s), n^*(s^*) > +\mu < b(s), n^*(s^*) > +\alpha < n(s), n^*(s^*) > = (s + c) < t(s), n^*(s^*) > +\mu < b(s), n^*(s^*) > +\alpha < \pm b^*(s^*), n^*(s^*) > = -\frac{1}{\tau^*(s^*)} < t(s), b^*(s^*) > -\frac{\mu}{\tau^*(s^*)} < b(s), b^*(s^*) > = \pm\frac{ds}{ds^*}\frac{1}{\tau^*(s^*)} \cdot [(s + c)k(s) - \mu\tau(s)] = 0$, by [1].

Therefore, ρ^* is always a rectifying curve.

Remark 2.5. $\alpha(s) = \alpha =$ constant implies that the distance between the corresponding points is constant.

Case 5) $b(s) = \pm n^*(s^*)$

$$\rho^*(s^*) = \rho(s) + \beta(s)b(s) \implies$$
$$\frac{d\rho^*(s^*)}{ds^*}\frac{ds^*}{ds} = \dot{\rho}(s) + \dot{\beta}(s)b(s) + \beta(s)\dot{b}(s) =$$
$$= t(s) + \dot{\beta}(s)b(s) - \beta(s)\tau(s)n(s).$$

But $t^*(s^*) \perp n^*(s^*) \implies t^*(s^*) \perp b(s) \implies \dot{\beta}(s) = 0 \implies \beta(s) = \beta = \mathit{constant}.$

This implies $\rho^*(s^*) = (s+c)t(s)+\mu b(s)+\beta b(s) = (s+c)t(s)+(\mu+\beta)b(s) = (s+c)t(s)\pm(\mu+\beta)n^*(s^*)$.

Therefore, $< \rho^*(s^*), n^*(s^*) > = \pm(\mu + \beta)$.

It follows that ρ^* is rectifying if and only if $\beta = -\mu \Leftrightarrow \rho^*(s) = (s + c)t(s)$.

Case 8) $t(s) = \pm n^*(s^*)$

Then

$$\rho^*(s^*) = \rho(s) + \gamma(s)t(s) = (s + c + \gamma(s))t(s) + \mu b(s)$$

and

$$\frac{d\rho^*(s^*)}{ds^*}\frac{ds^*}{ds} = (1 + \dot{\gamma}(s))t(s) + \gamma(s)k(s)n(s) = 0.$$

$$= (1 + \dot{\gamma}(s))t(s)(\pm n^*(s^*)) + (sk(s) + ck(s) + \gamma(s)k(s) - \mu\tau(s))n(s)$$
$$\Longrightarrow\ 1 + \dot{\gamma}(s) = 0 \ \Longrightarrow\ \gamma(s) = -s + d.$$

Thus, $\rho^*(s^*) = (c + d)t(s) + \mu b(s)$.
Computing the inner product, we have $< \rho^*(s^*), n^*(s^*) > = \pm(c+d)+\mu < b(s), \pm n^*(s^*) > = \pm(c+d)$.
So, ρ^* is rectifying $\Leftrightarrow$ $c + d = 0$, which implies $\gamma(s) = -s - c$.

To conclude this section and give answers to our question, we summarize the results in the following classification theorem.

Theorem 2.6. *Let $\rho : I \longrightarrow \mathbf{E^3}$ be a rectifying curve. Then:*
i) *Its mate ρ^* is not rectifying in case* 1).
ii) *Its mate ρ^* is always rectifying in case* 2).
iii) *Its mate ρ^* is rectifying in case* 5) *if and only if $\rho^*(s) = (s + c)t(s)$, with c a real constant.*
iv) *Its mate ρ^* is rectifying in case* 8) *if and only if $\rho^*(s^*) = \mu b(s)$, with μ a real constant.*

3. Rectifying Bertrand curves

As we have seen in the previous section, if ρ is rectifying then its Bertrand mate ρ^* is not rectifying, i.e. they can not be both rectifying.

A natural question is the following: **if ρ and ρ^* are Bertrand curves, is it possible for one of them to be rectifying?**

To answer this, we use once more Theorem 2 from [1] (see the section 2, proof of case 4), for its statement).

On the other hand, it is known (see [3]) that ρ and ρ^* are Bertrand if there exist α, β constants such that $\alpha k(s) + \beta\tau(s) = 1$, with $\alpha \neq 0$.

Then $\frac{1}{k(s)} = \alpha + \beta(c_1 s + c_2) = \beta c_1 s + \alpha + \beta c_2$. Therefore,

$$k(s) = \frac{1}{As + \beta} \ \Longrightarrow\ \tau(s) = \frac{c_1 s + c_2}{\beta c_1 s + \alpha + \beta c_2},$$

where $c_1 = \frac{1}{\mu}$, $c_2 = \frac{c}{\mu}$, $\mu \neq 0$.

Without loosing the generality, one can choose $\mu = 1$; this implies $c_1 = 1$; $c = 1 \implies c_2 = 1$ and $\alpha = \beta = 1$. Then $k(s) = \frac{1}{s+2}$ and $\tau(s) = \frac{s+1}{s+2}$.

It follows that we are looking for $\rho(s)$ with $\dot{\rho}(s) = t(s)$ and such that $t(s), n(s), b(s)$ are related by:

$$\begin{cases} \dot{t}(s) = \frac{1}{s+2} n(s), \\ \\ \dot{n}(s) = -\frac{1}{s+2} t(s) + (1 - \frac{1}{s+2}) b(s), \\ \\ \dot{b}(s) = (-1 + \frac{1}{s+2}) n(s). \end{cases} \tag{3.1}$$

By using the fundamental theorem of theory of curves, it follows that this system has an unique solution $\{t(s), n(s), b(s)\}$ up to some initial conditions (we also refer to the existence and uniqueness of the solutions of a system of differential equations).

Subtracting the first and second equation, we obtain

$$\dot{t}(s) - \dot{b}(s) = n(s), \tag{3.2}$$

and then

$$\dot{n}(s) = \ddot{t}(s) - \ddot{b}(s), \tag{3.3}$$

where double dots indicate the second derivative.

From the first equations of the system and from the relations (3.2) and (3.3), we obtain:

$$(3.4)\qquad \begin{cases} (1-\frac{1}{s+2})\dot{t}(s)+\frac{1}{s+2}\dot{b}(s)=0, \\ \ddot{t}(s)-\ddot{b}(s)=-\frac{1}{s+2}t(s)+(1-\frac{1}{s+2})b(s). \end{cases}$$

From the first equation of (3.4), we get

$$(3.5)\qquad \dot{t}(s)-\dot{b}(s)=(s+2)\dot{t}(s).$$

Using (3.5) in the second equation of the system (3.4), we have

$$(3.6)\qquad \dot{t}(s)+(s+2)\ddot{t}(s)=-\frac{1}{s+2}t(s)+\frac{s+1}{s+2}b(s)\Rightarrow b(s)=\frac{(s+2)^2}{s+1}\ddot{t}(s)+\frac{s+2}{s+1}\dot{t}(s)+\frac{1}{s+1}t(s).$$

By using (3.6) in the first equation of the system (3.4), we get

$$\frac{s+1}{s+2}\dot{t}(s)+\frac{1}{s+2}\left[\left(\frac{(s+2)^2}{s+1}\right)'\ddot{t}(s)+\frac{(s+2)^2}{s+1}\dddot{t}(s)+(\frac{s+2}{s+1})'\dot{t}(s)+\frac{s+2}{s+1}\ddot{t}(s)+(\frac{1}{s+1})'t(s)+\frac{1}{s+1}\dot{t}(s)\right]=0,$$

where three dots denote the third derivative.

By simplifying the terms, one obtains

$$(3.7)\qquad (s+1)(s+2)^2\dddot{t}(s)+(2s+1)(s+2)\ddot{t}(s)+\left[(s+1)^3+s\right]\dot{t}(s)-t(s)=0.$$

As a conclusion, the answer of the question posed at the beginning of this section is given by the following

Theorem 3.1 *Let $\rho : I \longrightarrow \mathbf{E^3}$ be a Bertrand curve. Then it is rectifying if and only if its tangent unit vector field $t(s)$ satisfies the differential equation (3.7).*

Remark 3.2. The solutions of the equation (3.7) determine a 3-dimensional linear space. The components of the vector t belong to this linear space.

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