NON-EXISTENCE OF A PARALLEL 2-FORM ON A REGULAR LORENTZIAN α-SASAKIAN MANIFOLD WITH COEFFICIENT α ENDOWED WITH RICCI SOLITON

LOVEJOY S. DAS

ABSTRACT. In 1926, Levy [11] had proved that a second order parallel non-singular tensor on a space of constant curvature is a constant multiple of the metric tensor. Sharma [14] has proved that a second order parallel tensor in a Kaehler space of constant holomorphic sectional curvature is a linear combination with constant coefficient of the Kaehlerian metric and the fundamental 2-form. In this paper, we have shown that a second order symmetric parallel tensor on a regular Lorentzian α-Sasakian manifold (briefly Lα-Sasakian) with coefficient α (non zero scalar function) is a constant multiple of the associated metric tensor and we have also proved that there does not exist a non zero parallel 2-form on a regular Lorentzian α-Sasakian manifold with a coefficient α.

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1. INTRODUCTION

On 1923, Eisenhart [9] showed that a Riemannian manifold admitting a second order symmetric parallel tensor other than a constant multiple of metric is reducible. In 1926 Levy [11] obtained the necessary and sufficient conditions for the existence of such tensors. Sharma [14] has generalized Levy's result by showing that a second order parallel (not necesarily symmetric and non-singular) tensor on an n-dimensional ($n > 2$) space of constant curvature is a constant multiple of the metric tensor. Recently the author [5] has proved that on a Para r-Sasakian manifold with a coefficient α, a second order symmetric parallel tensor is a constant multple of the associated positive definite Riemannian metric tensor. In this paper, we have defined a regular Lorentzian α-Sasakian manifold with a coefficient α (non-zero scalar function) and have proved the following theorems:

Theorem 1.1. *On a regular Lorentzian α-Sasakian manifold with a coefficient α, a second order symmetric parallel tensor is a constant multple of the associated metric tensor.*

Theorem 1.2. *On a regular Lorentzian α-Sasakian manifold with coefficient α, there is no non zero parallel 2-forms.*

Motivated by the works of Hamilton [10] towards the solution of the Poincare conjecture about the characterization of 3-sphere, many geometers have engaged themselves in providing the solutions of solitons of the Ricci flow.

The notion of a soliton structure on the Riemannian manifold (M,g) is the choice of a smooth vector field V on M and a real constant λ satisfying the structural requirement.

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{1.1}$$

where S is the Ricci tensor of the metric g and $\pounds_V g$ is the Lie Derivative in the direction of V and λ is referred to as the solition constant. The Ricci soliton is called expanding, steady or shinking if $\lambda > 0, \lambda = 0$ or $\lambda < 0$ respectively. In this paper, we prove that the tensor field $\pounds_V g + 2S$ on a Lorentzian α-Sasakian manifold with constant α is parallel then (g,V,λ) is a Ricci soliton.

2. Preliminaries:

Let C^∞ manifold M of dimension $2n+1$ is called a contact manifold if it carries a global 1-form A such taht $A \bigwedge (dA)^n \neq 0$. Let a contact manifold be endowed with (1,1) tensor field ϕ, a contravariant vector field T, a covarinant vector field A and a Lorentzian metric g on M, which makes T, a time like unit vector field such that the following conditions are satisfied [9]

$$A(T) = -1 \tag{2.1}$$

$$\phi(T) = 0 \tag{2.2}$$

$$A(\phi X) = 0 \tag{2.3}$$

$$\phi^2 X = X + A(X)T \tag{2.4}$$

$$A(X) = g(X,T) \tag{2.5}$$

$$g(\phi X, \phi Y) = g(X,Y) + A(X)A(Y) \tag{2.6}$$

$$\phi(X,Y) = g(X,\phi Y) = g(Y,\phi X) = \phi(Y,X) \tag{2.7}$$

$$\phi(X,T) = 0 \tag{2.8}$$

Definition 2.1. *If on a Lorentzian alpha α-Sasakian manifold, the following relations*

$$\phi X = -\frac{1}{\alpha}(\nabla_X T) \tag{2.9}$$

$$\nabla_X A(Y) = -\alpha g(\phi X, Y) = -\alpha \phi(X,Y) \tag{2.10}$$

$$\alpha(X) = \nabla_X \alpha = g(X, \overline{\alpha}) \tag{2.11}$$

$$(\nabla_X \phi)(Y,Z) = \alpha[\{g(X,Y) + A(X)A(Y) + g(X,Z) + A(X)A(Z)\}A(Y)] \tag{2.12}$$

hold, where ∇ denotes the Riemannian connection of the metric tensor g then M satisfying conditions (2.1) - (2.12) *is called a Lorentzian α-Sasakian manifold with a coefficient α.*

3. Proofs of Theorems 1.1 and 1.2:

In proving theorems 1.1 and 1.2, we need the following theorems.

Theorem 3.1. *On a Lorentzian α-Sasakian manifold with coefficient α, the following holds*

$$\begin{aligned} A(R(X,Y)Z) = {} & \alpha^2[g(Y,Z)A(X) - g(X,Z)A(Y)] \\ & -[\alpha(X)\phi(Y,Z) - \alpha(Y)\phi(X,Z)] \end{aligned} \tag{3.1}$$

Proof. On differentiating (2.10) covariantly and using (2.11) and (2.12) the proof follows immediately. □

Theorem 3.2. *For a Lorentzian α-Sasakian manifold with coefficient α, we have*

$$R(T,X)Y \quad = \alpha^2[g(X,Y)T - A(Y)X] + \alpha(Y)\phi X - \overline{\alpha}\phi(X,Y), \tag{3.2}$$

where $g(X,\overline{\alpha}) = \alpha(X)$.

Proof. The proof follows in an obvious manner after making use of (2.11) and (3.1). □

Theorem 3.3. *For a Lorentzian α-Sasakian manifold with coefficient α, the following holds*

$$R(T,X)T \quad = \beta\phi X + \alpha^2[X + A(X)T], \tag{3.3}$$

where $\alpha(T) = \beta$.

Proof. In view of equation (3.2), the proof follows immediately. □

4. Ricci Solitons and Second Order Parallel Symmetric Tensors

Proof of Theorem 1.1: Let h denote a $(0,2)$ tensor field on a Lorentzian α-Sasakian manifold M with coefficient α such that

$$h(R(W,X)Y,Z) + h(Y,R(W,X)Z) = 0, \tag{4.1}$$

for arbitrary vector fields X,Y,Z,W on M. Substituting $W = Y = Z = T$ in (4.1), we get

$$g(R(T,X)T,T) + g(T,R(T,X)T) = 0. \tag{4.2}$$

In view of Theorem 3.3, the above equation becomes

$$2\beta h(\phi X,T) + 2\alpha^2 h(X,T) + 2\alpha^2 g(X,T)h(T,T) = 0. \tag{4.3}$$

On simplifying (4.3), we get

$$g(X,T)h(T,T) + h(X,T) + \frac{\beta}{\alpha^2}h(\phi X,T) = 0 \tag{4.4}$$

Replacing X by ϕXin (4.4), we get

$$h(\phi Y,T) = -\frac{\alpha^2}{\beta}[A(Y)h(T,T) + h(Y,T)]. \tag{4.5}$$

Using (4.4) and (4.5), we get

$$h(T,T)A(Y) + h(Y,T) = 0 \ \text{ if } \ \alpha^4 - \beta^2 \neq 0. \tag{4.6}$$

Differentiating (4.6) covariantly with respect to Y, we get

$$h(T,T)g(X,\phi Y) + 2g(X,T)h(\phi X,T) + h(X,\phi Y) = 0. \tag{4.7}$$

In view of (2.9), the equation (4.7) assumes the following form

$$h(T,T)g(X,\phi Y) = -h(X,Y). \tag{4.8}$$

In view of the fact that $h(T,T)$ is constant along any vector on M, we have proved the theorem unless $\alpha^4 - \beta^2 \neq 0$.

Suppose that the $(0,2)$ type symmetric tensor field $\pounds_V g + 2S$ is parallel for any vector field V on a Lorentzian α- Sasakian manifold with coefficient α. Then by theorem 1.1 it follows that $\pounds_V g + 2S$ is a constant multiple of the metric tensor g since $\pounds_V g + 2S = -2\lambda g$ for all X,Y on M, where λ is a constant. Hence (1.1) holds. This shows that (g,V,λ) yields a Ricci Soliton. Hence we have the following theorem.

Theorem 4.1. *If the tensor field $\pounds_V g + 2S$ on a Lorentzian α-Sasakian manifold with a coefficient α, is parallel for any vector field V, then (g,V,λ) is a Ricci Soliton.*

Proof. Let (g,V,λ) be a Ricci Soliton on Lorentzian α-Sasakian manifold with a coefficient α. Then we have

$$(\pounds_T g)(Y,Z) + 2S(Y,Z) + 2\lambda g(Y,Z) = 0, \tag{4.9}$$

where $\pounds_T$ is the Lie Derivative along the vector field T on M. From (2.9), it follows that

$$\begin{aligned} (\pounds_T g)(Y,Z) &= g(\nabla_Y T,Z) + g(Y,\nabla_Z T) \\ &= -\alpha[g(\phi Y,Z) + g(Y,\phi Z)] \\ &= -2\alpha\phi(Y,Z) \end{aligned} \tag{4.10}$$

Using (4.10) in (4.9) we get

$$S(Y,Z) = \alpha\phi(Y,Z) - \lambda g(Y,Z), \tag{4.11}$$

where α and λ are non zero scalars. This shows that the manifold under consideration is nearly quasi-Einstein manifold [8]. Thus, we have the follwing theorem: □

Theorem 4.2. *If (g,T,λ) is Ricci Soliton on a Lorentzian α- Sasakian manifold M with a coefficient α, then M is nearly quasi-Einstein manifold.*

Proof of Theorem 1.2: Let h be a parallel 2-form on a Lorentzian α- Sasakian manifold M with a coefficient α. Then putting $W = Y = T$ in (4.1) and using theorem 3.3 and equations (2.1)-(2.12), we get

$$\begin{aligned} \beta h(Z,\phi X) + \alpha^2 h(X,Z) - \alpha^2 h(T,Z)A(X) + \alpha^2 h(T,X)A(Z) \\ + h(T,\phi X)\alpha(Z) + h(\overline{\alpha},T)\phi(X,Z) = 0 \end{aligned} \tag{4.12}$$

Let ϕ^* to be a (2,0) tensor field metrically equivalent to ϕ then contracting (4.12) with ϕ^* and using antisymmetric property of h and the symmetry property of ϕ^*, we obtain, in view of equations (2.3)-(2.6) and after simplifying, we get

$$h(\overline{\alpha},T) = 0. \tag{4.13}$$

Substituting (4.13) in (4.12) we get

$$\begin{aligned} \beta h(\phi X,Z) + \alpha^2[h(X,Z) - h(T,Z)A(X) + h(T,X)A(Z)] \\ + h(T,\phi X)\alpha(Z) = 0. \end{aligned} \tag{4.14}$$

On simplifying (4.14) we get

$$\beta h(\phi Z,X) - \alpha^2[h(Z,X) + h(T,X)A(Z) - h(T,Z)A(X)] + h(T,\phi Z)\alpha(X) = 0. \tag{4.15}$$

On simplifying (4.14) and (4.15) we get

$$\beta[h(Z,\phi X) + h(X,\phi Z)] + \alpha(X)h(\phi Z,T) + \alpha(Z)h(\phi X,T) = 0. \tag{4.16}$$

On replacing X by ϕY in (4.16), we get

$$\beta[h(Z,\phi^2 Y) + h(\phi Y,\phi Z)] + \alpha(\phi Y)h(\phi Z,T) + \alpha(Z)h(\phi^2 Y,T) = 0. \tag{4.17}$$

On making use of (2.4) in (4.17), we get

$$\beta[h(Z,Y) + h(Z,T)A(Y) + h(\phi Y,\phi Z)] + \alpha(Z)h(Y,T) + \alpha(\phi Y)h(\phi Z,T) = 0. \tag{4.18}$$

On simplifying (4.18), we get

$$\begin{aligned} \beta[h(Y,Z) + h(Y,T)A(Z) + h(\phi Z,\phi Y)] + \alpha(Y)h(Z,T) \\ + \alpha(\phi Z)h(\phi Y,T) = 0. \end{aligned} \tag{4.19}$$

In view of (4.18) and (4.19) on simplifying we obtain

$$\beta[h(T,Z)A(Y) + h(T,Y)A(Y)] - \alpha(Z)h(T,Y) - h(T,\phi Z)\alpha(\phi Y) - \alpha(Y)h(Z,T) - \alpha(\phi Z)h(T,\phi Y) = 0. \tag{4.20}$$

Putting $Y = \overline{\alpha}$ in (4.20) and using (4.13), we get

$$\beta[h(T,Z)A(\overline{\alpha}) - h(T,\phi Z)\alpha(\phi\overline{\alpha}) - \alpha(\overline{\alpha})h(Z,T) = 0 \tag{4.21}$$

Let us put $\alpha\overline{\alpha} = \hat{\alpha}$ and $\alpha(\phi\overline{\alpha}) = \hat{\beta}$ in (4.21), we get

$$h(Z,T)[\beta A(\overline{\alpha}) - \alpha(\overline{\alpha})] = h(T,\phi Z)\hat{\beta} \tag{4.22}$$

Replacing Z by ϕZ in (4.22), we get

$$h(\phi Z,T)[\beta^2 - \overline{\alpha}] = \hat{\beta}h(T,Z). \tag{4.23}$$

On simplifying (4.23) and replacing Z by ϕZ, we obtain

$$h(\phi^2 Z, T) = \frac{\hat{\beta}}{\overline{\alpha} - \beta^2} h(\phi Z, T). \tag{4.24}$$

On making use of (2.4) in (4.24), we get

$$\frac{\overline{\alpha} - \beta^2}{\hat{\beta}} h(Z, T) = \frac{\hat{\beta}}{\overline{\alpha} - \beta^2} h(Z, T). \tag{4.25}$$

From (4.25), it follows immediately that

$$h(Z, T) = 0 \text{ unless } (\overline{\alpha} - \beta^2)^2 - (\hat{\beta})^2 \neq 0. \tag{4.26}$$

Using (4.26) in (4.14), we get

$$\beta h(Z, \phi X) + \alpha^2 h(Z, X) = 0. \tag{4.27}$$

Differentiating (4.26) covariantly along Y and using the fact that $\nabla h = 0$, we get

$$h(Z, \phi Y) = 0. \tag{4.28}$$

In view of (4.28) and (4.27), we see that $h(Y, Z) = 0$, which completes the proof.

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DEPARTMENT OF MATHEMATICS, KENT STATE UNIVERSITY, NEW PHILIADEPHIA, OHIO, 44663, USA
Email address: ldas@kent.edu