HEURISTIC APPROACH TO NUMERICAL APPROXIMATIONS OF FUNCTIONS

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ABSTRACT. Numerical approximation of function is a classical mathematical problem and is most commonly obtained through series expansions. We investigate application of heuristic algorithms to finding optimal polynomial and rational approximations of elementary functions. Genetic algorithm, particle swarm optimization and differential evolution are implemented to find values of coefficients in such expansions which minimize maximum absolute error. Numerical precision of heuristic algorithms is compared with mathematical methods in terms of Chebysev and Chebysev-Padé approximations and we give advantages of applying heuristic approach in specific cases of function approximation.

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1. Introduction and motivation

Expressing functions in terms of basic arithmetic operations as addition and multiplication is a standard problem in mathematics. This is usually done by expanding function in power series and then, by taking finite number of terms in this expansion, we can obtain polynomial approximation of function with desired numerical precision. Theory of function approximation advanced rapidly in the second half of 20th century by development of modern computers. Necessity for implementing elementary mathematical functions (trigonometric functions, logarithmic functions etc.) in digital environment emphasized not only good numerical approximation, but also fast computation and short execution time. It was shown that expressing functions in terms of rational functions and continued fractions requires less arithmetic operations for value computation than in terms of polynomials. Hence, today there are many known algorithms and methods for expressing mathematical functions in terms of polynomials, rational functions and continued fractions, see e.g. [3, 6].

In the last few decades, heuristic algorithms gain much significance in computer science and are widely used in solving various complex (mathematical) problems, especially when exact algorithms are not known or the exhaustive search for the solution is too slow and impossible to implement in reasonable amount of time. Although they are mostly used in other areas of mathematics such as graph theory, there also may be application of heuristics in numerical approximation of functions. One can argue there is no point in using heuristics when the exact approximations are known and mathematically proved, but there is one notable advantage. Exact approximations rely on mathematical properties of functions, like continuity and differentiability, and they are not valid if these properties are not satisfied. Also, these approximations usually require lot of terms in the expansions to achieve desired precision and sometimes it can be impractical. On the other hand, heuristic algorithms find their solution based on the gained information of how close are they to the optimal solution and not on the properties of the functions. Therefore, they may be used for approximation of functions which do not have some nice properties like continuity and differentiability. Another common advantage of heuristics could be shorter computation time than with some exact methods.

Main aim of this paper is too apply heuristic approach to finding polynomial and rational approximation of mathematical functions and compare them to exact approximations. We will show advantages and disadvantages of this approach for exponential and logarithmic function, for trigonometric and inverse trigonometric functions, and also for absolute value function which is not differentiable on its domain.

2. Preliminaries and theoretical background

For a continuous function f on $[a,b]$, there exists a unique polynomial and unique rational function which is the best approximation of f in terms of absolute error. Approximations of this type are called minimax approximations and are described by the following equioscillation theorem, attributed to Chebyshev.

Theorem 2.1 ([3])**.** *Let $f(x)$ be a continuous function on $[a,b]$, and let V_n be set of all polynomials of order $\leq n$. Then there exists the unique polynomial $P_n^*(x) \in V_n$ such that*

$$\max_{[a,b]}\Big|P_n^*(x) - f(x)\Big| = \min_{P_n(x)\in V_n}\max_{[a,b]}\Big|P_n(x) - f(x)\Big|,$$

if and only if there are $n+2$ points

$$a \leq x_1^* < x_2^* < x_3^* < ... < x_{n+2}^* \leq b$$

such that

$$P_n(x_k^*) - f(x_k^*) = (-1)^k\mu^*, \quad k = 1,2,3,...,n+2,$$

where

$$|\mu^*| = \max_{[a,b]}\Big|P_n(x) - f(x)\Big|.$$

There is a similar statement for the minimax approximation with the unique rational function $R_{m,n}(x)$ if and only if there are $m+n+2-d$ points with the same property as above, where $d = \min(m-m', n-n')$, and m' and n' are degrees of polynomials in numerator and denominator after reducing the quotient (usually $d = 0$ since it is rarely possible to simplify the obtained rational function). But this theorem only says when the best approximation exists and not have to derive it. There are not any general algorithms for computing coefficients of the polynomials and rational functions in minimax approximations, but there are many iterative methods for finding these coefficients, the most famous one being the Remez algorithm. Since this algorithm solves the linear systems of equations in each step, complexity increases with the higher order of matrices and the problem for numerical stability is that its determinant is close to zero. Therefore, in this paper we will consider simpler methods which give solutions which are very close or even exact to minimax approximations of functions.

One of such methods is based on the Chebyshev polynomials which have an important role in approximation theory. The Chebysev polynomials $T_n(x)$ can be defined in various ways, for example through trigonometric functions as

$$T_n(x) = \cos(n \arccos x),$$

or via recursive relation:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n \geq 1,$$
$$T_0(x) = 1,\ T_1(x) = x.$$

The main property of Chebysev polynomials is that among all the polynomials of degree $n \geq 1$, with the leading coefficient 1, the polynomial $\frac{1}{2^{n-1}}T_n(x)$ is the one of which the maximal absolute value on $[-1,1]$ is minimal. Hence, they provide the closest polynomial approximation under the minimax criterion from Theorem 2.1. Another significant property of Chebysev polynomials is that they are orthogonal with respect to the weight $1/\sqrt{1-x^2}$ on $[-1,1]$. Therefore, they form a basis for expressing continuous function through expansion which is usually called Chebysev series:

$$f(x) = \sum_{n=0}^{\infty} c_n T_n(x),$$

where coefficients are obtained by inner product:

$$c_n = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x)T_n(x)}{\sqrt{1-x^2}}\, dx.$$

The Chebyshev series is an important tool in numerical analysis, especially in the spectral methods, and are often more favourable than Fourier trigonometric series due to generally faster convergence.

As we mentioned before, second approach we will investigate is approximation via rational functions. The technique of deriving rational function of a given order which agrees with the power series it is approximating, was developed by Henri Padé and obtained ration function is called Padé approximant. It often gives better and computationally faster approximation than polynomial approximations and therefore it is used extensively in computer calculations.

In general, let $f(x)$ be a given function, and let $m \geq 0$, $n \geq 1$. Then Padé approximant of order $[m,n]$ is a rational function of the form:

$$R_{m,n}(x) = \frac{P_m(x)}{Q_n(x)} = \frac{p_0 + p_1 x + p_2 x^2 + ... + p_m x^m}{1 + q_1 x + q_2 x^2 + ... + q_n x^n}, \tag{2.1}$$

which agrees with Taylor series of $f(x)$ in $m+n+1$ terms, that is

$$R(0) = f(0),\ R'(0) = f'(0),\ \ldots,\ R^{m+n}(0) = f^{m+n}(0).$$

It means that in order to find coefficients of Padé approximant, we need to solve $n+m+1$ system of linear equations. But to achieve even better approximation close to minimax approximation, in definition (2.1) we may replace powers x^k with Chebysev polynomials $T_k(x)$. This method is developed by Maehyl [5] and obtained rational function is called Chebysev-Padé approximant. It also acquires solving system of linear equations to find coefficients in this expansion, for details see [5].

To conclude, in this paper we will use Chebyshev series for polynomial approximation and Chebysev-Padé approximant for rational approximation and compare both of them with heuristic determination of the coefficients in their expansions.

3. Heuristic algorithms

For the heuristic approach we decided to use three methods: genetic algorithm, particle swarm optimization and differential evolution. All three are population-based optimization algorithms which are naturally more suitable for problems in continuous domain. Solution of algorithms is an (one-dimensional) array of real numbers which represents the coefficients in either polynomial or rational approximation and objective is to find coefficients which give approximation as close as possible to minimax approximation from Theorem 2.1. Hence, in all three cases the fitness function is maximal absolute error between the exact value of the observed function and the value of polynomial or rational approximation evaluated at the 100 points from the given interval, and the search for optimal solution is guided to minimize this error. Pseudocodes and description of each step in these algorithms will not be stated here since they are well-known, but we will mention some other specifics related to each algorithm.

Genetic algorithm (GA in the sequel). In the selection process, we have achieved the best results with 3-way tournament selection. Out of three randomly picked individuals, better two are chosen for crossover, and the third one is replaced with new offspring if it gives better fitness function value. For crossover operation we used combination of uniform and arithmetic crossover, and then on the newborn we have performed simple mutation by adding Gaussian noise. More precisely, for each real number in the array we added random number from Gaussian distribution controlled by standard deviation.

Particle swarm optimization (PSO in the sequel). For the neighbourhood topology we have implemented the ring topology due to its better exploration abilities and smaller chance of being trapped in local minima. This variant is also called the local-best PSO since the particle does not compare itself to the overall performance of the swarm, instead it is compared with its nearest neighbours. In our case, size of neighbourhood is 5% of the population. Optimization performance of the PSO depends on the two main parameters, the cognitive coefficient and social coefficient which usually have values in $[1, 3]$. For both coefficients we have used value 2.

Differential evolution (DE in the sequel). We have chosen random strategy for differentiation operator. For every base vector $\vec{a}$, we randomly pick two other agents $\vec{b}$ and $\vec{c}$ from the population. New agent is derived by linear combination $\vec{x} = \vec{a} + \lambda(\vec{b} - \vec{c})$, where we have used 0.5 for the value of parameter λ. Then we did an uniform crossover between $\vec{a}$ and $\vec{x}$ and compared the fitness function value of obtained agent. If it is better than of base vector, we replace them in the new population.

4. Results

In this section we will present results obtained by mathematical and heuristic methods for various classes of elementary functions. Aim is to derive coefficients of the Chebysev polynomial approximation and Chebysev-Padé rational approximation in two ways, mathematically as explained in Section 2., and heuristically by implementing algorithms (GA, PSO, DE) explained in Section 3. We will then numerically compare absolute error of obtained approximations.

Heuristic algorithms are implemented in *Java* using 128-bit precision for data. Mathematical methods and numerical errors are computed in *SageMath*.

4.1. **Trigonometric functions.** We will start with function $\cos x$ and approximate it with polynomial approximation of order 10 and with rational function of corresponding order $[m, n]$, where $m + n \leq 10$. Cosine function is defined on $\mathbb{R}$, but we may only study behaviour on interval $[-\pi/2, \pi/2]$ since other values can be easily obtained by trig. properties. Also values of the other trigonometric functions can be easily obtained by trig. identities.

Since cosine is even, polynomial approximation has only even powers:

$$P_{10}(x) = c_0 + c_2x^2 + c_4x^4 + c_6x^6 + c_8x^8 + c_{10}x^{10}. \tag{4.1}$$

Table 1. Coefficients in the polynomial approximation for $\cos x$

coeff	Chebysev	PSO	DE
c_0	0.99999999978	0.99999922570	0.999999999782
c_2	-0.4999999935695	-0.49999429105	-0.49999999362
c_4	0.04166663620893	0.041659842494	0.04166663639674
c_6	-0.0013888360842	-0.001385897682	-0.001388836306
c_8	$2.476013551 \cdot 10^{-5}$	$2.420513425 \cdot 10^{-5}$	$2.476024152 \cdot 10^{-5}$
c_{10}	$-2.60510763 \cdot 10^{-7}$	$-2.19819212 \cdot 10^{-7}$	$-2.60528259 \cdot 10^{-7}$

In Table 1 we show coefficients of the Chebysev approximation and also ones obtained by PSO and DE algorithms. Genetic algorithm did not give good enough results so we will not mention them here. Problem is that coefficients have a significant difference in order of magnitude, hence mutation operator with the Gaussian noise of the same order does not work well for coefficients of the very small order. Solution is to have separate parameters of mutations for each coefficient, but this is impractical to use, especially since other heuristic algorithms without mutation do not have this problem.

Maximum absolute error of Chebysev approximation is $9.2 \cdot 10^{-11}$ and DE gave the error of the same order $9.8 \cdot 10^{-11}$. Both errors are achieved in 12 points, therefore they are minimax polynomials which satisfy the Theorem 2.1. Maximum error of the PSO is $7.7 \cdot 10^{-7}$ and it is satisfied in only 5 points. This approximation is far from minimax, and also it is even greater than the error of the classic Taylor approximation. This is a common recurrence in the heuristic algorithms that they get stuck in the local optimum and cannot find better solution.

Corresponding rational approximation is of order $[5,5]$, but odd terms vanish and therefore we have:

$$R_{4,4}(x) = \frac{p_0 + p_2 x^2 + p_4 x^4}{1 + q_2 x^2 + q_4 x^4}. \tag{4.2}$$

Coefficients obtained by Chebysev-Padé approximation and by heuristic algorithms are in Table 2. Note that genetic algorithm did not have any problems for finding appropriate solution as for polynomial. It is interesting that differential algorithm gave the same coefficients as Chebysev-Padé up to 10 decimal places so we will not write them down separately.

Table 2. Coefficients in the rational approximation for $\cos x$

coeff	Chebysev-Padé/DE	GA	PSO
p_0	0.9999999277	0.99999335987	1.00000042477
p_2	-0.455247764051	-0.44530174108	-0.45471229598
p_4	0.02024927711	0.0162181212791	0.020032060847
q_2	0.0447507446063	0.05460608919985	0.0452852335741
q_4	$9.6290873 \cdot 10^{-4}$	0.0020532093602	0.0010193736010

As for polynomial approximation, Chebysev-Padé and DE gave the minimax solution with maximal error $6.5 \cdot 10^{-8}$ achieved at 11 points. Error of the GA is $6.4 \cdot 10^{-6}$ and achieved at 9 points, therefore this rational approximation does not satisfy minimax Theorem 2.1. PSO gave a slightly better error than GA, $6 \cdot 10^{-7}$, but worse than DE.

To conclude, because of the behaviour of $\cos x$, better approximations are obtained by polynomials where both Chebysev and heuristic DE gave minimax approximations with the error of the same order. Other algorithms could not find these approximations.

4.2. **Exponential function.** Next function we will analyse is e^x. Its domain is $\mathbb{R}$, but we will consider interval $[0, \log 2]$ which is usually enough for computer implementations since it can be easily expanded on $\mathbb{R}$, see [3] for details. And to simplify our calculations, we will approximate e^{-x} because of nicer numerical properties. This is also used to approximate hyperbolic functions which are defined through exponential function.

Since its series converges very quickly, we will find polynomial approximation of just order 6:

$$P_6(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6, \tag{4.3}$$

and the coefficients are in Table 3. Out of heuristic methods, genetic algorithm could not find a good approximation because of the same reasons as for trigonometric functions so we will not write it down. Differential evolution again gave the same coefficients as Chebysev polynomial up to 10 decimal places.

Table 3. Coefficients in the polynomial approximation for e^{-x}

coeff	Chebysev/DE	PSO
c_0	0.9999999986493	1.0000000135686
c_1	-0.99999980774	-1.000000933867
c_2	0.4999955219477	0.500010943557
c_3	-0.16662741017	-0.16671748379
c_4	0.041501740906	0.041753622103
c_5	-0.007973091922	-0.008303554984
c_6	0.0009863136	0.001149491058

Same as for the $\cos x$, Chebysev and De gave the minimax polynomial approximation with maximum error $1.33 \cdot 10^{-9}$ (at 8 points). PSO gave the error $1.36 \cdot 10^{-8}$ at 6 points, hence it does not satisfy Theorem 2.1.

Corresponding Chebysev-Padé rational approximation is of order $[3, 3]$:

$$R_{3,3}(x) = \frac{p_0 + p_1x + p_2x^2 + p_3x^3}{1 + q_1x + q_2x^2 + q_3x^3}, \tag{4.4}$$

and the coefficients are in the Table 4.

TABLE 4. Coefficients in the rational approximation for e^{-x}

coeff	Chebysev-Padé	GA	PSO	DE
p_0	1.000000000078	0.999999996858	1.000000006032	1.000000000065
p_1	-0.48290214352	-0.49999850358	-0.499999963001	-0.48311395737
p_2	0.091717576902	0.099992315061	0.099999924569	0.091816369409
p_3	-0.007032549362	-0.008320102777	-0.00832133324	-0.007047026335
q_1	0.51709786812	0.50000158707	0.50000046892	0.51688605288
q_2	0.1088151618047	0.0999914268	0.09999546267	0.108702164426
q_3	0.009902954158	0.0083497031833	0.0083596664672	0.0098812160168

Chebysev-Padé and DE gave the maximum absolute error $6.56 \cdot 10^{-11}$ at 8 points and represent the minimax rational approximation. Other two algorithms achieved the maximum error at 6 points, where PSO got a slightly better result with error $6.04 \cdot 10^{-9}$. It is better than polynomial approximation, but not as good as other rational approximations.

As it was expected, rational approximations are better for exponential function and again DE heuristic algorithm was able to find minimax solution which corresponds with the theoretical Chebysev-Padé approximation.

4.3. **Logarithmic function.** Function $\log x$ is defined on $(0, \infty)$ and there are several reducing techniques to smaller interval which is appropriate for approximation. The most common method is to approximate $\log(1 + x)$ on $[0, 1]$ which can be easily expanded to other arguments, see cited literature for details.

Coefficients for the polynomial approximations of order 8 are in Table 5.

TABLE 5. Coefficients in the polynomial approximation for $\log(1 + x)$

coeff	Chebysev	GA	PSO	DE
c_0	$3.386 \cdot 10^{-8}$	$-9.5796 \cdot 10^{-6}$	$-9.2049 \cdot 10^{-6}$	$2.9256 \cdot 10^{-6}$
c_1	0.9999942724	1.0004170364	1.00058212	0.99999485
c_2	-0.4998385618	-0.50268858	-0.50377552	-0.499850699
c_3	0.331548616	0.337736447	0.337691762	0.331645788
c_4	-0.23982616	-0.249431129	-0.241912765	-0.24020813
c_5	0.165822752	0.201888139	0.200243508	0.1666380807
c_6	-0.09325203	-0.179229191	-0.2109754564	-0.09421608
c_7	0.03484971247	0.12110375	0.165417875	0.035443233
c_8	-0.00615147096	-0.03664902567	-0.054122983	-0.006299909

We can see that Chebysev and DE gave similar result, they both maximal error at 10 points which corresponds to the minimax theorem, but value of maximum error for Chebysev is $3.4 \cdot 10^{-8}$ and for DE $2.95 \cdot 10^{-8}$. Hence, this is the first example that heuristic algorithm gave a slightly better solution closer to the minimax polynomial. Other two heuristic methods were not able to find minimax solution, they both have absolute error up to order 10^{-5} which is achieved at 6 points. It seems that they both get stuck in the local optimum because of the first term which is of much different order than the other coefficients. We may consider algorithms with different parameters for the first coefficient, but this leads to impractical implementations and we will stay consistent since we also did not do this in previous examples.

In the Table 6 we present the coefficients for the rational approximation of the corresponding order $[4,4]$.

TABLE 6. Coefficients in the rational approximation for $\log(1+x)$

coeff	Chebysev-Padé	GA	PSO	DE
p_0	$1.33572 \cdot 10^{-11}$	$1.34558 \cdot 10^{-9}$	$1.05377 \cdot 10^{-8}$	$6.22918 \cdot 10^{-12}$
p_1	0.99999999738	1.00000035565	0.99999956776	0.99999999855
p_2	1.29467105603	1.49999105861	1.50000054559	1.30313368728
p_3	0.43161313791	0.61898152449	0.61903091558	0.43854173027
p_4	0.02995600716	0.05949983100	0.05951397119	0.03085641103
q_1	1.794670970667	1.99999725465	1.99999870948	1.80313363351
q_2	0.995616395481	1.28561461762	1.28569373343	1.00677598757
q_3	0.179532990910	0.28571249014	0.28571830126	0.18319410464
q_4	0.006593708860	0.01424253445	0.01426045823	0.00681419203

GA and PSO algorithm again got stuck in the local optimum and could not find minimax solution, probably because of the same reason as for polynomial approximations. Their maximal error is $6.9 \cdot 10^{-9}$ at 7 points. Chebysev-Padé and DE found minimax approximations with maximum achieved at 10 points. Chebysev-Padé has maximum error of $1.34 \cdot 10^{-11}$, but DE is even better and has absolute error of $6.22 \cdot 10^{-12}$.

Of course, rational approximations are better than polynomial because of non-polynomial behaviour of $\log x$. Logarithmic function is the first example where heuristic approach gave better approximation than the mathematical methods, and DE was superior in finding optimal value than other heuristic algorithms.

4.4. **Inverse trigonometric function.** This subsection will be separated into two parts. First we will consider inverse tangent function and in the second part we will analyse inverse sine function.

4.4.1. Inverse tangent function $\arctan x$ is defined on $\mathbb{R}$ and the most common interval for approximation is $[-1,1]$ which can be expanded to whole domain via trig. identities.

Function is odd so polynomial approximation has only odd powers, coefficients up to order 9 are in Table 7.

TABLE 7. Coefficients in the polynomial approximation for $\arctan x$

coeff	Chebysev	GA	PSO	DE
c_1	0.999851323	0.999847356	0.999974819	0.999866672
c_3	-0.33010495	-0.329606723	-0.331413106	-0.330310479
c_5	0.179440488	0.177852666	0.1836193478	0.180181447
c_7	-0.08419846	-0.082864547	-0.089374757	-0.085186205
c_9	0.020419554	0.020188758	0.022607643	0.020858105

All the polynomial approximations have the maximum absolute error of the same order 10^{-5}, but only DE managed to find minimax polynomial. Its maximum absolute error is $1.13 \cdot 10^{-5}$, achieved at 11 points. GA and PSO have the slightly bigger error of the $2.9 \cdot 10^{-5}$ and $1.6 \cdot 10^{-5}$ respectively, but they are not minimax approximations satisfying Theorem 2.1.

Corresponding approximations in terms of rational functions are of order $[5,4]$ with the coefficients in next Table 8.

TABLE 8. Coefficients in the rational approximation for $\arctan x$

coeff	Chebysev-Padé	GA	PSO	DE
p_1	0.99999640279	1.00004875871	1.00002077458	0.999997532783
p_3	0.6506772997	0.834417664847	0.777699176656	0.655937372501
p_5	0.0396499964	0.0699589568	0.0617046327	0.0405569329
q_2	0.98392258974	1.1685114627	1.11137926916	0.98920187708
q_4	0.16826406867	0.25629312180	0.230642945032	0.170838103264

Chebysev-Padé approximation is close to minimax with maximum absolute value $2.95 \cdot 10^{-7}$, but DE was able to achieve minimax approximation with error $1.88 \cdot 10^{-7}$ (at 11 points). PSO has the maximum error of $2.02 \cdot 10^{-6}$ at 10 points, and GA $4.85 \cdot 10^{-6}$ at 8 points, hence both algorithms were not able to find minimax approximation. In case of $\arctan x$ rational functions gave better approximations than polynomial, and as for logarithmic function, DE found the best minimax approximation.

4.4.2. We will now analyse inverse sine function $\arcsin x$ which is defined on $[-1, 1]$. It is odd so it is enough to consider interval $[0, 1]$. Here for the first time we encounter problem with classic approximations since derivative of function is $1/\sqrt{1-x^2}$ and we have a bad behaviour about point $x = 1$. Coefficients have a big value and expansions converge slowly. For example, here is a Chebysev polynomial of order 6:

$$\begin{aligned} P_6(x) &= 0.00473644261 + 0.5928819698x + 5.57590370013x^2 - 27.39469819688x^3 \\ &\quad + 60.9998617130x^4 - 61.7822812948x^5 + 23.4980331938x^6. \end{aligned} \tag{4.5}$$

Heuristic algorithms does not consider properties of functions and they can found a better solution. For example, here is a polynomial obtained by differential evolution:

$$\begin{aligned} P_6(x) &= 0.0328289279966 - 1.4581931942x + 29.726925136x^2 - 131.3603214837x^3 \\ &\quad + 263.254833263x^4 - 243.092811112x^5 + 84.4347058615x^6. \end{aligned} \tag{4.6}$$

Chebysev polynomial has a maximum absolute error of $7.7 \cdot 10^{-2}$ which shows that this is far from approximations in previous cases. DE managed to found minimax approximation but with error $3.3 \cdot 10^{-2}$. This is better than Chebysev, but of course not good enough.

Same problems appear with rational approximations. Mathematical software cannot find Chebysev-Padé approximant because of insufficient numerical precision, and similar problem is even with superior Remez algorithm which does not converge because of numerical instability. Heuristic algorithms will always give some kind of solution which can vary in quality. For example, differential evolution gives following rational approximation:

$$R_{3,3}(x) = \frac{0.0037109394 + 0.92327391083x - 1.77875691x^2 + 0.8518702832x^3}{1 - 2.11448573440x + 1.2822337391x^2 - 0.1676853256x^3}. \tag{4.7}$$

This approximation has maximum absolute error of $3.8 \cdot 10^{-3}$ achieved at 8 points, therefore it is minimax approximation, and by one order it is better than polynomial. But the precision is still not satisfactory.

One solution is to add more terms in the polynomial or rational approximations, but the precision increases slowly and this becomes impractical to implement. The most common method to solve this problem is to approximate some surrogate function which behaves more nicely in the neighbourhood of the problematic points. It is usually done by adding some terms which cancel the problematic parts of derivative of original function. Therefore, one of the suitable functions to consider in our case is $\arcsin x + \sqrt{1-x^2}$. After finding its approximation, we just subtract the value of square root (which is easy to implement) and then obtain the wanted value of $\arcsin x$.

Hence, in the Table 9 are the coefficients in the polynomial approximation of order 6 for function $\arcsin x + \sqrt{1-x^2}$ on interval $[0, 1]$.

Again, DE find the minimax approximation with maximum absolute error $1.58 \cdot 10^{-4}$, and the Chebysev polynomial has error $2.85 \cdot 10^{-4}$ close to the minimax. We see that this is a few orders better than the

TABLE 9. Coefficients in the polynomial approximation for $\arcsin x$

coeff	Chebysev	GA	PSO	DE
c_0	0.9999390716	0.99997912529	0.99978076898	0.99984287172
c_1	1.00534378169	0.99569486745	1.00049133587	1.01233407152
c_2	-0.574678068393	-0.470181330810	-0.500086139120	-0.656872309003
c_3	0.54377349263	0.14570025519	0.23572723148	0.89704083768
c_4	-0.97788911207	-0.27621811353	-0.43524455768	-1.66387029799
c_5	0.953051107178	0.3709391412825	0.5192046447327	1.566739745999
c_6	-0.37845979828	-0.1948596410	-0.2488577264	-0.58426146486

original approximation of $\arcsin x$. Surprisingly, other two heuristic algorithms gave error even little bit smaller than the Chebysev polynomial, for GA is $2.58 \cdot 10^{-4}$ and for PSO $2.20 \cdot 10^{-4}$, but it is achieved at less points and it is not minimax approximation. Here we may also notice that despite having the similar error, the coefficients in the GA and PSO quite vary and such difference was not common in the previous examples where they found similar local optimum.

For the rational approximations of the corresponding order $[3, 3]$, we will present coefficients obtained by Chebysev-Padé (4.8) and differential evolution (4.9).

$$R_{3,3}(x) = \frac{1.00000280309 - 0.016548309632x - 1.6237586494x^2 + 0.7223553379x^3}{1 - 1.01629918321x - 0.11115080652x^2 + 0.17968399621x^3} \tag{4.8}$$

$$R_{3,3}(x) = \frac{0.99998278877 - 0.043012591713x - 1.65432314515x^2 + 0.7613108324x^3}{1 - 1.04340704675x - 0.11052959327x^2 + 0.194653047640x^3} \tag{4.9}$$

Genetic algorithm and PSO could not find approximation near the minimax approximation and the error was too large so we will not mention them here. Rational functions gave better approximation than polynomials and only DE found minimax approximation with maximum absolute error $1.73 \cdot 10^{-5}$.

Except this approach, there are also some other surrogate functions we may consider, see cited literature.

4.5. **Absolute value function.** As the last example, we will show application of heuristic methods to the function $|x|$ on interval $[-1, 1]$. Note that this function is not differentiable at $x = 0$ and we cannot directly find a minimax approximation that fits the Chebysev equioscillation theorem. Of course, calculations and implementations can be easily obtained separately for two cases, $-x$ on $[-1, 0]$ and x on $[0, 1]$, i.e. just for one of these cases since function is even. But this simple example is to show advantages of heuristic approach when function does not have nice properties on observed interval. Therefore, we will find polynomial approximation of the order 8 and rational approximation of the order $[4, 4]$ on the interval $[-1, 1]$.

In general, if function is not differentiable or has some discontinuity, it is not possible to apply exact mathematical methods, like Taylor series. But even superior method like Remez algorithm which may be applied, leads to numerical instability and does not give nice results. As already mentioned before, heuristic methods will not be concerned with this and they will always find some kind of solution.

Function is even, hence we only have even power in the expansion. Coefficients of the polynomial approximation are in Table 10 and for rational in Table 11. Heuristic algorithms gave the same coefficients up to 10 decimal places so we will not write them separately.

Chebysev polynomial is far from minimax approximation and has error of $7.08 \cdot 10^{-2}$. As expected, the largest error is about point $x = 0$. Heuristic algorithms succeed to find minimax approximation and the error $3.46 \cdot 10^{-2}$ is better than Chebysev.

For rational functions is similar as for polynomials. Chebysev-Padé gave the error $4.24 \cdot 10^{-2}$ which is only achieved at one point $x = 0$. Heuristic algorithms found the minimax approximation, for GA error is $8.92 \cdot 10^{-3}$ and for DE is $8.48 \cdot 10^{-3}$.

Table 10. Coefficients in the polynomial approximation for $|x|$

coeff	Chebysev	GA/PSO/DE
c_0	0.070733313	0.034574675
c_2	2.829450768	3.813185032
c_4	-5.658948828	-10.38764895
c_6	6.338045388	13.76790842
c_8	-2.586962449	-6.262593849

Table 11. Coefficients in the rational approximation for $|x|$

coeff	Chebysev-Padé	GA	PSO/DE
p_0	0.042437623430	0.008914482648	0.008471137273
p_2	5.941956965424	15.99923768712	16.77128359549
p_4	14.261027774	122.864808819	139.780241308
q_2	14.00060939647	74.77459443073	81.83190175923
q_4	5.25030469949	63.99677596166	75.0656662558

In this simple example we have shown that heuristic algorithms can be used to find better approximations of functions than applying exact mathematical approach.

5. Conclusion and final remarks

Numerical approximation of functions is one of basic problems in mathematics and its significance greatly increased by development of modern computers and necessity to implement functions in digital environment. Therefore various polynomial and rational expansion formulas for function approximation were obtained and studied. Their precision depends on how many terms we take, but calculating coefficients in these terms is not always an easy task. It usually involves iterative process with matrices of big order and may lack numerical stability. This especially happens when we try to approximate functions which have discontinuity or are not differentiable at all points on the observed interval.

Heuristic algorithms are commonly not applied to mathematical problems of these type where exact formulas exist, but they have a potential in numerical approximation of functions. As we have seen, heuristic approach was able to found the coefficients in the polynomial and rational minimax approximation for all classes of elementary functions, even when exact methods did not succeed. For the trigonometric and exponential function, differential evolution found the minimax solution with the maximum absolute error of the same order as Chebysev and Chebysev-Padé approximations, but for the logarithmic and the inverse trigonometric functions, differential evolution found the better solution with the smaller error. It was also able to find solution which satisfies minimax approximation properties when exact methods could not find it. Other heuristic algorithms sometimes found the minimax approximation but with greater error than differential evolution. Genetic algorithm and particle swarm optimization usually got stuck in the local optimum and they can be improved by implementing different parameters for each coefficient, but this is not practical in specific use with large number of coefficients. Advantages of the heuristic algorithms were especially observed in the case where function is not differentiable or does not have some other mathematical properties required for exact approach.

With the further development of the computer performance characteristics and further research in computer science and heuristic algorithms, we believe that application of heuristics might have significant application and importance in area of numerical approximation of functions.

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