COEFFICIENT ESTIMATES FOR SOME SUBCLASSES OF m-FOLD SYMMETRIC BI-UNIVALENT FUNCTIONS

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ABSTRACT. In the present paper, we considered two general subclasses $\mathcal{H}_{\Sigma,m}(\gamma,\mu,\lambda,\eta;\alpha)$ and $\mathcal{H}_{\Sigma,m}(\gamma,\mu,\lambda,\eta;\beta)$ of Σ_m, consisting of analytic and m-fold symmetric bi-univalent functions in the open unit disk $\mathbb{U}$. For functions belonging to the two classes introduced here, we derived estimates on the modulus of the initial coefficients a_{m+1} and a_{2m+1}. Several related classes were also considered and connections to earlier known results were made.

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1. Introduction and Definitions

Let $\mathcal{F}$ be the class of analytic functions defined on the open unit disk $\mathbb{U}=\{z:|z|<1\}$ and normalized under the condition $f(0)=0$ and $f'(0)=1$ in $\mathbb{U}$. An analytic function $f\in\mathcal{F}$ has Taylor's series expansion of the form:

$$f(z)=z+\sum_{j=2}^{\infty}a_jz^j \quad (z\in\mathbb{U}). \tag{1.1}$$

Further, let $\mathcal{S}$ denote the class of all functions in $\mathcal{F}$ consisting of form (1.1) which are univalent in $\mathbb{U}$. Some of the important and well-investigated subclasses of the univalent function class $\mathcal{S}$ include (for example) the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α in $\mathbb{U}$ and the class $\mathcal{K}(\alpha)$ of convex functions of order α $(0\le\alpha<1)$ in $\mathbb{U}$.

From Koebe one quarter theorem [5], it is well known that every function $f\in\mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z))=z \quad (z\in\mathbb{U})$$

and

$$f(f^{-1}(w))=w \quad (|w|<r_0(f);r_0(f)\ge 1/4),$$

where

$$f^{-1}(w)=g(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\cdots. \tag{1.2}$$

A function $f\in\mathcal{F}$ is said to be bi-univalent in $\mathbb{U}$ when both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). For each function $f\in\mathcal{S}$, the function

$$h(z)=\sqrt[m]{f(z^m)} \qquad (z\in\mathbb{U}\ ,\ m\in\mathbb{N}=\{1,2,3,...\}) \tag{1.3}$$

is univalent and maps the unit disk $\mathbb{U}$ in to a region with m-fold symmetry. A function is said to be m-fold symmetric [6, 8] if it has the following normalized from:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \qquad (z \in \mathbb{U}\ ,\ m \in \mathbb{N} = \{1, 2, 3, ...\}). \tag{1.4}$$

We denote by $\mathcal{S}_m$ the class of m-fold symmetric univalent function in $\mathbb{U}$, which are normalized by the series expansion (1.4). In fact, the functions in class $\mathcal{S}$ are one-fold symmetric. Brannan and Taha [3] introduced certain subclasses of bi-univalent function class Σ, namely bi-starlike functions of order α denoted by $\mathcal{S}^*_\Sigma(\alpha)$ and bi-convex functions of order α denoted by $\mathcal{K}_\Sigma(\alpha)$ corresponding to the function classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ respectively. Also, they determined non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. In [11], Srivastava et al. specified that m-fold symmetric bi-univalent function analogues to the concept of m-fold symmetric univalent function and these gave some important results, such as each function $f \in \Sigma$ generates an m-fold symmetric bi-univalent function for each $m \in \mathbb{N}$, in their study. Furthermore, for the normalized from of f given by (1.4) is concerned, they obtained the series expansion the expansion for f^{-1} as follows:

$$\begin{aligned} f^{-1}(w) = g(w) = w - a_{m+1}w^{m+1} + \left[(m+1)a_{m+1}^2 - a_{2m+1}\right] w^{2m+1} \\ - \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right] w^{3m+1} + \cdots . \end{aligned} \tag{1.5}$$

We denote by Σ_m the class of m-fold symmetric bi-univalent functions in $\mathbb{U}$. For $m = 1$, the formula (1.5) coincides with formula (1.2).

The functions $[-\log(1 - z^m)]^{\frac{1}{m}}$ and $\left(\frac{z^m}{1-z^m}\right)^{\frac{1}{m}}$ with the corresponding inverse functions $\left(\frac{e^{wm}-1}{e^{wm}}\right)^{\frac{1}{m}}$ and $\left(\frac{w^m}{1+w^m}\right)^{\frac{1}{m}}$, respectively, are some examples of m-fold symmetric bi-univalent functions.

Recently, many authors have obtained remarkable coefficient bounds for various subclasses of m-fold bi-univalent functions (see [1, 2, 4, 7, 10, 12, 13]).

In this paper, we derive estimates on the initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions belonging to the general subclasses $\mathcal{H}_{\Sigma,m}(\gamma, \mu, \lambda, \eta; \alpha)$ and $\mathcal{H}_{\Sigma,m}(\gamma, \mu, \lambda, \eta; \beta)$ of Σ_m, consisting of analytic and m-fold symmetric bi-univalent functions in the open unit disk $\mathbb{U}$. Also, some interesting applications of the results presented here are discussed.

Now, we introduce the following general subclasses of m-fold symmetric bi-univalent functions.

Definition 1.1. *A function f given by (1.4) is said to be in the class $\mathcal{H}_{\Sigma,m}(\gamma, \mu, \lambda, \eta; \alpha)$* $(\gamma \in \mathbb{C} - \{0\}, 0 \le \mu \le 1, 0 \le \lambda \le 1, 0 \le \eta \le 1, 0 < \alpha \le 1)$ *if the following conditions are satisfied: $f, g \in \Sigma$,*

$$\left| arg\left(1 + \frac{1}{\gamma}\left[\frac{zf'(z) + \mu z^2 f''(z)}{(1-\lambda)z + \lambda(1-\eta)f(z) + \eta z f'(z)} - 1\right]\right)\right| < \frac{\pi\alpha}{2} \ \ (z \in \mathbb{U}) \tag{1.6}$$

and

$$\left| arg\left(1 + \frac{1}{\gamma}\left[\frac{wg'(w) + \mu w^2 g''(w)}{(1-\lambda)w + \lambda(1-\eta)g(w) + \eta w g'(w)} - 1\right]\right)\right| < \frac{\pi\alpha}{2} \ \ (w \in \mathbb{U}) \tag{1.7}$$

where the funtion g is given by (1.5).

Definition 1.2. *A function f given by (1.4) is said to be in the class $\mathcal{H}_{\Sigma,m}(\gamma, \mu, \lambda, \eta; \beta)$* $(\gamma \in \mathbb{C} - \{0\}, 0 \le \mu \le 1, 0 \le \lambda \le 1, 0 \le \eta \le 1, 0 \le \beta < 1)$ *if the following conditions are satisfied: $f, g \in \Sigma$,*

$$Re\left(1 + \frac{1}{\gamma}\left[\frac{zf'(z) + \mu z^2 f''(z)}{(1-\lambda)z + \lambda(1-\eta)f(z) + \eta z f'(z)} - 1\right]\right) > \beta \ \ (z \in \mathbb{U}) \tag{1.8}$$

and

$$\mathrm{Re}\left(1+\frac{1}{\gamma}\left[\frac{wg'(w)+\mu w^2 g''(w)}{(1-\lambda)w+\lambda(1-\eta)g(w)+\eta w g'(w)}-1\right]\right)>\beta \ \ (w\in\mathbb{U}), \tag{1.9}$$

where the function g is given by (1.5).

Now we give the following lemma which will use in proving.

Lemma 1.3. *([5]) Let $p\in\mathcal{P}$ with $p(z)=1+p_1z+p_2z^2+\cdots$. Then*

$$|p_k|\leq 2 \qquad for \qquad k\geq 1. \tag{1.10}$$

Thus the m- fold symmetric function p in the class $\mathcal{P}$ is of the form

$$p(z)=1+p_mz^m+p_{2m}z^{2m}+p_{3m}z^{3m}+\cdots$$

(see [8]).

2. Coefficient estimates for the function classes $\mathcal{H}_{\Sigma,m}(\gamma,\mu,\lambda,\eta;\alpha)$

The first original results present the non-sharp estimates on the initial coefficient of functions in the class $\mathcal{H}_{\Sigma,m}(\gamma,\mu,\lambda,\eta;\alpha)$.

Theorem 2.1. *Let f given by (1.4) be in the class $\mathcal{H}_{\Sigma,m}(\gamma,\mu,\lambda,\eta;\alpha)$ $(\gamma\in\mathbb{C}-\{0\},0\leq\mu\leq 1,0\leq\lambda\leq 1,0\leq\eta\leq 1,0<\alpha\leq 1)$. Then*

$$|a_{m+1}|\leq\frac{2\alpha|\gamma|((\lambda-1)\eta-1)^2}{\sqrt{|\gamma\alpha((\lambda-1)\eta-1)\vartheta(\mu,\lambda,\eta)+(1-\alpha)\Theta^2(\mu,\lambda,\eta)|}} \tag{2.1}$$

and

$$|a_{2m+1}|\leq\frac{2(m+1)\alpha^2|\gamma|^2((\lambda-1)\eta-1)^4}{\vartheta^2(\mu,\lambda,\eta)}+\frac{2\alpha|\gamma|((\lambda-1)\eta-1)^2}{|\chi(\mu,\lambda,\eta)|}, \tag{2.2}$$

where

$$\begin{aligned}\Theta(\mu,\lambda,\eta)&=m\mu\left((\lambda-1)\eta-1\right)(m+1)+m(\lambda\eta-1)+\lambda-1,\\ \chi(\mu,\lambda,\eta)&=-2m\mu\left((\lambda-1)\eta-1\right)(2m+1)-2m(\lambda\eta-1)-\lambda+1,\\ \vartheta(\mu,\lambda,\eta)&=-2m\mu(m+1)\left((\lambda-1)\eta-1\right)\left[(\lambda-1)+m(\eta+2\left((\lambda-1)\eta-1\right))\right]\\ &\quad-2m^2\left[\left((\lambda-1)\eta-1\right)^2+\eta(\eta+2\left((\lambda-1)\eta-1\right))\right]\\ &\quad-m\left[4+\left((\lambda-1)\eta-1\right)(3\lambda+1)\right]+(\lambda-1)\left[\left((\lambda-1)\eta-1\right)-2(\lambda-1)\right].\end{aligned}$$

Proof. If $f\in\mathcal{H}_{\Sigma,m}(\gamma,\mu,\lambda,\eta;\alpha)$. Then

$$1+\frac{1}{\gamma}\left[\frac{zf'(z)+\mu z^2f''(z)}{(1-\lambda)z+\lambda(1-\eta)f(z)+\eta zf'(z)}-1\right]=[p(z)]^\alpha \tag{2.3}$$

and

$$1+\frac{1}{\gamma}\left[\frac{wg'(w)+\mu w^2g''(w)}{(1-\lambda)w+\lambda(1-\eta)g(w)+\eta wg'(w)}-1\right]=[q(w)]^\alpha, \tag{2.4}$$

where the function g is given by (1.5) and the functions $p(z),q(w)$ are in $\mathcal{P}$ which have the forms:

$$p(z)=1+p_mz^m+p_{2m}z^{2m}+p_{3m}z^{3m}+\cdots \tag{2.5}$$

and

$$q(w)=1+q_mw^m+q_{2m}w^{2m}+q_{3m}w^{3m}+\cdots . \tag{2.6}$$

Now, equating the coefficients in (2.3) and (2.4), we obtain

$$\left(\frac{m\mu\left((\lambda-1)\eta-1\right)(m+1)+m(\lambda\eta-1)+\lambda-1}{\gamma\left((\lambda-1)\eta-1\right)^2}\right)a_{m+1}^2=\alpha p_m \tag{2.7}$$

$$\left(\frac{(\lambda\eta-\lambda-\eta(m+1))(m\mu\left((\lambda-1)\eta-1\right)(m+1)+m(\lambda\eta-1)+\lambda-1)}{\gamma\left((\lambda-1)\eta-1\right)^3}\right)a_{m+1}^2$$

$$+\left(\frac{-2m\mu\left((\lambda-1)\eta-1\right)(2m+1)-2m(\lambda\eta-1)-\lambda+1}{\gamma\left((\lambda-1)\eta-1\right)^2}\right)a_{2m+1}=\alpha p_{2m}+\frac{\alpha(\alpha-1)}{2}p_m^2 \tag{2.8}$$

and

$$-\left(\frac{m\mu\left((\lambda-1)\eta-1\right)(m+1)+m(\lambda\eta-1)+\lambda-1}{\gamma\left((\lambda-1)\eta-1\right)^2}\right)a_{m+1}^2=\alpha q_m \tag{2.9}$$

$$\left(\frac{(\lambda\eta-\lambda-\eta(m+1))(m\mu\left((\lambda-1)\eta-1\right)(m+1)+m(\lambda\eta-1)+\lambda-1)}{\gamma\left((\lambda-1)\eta-1\right)^3}\right)a_{m+1}^2$$

$$+\left(\frac{-2m\mu\left((\lambda-1)\eta-1\right)(2m+1)-2m(\lambda\eta-1)-\lambda+1}{\gamma\left((\lambda-1)\eta-1\right)^2}\right)\left[(m+1)a_{m+1}^2-a_{2m+1}\right]=\alpha q_{2m}+\frac{\alpha(\alpha-1)}{2}q_m^2. \tag{2.10}$$

From (2.7) and (2.9), we get

$$a_{m+1}=\frac{\gamma\left((\lambda-1)\eta-1\right)^2\alpha p_m}{m\mu\left((\lambda-1)\eta-1\right)(m+1)+m(\lambda\eta-1)+\lambda-1}$$

$$=-\frac{\gamma\left((\lambda-1)\eta-1\right)^2\alpha q_m}{m\mu\left((\lambda-1)\eta-1\right)(m+1)+m(\lambda\eta-1)+\lambda-1}, \tag{2.11}$$

which implies

$$p_m=-q_m \tag{2.12}$$

and

$$2\left(\frac{m\mu\left((\lambda-1)\eta-1\right)(m+1)+m(\lambda\eta-1)+\lambda-1}{\gamma\left((\lambda-1)\eta-1\right)^2}\right)^2a_{m+1}^2=\alpha^2(p_m^2+q_m^2). \tag{2.13}$$

Now, from (2.8) and (2.10), we have

$$\frac{1}{\gamma((\lambda-1)\eta-1)^3}\left[-2m\mu(m+1)\left((\lambda-1)\eta-1\right)\left[(\lambda-1)+m(\eta+2\left((\lambda-1)\eta-1\right))\right]\right.$$
$$-2m^2\left[\left((\lambda-1)\eta-1\right)^2+\eta(\eta+2\left((\lambda-1)\eta-1\right))\right]$$
$$\left.-m\left[4+\left((\lambda-1)\eta-1\right)(3\lambda+1)\right]+(\lambda-1)\left[\left((\lambda-1)\eta-1\right)-2(\lambda-1)\right]\right]a_{m+1}^2$$
$$=\alpha(p_{2m}+q_{2m})+\frac{\alpha(\alpha-1)}{2}\left(p_m^2+q_m^2\right)$$
$$=\alpha(p_{2m}+q_{2m})+\frac{(\alpha-1)}{2}\left(\frac{2(m\mu\left((\lambda-1)\eta-1\right)(m+1)+m(\lambda\eta-1)+\lambda-1)^2}{\alpha\gamma^2\left((\lambda-1)\eta-1\right)^4}\right)a_{m+1}^2. \tag{2.14}$$

Therefore, we obtain

$$a_{m+1}^2=\frac{\alpha^2\gamma^2((\lambda-1)\eta-1)^4(p_{2m}+q_{2m})}{\gamma\alpha((\lambda-1)\eta-1)\vartheta(\mu,\lambda,\eta)+(1-\alpha)\Theta^2(\mu,\lambda,\eta)}. \tag{2.15}$$

Applying Lemma 1.3 for the coefficients p_{2m} and q_{2m}, we immediately have

$$|a_{m+1}|\leq\frac{2\alpha|\gamma|((\lambda-1)\eta-1)^2}{\sqrt{\gamma\alpha((\lambda-1)\eta-1)\vartheta(\mu,\lambda,\eta)+(1-\alpha)\Theta^2(\mu,\lambda,\eta)}}. \tag{2.16}$$

This gives the desired bound for $|a_{m+1}|$ as asserted in (2.1).

Next, in order to find the bound on $|a_{2m+1}|$ by subtracting (2.10) from (2.8), we get

$$\frac{\chi(\mu,\lambda,\eta)}{\gamma((\lambda-1)\eta-1)^2}[2a_{2m+1}-(m+1)a_{m+1}^2]=\alpha(p_{2m}-q_{2m})+\frac{\alpha(\alpha-1)}{2}(p_m^2-q_m^2). \tag{2.17}$$

It follows from (2.11), (2.12) and (2.16), that

$$a_{2m+1}=\frac{(m+1)\alpha^2\gamma^2((\lambda-1)\eta-1)^4(p_{2m}^2+q_{2m}^2)}{4\vartheta^2(\mu,\lambda,\eta)}+\frac{\alpha\gamma((\lambda-1)\eta-1)^2(p_{2m}-q_{2m})}{2\chi(\mu,\lambda,\eta)}. \tag{2.18}$$

Applying Lemma 1.3 for the coefficients p_m, p_{2m} and q_m, q_{2m}, we readily obtain

$$|a_{2m+1}|\leq\frac{2(m+1)\alpha^2|\gamma|^2((\lambda-1)\eta-1)^4}{\vartheta^2(\mu,\lambda,\eta)}+\frac{2\alpha|\gamma|((\lambda-1)\eta-1)^2}{|\chi(\mu,\lambda,\eta)|}. \tag{2.19}$$

So, the proof of theorem is completed. ☐

3. Coefficient estimates for the function classes $\mathcal{H}_{\Sigma,m}(\gamma,\mu,\lambda,\eta;\beta)$

This section is devoted to finding the non-sharp estimates on the coefficients a_{m+1} and a_{2m+1} for functions in the class $\mathcal{H}_{\Sigma,m}(\gamma,\mu,\lambda,\eta;\beta)$.

Theorem 3.1. *Let f given by (1.4) be in the class $\mathcal{H}_{\Sigma,m}(\gamma,\mu,\lambda,\eta;\beta)$ $(\gamma\in\mathbb{C}-\{0\}, 0\leq\mu\leq1, 0\leq\lambda\leq1, 0\leq\eta\leq1, 0\leq\beta<1)$. Then*

$$|a_{m+1}|\leq\sqrt{\frac{4|\gamma|((\lambda-1)\eta-1)^3(1-\beta)}{|\vartheta(\mu,\lambda,\eta)|}} \tag{3.1}$$

and

$$|a_{2m+1}|\leq\frac{2(m+1)|\gamma|^2((\lambda-1)\eta-1)^4(1-\beta)^2}{\Theta^2(\mu,\lambda,\eta)}+\frac{2|\gamma|((\lambda-1)\eta-1)^2(1-\beta)}{|\chi(\mu,\lambda,\eta)|}, \tag{3.2}$$

where

$$\begin{aligned}\Theta(\mu,\lambda,\eta)&=m\mu\left((\lambda-1)\eta-1\right)(m+1)+m(\lambda\eta-1)+\lambda-1,\\ \chi(\mu,\lambda,\eta)&=-2m\mu\left((\lambda-1)\eta-1\right)(2m+1)-2m(\lambda\eta-1)-\lambda+1,\\ \vartheta(\mu,\lambda,\eta)&=-2m\mu(m+1)\left((\lambda-1)\eta-1\right)\left[(\lambda-1)+m(\eta+2\left((\lambda-1)\eta-1\right))\right]\\ &\quad-2m^2\left[\left((\lambda-1)\eta-1\right)^2+\eta(\eta+2\left((\lambda-1)\eta-1\right))\right]\\ &\quad-m\left[4+\left((\lambda-1)\eta-1\right)(3\lambda+1)\right]+(\lambda-1)\left[\left((\lambda-1)\eta-1\right)-2(\lambda-1)\right].\end{aligned}$$

Proof. If $f\in\mathcal{H}_{\Sigma,m}(\gamma,\mu,\lambda,\eta;\beta)$. Then

$$1+\frac{1}{\gamma}\left[\frac{zf'(z)+\mu z^2f''(z)}{(1-\lambda)z+\lambda(1-\eta)f(z)+\eta zf'(z)}-1\right]=\beta+(1-\beta)p(z) \tag{3.3}$$

and

$$1+\frac{1}{\gamma}\left[\frac{wg'(w)+\mu w^2g''(w)}{(1-\lambda)w+\lambda(1-\eta)g(w)+\eta wg'(w)}-1\right]=\beta+(1-\beta)q(w), \tag{3.4}$$

where the function g is given by (1.5) and the functions $p(z), q(w)$ are in $\mathcal{P}$ which have the forms (2.5) and (2.6), respectively.

Now, equating the coefficients in (3.3) and (3.4), we obtain

$$\left(\frac{m\mu\left((\lambda-1)\eta-1\right)(m+1)+m(\lambda\eta-1)+\lambda-1}{\gamma\left((\lambda-1)\eta-1\right)^2}\right)a_{m+1}^2=(1-\beta)p_m \tag{3.5}$$

$$\left(\frac{(\lambda\eta-\lambda-\eta(m+1))(m\mu\left((\lambda-1)\eta-1\right)(m+1)+m(\lambda\eta-1)+\lambda-1)}{\gamma\left((\lambda-1)\eta-1\right)^3}\right)a_{m+1}^2$$

$$+\left(\frac{-2m\mu\left((\lambda-1)\eta-1\right)(2m+1)-2m(\lambda\eta-1)-\lambda+1}{\gamma\left((\lambda-1)\eta-1\right)^2}\right)a_{2m+1}=(1-\beta)p_{2m} \tag{3.6}$$

and

$$-\left(\frac{m\mu\left((\lambda-1)\eta-1\right)(m+1)+m(\lambda\eta-1)+\lambda-1}{\gamma\left((\lambda-1)\eta-1\right)^2}\right)a_{m+1}^2=(1-\beta)q_m \tag{3.7}$$

$$\left(\frac{(\lambda\eta-\lambda-\eta(m+1))(m\mu\left((\lambda-1)\eta-1\right)(m+1)+m(\lambda\eta-1)+\lambda-1)}{\gamma\left((\lambda-1)\eta-1\right)^3}\right)a_{m+1}^2$$

$$+\left(\frac{-2m\mu\left((\lambda-1)\eta-1\right)(2m+1)-2m(\lambda\eta-1)-\lambda+1}{\gamma\left((\lambda-1)\eta-1\right)^2}\right)\left[(m+1)a_{m+1}^2-a_{2m+1}\right]=(1-\beta)q_{2m}. \tag{3.8}$$

From (3.5) and (3.7), we get

$$p_m=-q_m \tag{3.9}$$

and

$$2\left(\frac{m\mu\left((\lambda-1)\eta-1\right)(m+1)+m(\lambda\eta-1)+\lambda-1}{\gamma\left((\lambda-1)\eta-1\right)^2}\right)^2a_{m+1}^2=(1-\beta)^2(p_m^2+q_m^2). \tag{3.10}$$

Adding (3.6) and (3.8), we have

$$\frac{1}{\gamma\left((\lambda-1)\eta-1\right)^3}\vartheta(\mu,\lambda,\eta)a_{m+1}^2=(1-\beta)(p_{2m}+q_{2m}). \tag{3.11}$$

Therefore, we get

$$a_{m+1}^2=\frac{\gamma\left((\lambda-1)\eta-1\right)^4(1-\beta)(p_{2m}+q_{2m})}{\vartheta(\mu,\lambda,\eta)}. \tag{3.12}$$

Applying Lemma 1.3 for the coefficients p_{2m} and q_{2m}, we immediately have

$$|a_{m+1}|\leq\sqrt{\frac{4|\gamma|\alpha((\lambda-1)\eta-1)^3(1-\beta)}{|\vartheta(\mu,\lambda,\eta)|}} \tag{3.13}$$

This gives the desired bound for $|a_{m+1}|$ as asserted in (3.1).

Now, in order to find the bound on $|a_{2m+1}|$ by subtracting (3.8) from (3.6), we have

$$a_{2m+1}=\frac{(m+1)}{2}a_{m+1}^2+\frac{\gamma((\lambda-1)\eta-1)^2(p_{2m}-q_{2m})(1-\beta)}{2\chi(\mu,\lambda,\eta)}. \tag{3.14}$$

It follows from (3.12) that

$$a_{2m+1}=\frac{(m+1)\gamma^2((\lambda-1)\eta-1)^4(1-\beta)^2(p_m^2+q_m^2)}{4\Theta^2(\mu,\lambda,\eta)}+\frac{\gamma((\lambda-1)\eta-1)^2(1-\beta)(p_{2m}-q_{2m})}{2\chi(\mu,\lambda,\eta)}. \tag{3.15}$$

Applying Lemma 1.3 for the coefficients p_m, p_{2m} and q_m, q_{2m}, we easily obtain

$$|a_{2m+1}| \leq \frac{2(m+1)|\gamma|^2((\lambda-1)\eta-1)^4(1-\beta)^2}{\Theta^2(\mu,\lambda,\eta)} + \frac{2|\gamma|((\lambda-1)\eta-1)^2(1-\beta)}{|\chi(\mu,\lambda,\eta)|}. \tag{3.16}$$

So, the proof of theorem is completed. □

4. Applications of the Main Results

For one-fold symmetric bi-univalent functions, Theorems 2.1 and 3.1 reduce to the following corollaries, respectively, which were obtained by Srivastava et al. in [9].

Corollary 4.1. *Let f given by (1.1) be in the class $\mathcal{H}_{\Sigma,1}(\gamma,\mu,\lambda,\eta;\alpha) \equiv \mathcal{H}_{\Sigma}(\gamma,\mu,\lambda,\eta;\alpha)$ $(\gamma \in \mathbb{C}-\{0\}, 0 \leq \mu \leq 1, 0 \leq \lambda \leq 1, 0 \leq \eta \leq 1, 0 < \alpha \leq 1)$. Then*

$$|a_2| \leq \frac{2\alpha|\gamma|((\lambda-1)\eta-1)^2}{\sqrt{|\gamma\alpha((\lambda-1)\eta-1)\Omega(\mu,\lambda,\eta)+(1-\alpha)\Lambda^2(\mu,\lambda,\eta)|}} \tag{4.1}$$

and

$$|a_3| \leq \frac{4\alpha^2|\gamma|^2((\lambda-1)\eta-1)^4}{\Lambda^2(\mu,\lambda,\eta)} + \frac{2\alpha|\gamma|((\lambda-1)\eta-1)^2}{|\Xi(\mu,\lambda,\eta)|}, \tag{4.2}$$

where

$$\begin{aligned}
\Lambda(\mu,\lambda,\eta) &= 2\mu\left((\lambda-1)\eta-1\right)+(\lambda(\eta+1)-2,\\
\Xi(\mu,\lambda,\eta) &= -6\mu\left((\lambda-1)\eta-1\right)-\lambda(2\eta+1)+3,\\
\Omega(\mu,\lambda,\eta) &= -4\mu\left((\lambda-1)\eta-1\right)\left[(\lambda-1)+(\eta+2\left((\lambda-1)\eta-1\right))\right]\\
&\quad -2\left[\left((\lambda-1)\eta-1\right)^2+\eta(\eta+2\left((\lambda-1)\eta-1\right))\right]\\
&\quad -\left[4+\left((\lambda-1)\eta-1\right)(3\lambda+1)\right]+(\lambda-1)\left[\left((\lambda-1)\eta-1\right)-2(\lambda-1)\right].
\end{aligned}$$

Corollary 4.2. *Let f given by (1.1) be in the class $\mathcal{H}_{\Sigma,1}(\gamma,\mu,\lambda,\eta;\beta) \equiv \mathcal{H}_{\Sigma}(\gamma,\mu,\lambda,\eta;\beta)$ $(\gamma \in \mathbb{C}-\{0\}, 0 \leq \mu \leq 1, 0 \leq \lambda \leq 1, 0 \leq \eta \leq 1, 0 \leq \beta < 1)$. Then*

$$|a_2| \leq \sqrt{\frac{4|\gamma|(1-\beta)\left|((\lambda-1)\eta-1)^3\right|}{|\Omega(\mu,\lambda,\eta)|}} \tag{4.3}$$

and

$$|a_3| \leq \frac{4|\gamma|^2((\lambda-1)\eta-1)^4(1-\beta)^2}{\Lambda^2(\mu,\lambda,\eta)} + \frac{2|\gamma|((\lambda-1)\eta-1)^2(1-\beta)}{|\Xi(\mu,\lambda,\eta)|}, \tag{4.4}$$

where

$$\begin{aligned}
\Lambda(\mu,\lambda,\eta) &= 2\mu\left((\lambda-1)\eta-1\right)+(\lambda(\eta+1)-2,\\
\Xi(\mu,\lambda,\eta) &= -6\mu\left((\lambda-1)\eta-1\right)-\lambda(2\eta+1)+3,\\
\Omega(\mu,\lambda,\eta) &= -4\mu\left((\lambda-1)\eta-1\right)\left[(\lambda-1)+(\eta+2\left((\lambda-1)\eta-1\right))\right]\\
&\quad -2\left[\left((\lambda-1)\eta-1\right)^2+\eta(\eta+2\left((\lambda-1)\eta-1\right))\right]\\
&\quad -\left[4+\left((\lambda-1)\eta-1\right)(3\lambda+1)\right]+(\lambda-1)\left[\left((\lambda-1)\eta-1\right)-2(\lambda-1)\right].
\end{aligned}$$

If we set $\mu = 0$ and $\eta = 0$ in Theorem 2.1 and 3.1, we obtain the classes $\mathcal{H}_{\Sigma,m}(\gamma,0,\lambda,0;\alpha) \equiv \mathcal{W}(\gamma,\lambda;\alpha)$ and $\mathcal{H}_{\Sigma,m}(\gamma,0,\lambda,0;\beta) \equiv \mathcal{W}(\gamma,\lambda;\beta)$. Thus we can write the following corollaries.

Corollary 4.3. *Let f given by (1.4) be in the class $\mathcal{W}(\gamma,\lambda;;\alpha)$ $(\gamma\in\mathbb{C}-\{0\}, 0\le\lambda\le 1, 0<\alpha\le 1)$. Then*

$$|a_{m+1}| \le \frac{2\alpha|\gamma|}{\sqrt{|\alpha\gamma\,(2(m-\lambda)^2+(m-3)\lambda+3m+1)+(1-\alpha)(m+1-\lambda)^2|}} \tag{4.5}$$

and

$$|a_{2m+1}| \le \frac{2(m+1)\alpha^2|\gamma|^2}{(2(m-\lambda)^2+(m-3)\lambda+3m+1)^2}+\frac{2\alpha|\gamma|}{(1+2m-\lambda)} \tag{4.6}$$

Corollary 4.4. *Let f given by (1.4) be in the class $\mathcal{W}(\gamma,\lambda;\beta)$ $(\gamma\in\mathbb{C}-\{0\}, 0\le\lambda\le 1, 0\le\beta<1)$. Then*

$$|a_{m+1}| \le \sqrt{\frac{4|\gamma|(1-\beta)}{2(m-\lambda)^2+(m-3)\lambda+3m+1}} \tag{4.7}$$

and

$$|a_{2m+1}| \le \frac{2(m+1)|\gamma|^2(1-\beta)^2}{(m+1-\lambda)^2}+\frac{2|\gamma|(1-\beta)}{1+2m-\lambda}. \tag{4.8}$$

If we set $\mu=1$, $\lambda=0$ and $\eta=1$ in Theorem 2.1 and 3.1, we obtain the classes $\mathcal{H}_{\Sigma,m}(\gamma,1,0,1;\alpha)\equiv\mathcal{W}(\gamma;\alpha)$ and $\mathcal{H}_{\Sigma,m}(\gamma,1,0,1;\beta)\equiv\mathcal{W}(\gamma;\beta)$. Thus we can write the following corollaries.

Corollary 4.5. *Let f given by (1.4) be in the class $\mathcal{W}(\gamma;\alpha)$ $(\gamma\in\mathbb{C}-\{0\}, 0<\alpha\le 1)$. Then*

$$|a_{m+1}| \le \frac{8\alpha|\gamma|}{\sqrt{12m\gamma\alpha(2m^2+3m+1)+(1-\alpha)(2m^2+3m+1)^2}} \tag{4.9}$$

and

$$|a_{2m+1}| \le \frac{32(m+1)\alpha^2|\gamma|^2}{(2m^2+3m+1)^2}+\frac{8\alpha|\gamma|}{(8m^2+6m+1)^2}. \tag{4.10}$$

Corollary 4.6. *Let f given by (1.4) be in the class $\mathcal{W}(\gamma;\beta)$ $(\gamma\in\mathbb{C}-\{0\}, 0\le\beta<1)$. Then*

$$|a_{m+1}| \le \sqrt{\frac{16|\gamma|(1-\beta)}{12m(2m^2+3m+1)}} \tag{4.11}$$

and

$$|a_{2m+1}| \le \frac{32(m+1)|\gamma|^2(1-\beta)^2}{(2m^2+3m+1)^2}+\frac{8|\gamma|(1-\beta)}{(8m^2+6m+1)^2}. \tag{4.12}$$

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ORDERED Γ-SEMIRINGS WITH APARTNESS UNDER CO-ORDER

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Abstract. The settings of this article is the Bishop's constructive algebra including the Intuitionistic logic. In this paper we introduce the concept of Γ-semirings with apartness ordered under co-order as a continuation of our published article (*Γ-semirings with apartness*. Rom. J. Math. Comput. Sci., **9**(2)(2019), 108–112). Additionally, we analyze ordered co-ideals and co-filters in this ordered algebraic structure with apartness.

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1. Introduction

The settings of this article is the Bishop's constructive algebra **Bish** including the Intuitionistic logic **IL** in the sense of books [1, 2, 8] and articles [3, 4, 5]. Let $(S, =, \neq)$ be a relational system, where the relation $' \neq '$ is an apertness relation - a relation on a set S which is consistent, symmetric and co-transitive. This relation is an extensive relation with respect to the equality relation in S ([12]). This relational system is called '*set with apartness*' or, shortly, a *set* in **Bish** orientation. Since in this system the logical principle of the TND (Tercium non datur principle (Lat.) - the principle of exclusion of the third) is not an axiom in **IL** logical system, all formulas that (directly or indirectly) contain the equality have their own non-equivalent doubles. These specifics generate greater complexity in many algebraic structures than is the case in classical algebra. Algebraic structures with apartness relation have been the subject of study by a number of authors for more than twenty years (for example, [3, 4, 5, 16, 19]).

In this article, our intention is to recognize, understand and describe as precisely as possible these specificity on the example of one complex algebraic structure, Γ-semirings structure.

The concept of Γ-semirings were first introduced and studied by M. M. Krishna Rao [9, 10] as a generalization of notion of Γ-rings. Many authors have studies on these algebraic structures. For example: H. Hedayati and K. P. Shum [6] (2011), R. Jagatap1 and Y. Pawar [7] (2011) and M. M. Krishna Rao [11] (2018). There is an interest in the academic community to study and publish the results of these research on these algebraic structures, their internal organization as well as their substructures in general, as well as in many specific cases.

In our published article [20], we will deal with this algebraic structure within the specific environment offered by **Bish** orientation. So, we will observe the behavior of these algebraic structures, assuming that all carriers of algebraicity are sets with apartness relations, that all relations, operations, and functions that appear in they are strongly extensional with respect to apartneses. So, we introduced and analyzed the concept of Γ-semirings with apartness. We also analyzed the classes of doubles of the congruence relations, the ideals, and the filters in such introduced Γ-semirings with apartness. In this article, as a continuation of our article [20], we introduce and analyze the concept of 'Γ-semirings with apartness ordered under co-order relations' and demonstrate its main fundamental properties.

The notions and notations used in this article but not determined in it, we are take over from our previously published articles [4, 5, 16, 17, 19]. For example, for the element $x \in S$ and subset A of S we write $x \lhd A$ if the formula $(\forall a \in A)(x \neq a)$ is valid. In what follows the symbol $A^{\lhd} =: \{x \in S : x \lhd A\}$ will be used.

The article is designed so that the definitions and most important properties of the Γ-semigrngs concept introduced and analyzed in [20] are repeated in Section 2. In Section 3, which is a major part of the article, the concepts of co-ordered Γ-semiring with apartness is introduced as well as the concepts of ordered co-ideals and co-filters in such algebraic structures.

2. Preliminaries: Γ-semirings with apartness

Looking at the definition of Γ-semigring in the classical sense (for example [9, 10, 6]), we first introduced the concept of Γ-semirings with apartness which was used throughout paper [20]. Let $(R, +)$ and $(\Gamma, +)$ be commutative semigroups with apartness. Apartness relation $\neq$ is a consistent, symmetric and co-transitive relation extensive to the equality. About the 'apartness' a reader can consult the following books [1, 2, 8]. By this we mean that the sets $R :\equiv (R, =_R, \neq_R)$ and $\Gamma :\equiv (\Gamma, =_\Gamma, \neq_\Gamma)$ are supplied by apartness relations and that the internal operations in them are strongly extensional total functions. A function $f : X \longrightarrow Y$ between sets with apartness is strongly extensional if holds

$$(\forall u, v \in X)(f(u) \neq_Y f(v) \implies u \neq_X v).$$

In the following, we do not use indices in the equation relations and apartness relations, except in cases where it is necessary to distinguish them so as not to cause confusion. About the relations, functions and operations in the system **Bish** a reader can consult some of our previously published articles such as [12], or any of our bibliographic units listed in the literature of this article: [3, 4, 5, 13, 14, 16].

Definition 2.1 ([20], Definition 2.1)**.** We call R a *Γ-semiring with apartness* if there exists a map $R \times \Gamma \times R \longrightarrow R$, written image of (x, a, y) by xay, such that it satisfies the following axioms:
(1) $(\forall x, y, z \in R)(\forall a \in \Gamma)(xa(y + z) = xay + xaz$ and $(x + y)az = xaz + yaz)$,
(2) $(\forall x, y \in R)(\forall a, b \in \Gamma)(x(a + b)y = xay + xby)$,
(3) $(\forall x, y, z \in R)(\forall a, b \in \Gamma)((xay)bz = xa(ybz))$.

Remark 2.1. *As can be seen, the definition of Γ-semirings with apartness is completely identical to the definition of Γ-semiring in the classical case. However, they do not determine the same algebraic structure. The reader should always keep in mind that the logical setting are different and that the manipulation with them takes place with the previously acceptance of the various principles-philosophical orientations. In this environment, the following implication is valid*

$$(\forall x, y, u, v \in R)(\forall a, b \in \Gamma)(xay \neq ubv \implies (x \neq u \vee a \neq b \vee y \neq v)).$$

A Γ-semiring with apartness R is said to have a *zero element* if there exists an element $0 \in R$ such that the following

$$(\forall x \in R)(\forall a \in \Gamma)(0 + x = x = x + 0 \text{ and } 0ax = 0 = xa0)$$

is valid. Of course, we also have

$$(\forall x, y \in R)(x + y \neq 0 \implies (x \neq 0 \vee y \neq 0))$$

and

$$(\forall x, y \in R)(\forall a \in \Gamma)(xay \neq 0 \implies (x \neq 0 \wedge y \neq 0)).$$

Also, a Γ-semiring with apartness R is said to be *commutative* if the following holds

$$(\forall x, y \in R)(\forall a \in \Gamma)(xay = yax).$$

About the slogan '*a function f is an embedding*', which we will use in the following definition, a reader can consult with some of our previously published texts [12, 13, 14, 15, 16] : A function $f : X \longrightarrow Y$ between sets with apartness is an embedding if holds

$$(\forall u, v \in X)(u \neq v \implies f(u) \neq f(v)).$$

Definition 2.2 ([20], Definition 2.3)**.** Let R be a Γ-semiring and T a Λ-semiring. Then $(f, \varphi) : (R, \Gamma) \longrightarrow (T, \Lambda)$ is called a *homomorphism* if $f : R \longrightarrow T$ and $\varphi : \Gamma \longrightarrow \Lambda$ are strongly extensional homomorphisms of semigroups such that

$$(\forall x, y \in R)(\forall a \in \Gamma)((f, \varphi)(xay) = f(x)\varphi(a)f(y))$$

holds. The mapping (f, φ) is called an *epimorphism* if (f, φ) is a homomorphism and f and φ are epimorphisms of semigroups. Similarly, we can define a monomorphism. A homomorphism (f, φ) is an isomorphism if (f, φ) is an epimorphism and a monomorphism and f and φ are embeddings.

By the following definition we introduced the notion of cosub-Γ-semiring.

Definition 2.3 ([20], Definition 2.4)**.** Let R be a Γ-semiring with apartness

(4) A non-empty subset A of R is a *sub-Γ-semiring* of R if A is an additive sub-semigroup of R and the following holds

$$(\forall x, y \in R)(\forall a \in \Gamma)((x \in A \wedge y \in A) \implies xay \in A).$$

(5) A subset B of R is a *cosub-Γ-semiring* of R if B is an additive cosub-semigroup of R and the following holds

$$(\forall x, y \in R)(\forall a \in \Gamma)(xay \in B \implies (x \in A \vee y \in A)).$$

By the following definition we introduced the notion of co-ideals of Γ-semiring.

Definition 2.4 ([20], Definition 2.5)**.** Let R be a Γ-semiring with apartness.

(6) A subset B of R is a *right Γ-coideal* of R if B is a additive cosub-semigroup of R and the following holds

$$(\forall x, y \in R)(\forall a \in \Gamma)(xay \in B \implies y \in B).$$

(7) A subset B of R is a *left Γ-coideal* of R if B is a additive cosub-semigroup of R and the following holds

$$(\forall x, y \in R)(\forall a \in \Gamma)(xay \in B \implies x \in B).$$

(8) A subset B of R is a *Γ-coideal* of R if B is a additive cosub-semigroup of R and the following holds

$$(\forall x, y \in R)(\forall a \in \Gamma)(xay \in B \implies (x \in B \wedge y \in B)).$$

If R is a Γ-semiring with zero element 0, then it is mandatory to assume that $0 \lhd B$..

Proposition 2.2 ([20], Proposition 2.6)**.** *If B is (left, right) co-ideal of a Γ-semiring R, then the set $B^{\lhd}$ is a (letf, ringt) ideal of R.*

3. The main results

For relation $\nleqslant$ on a set $(R, =, \neq)$ we say that a is a co-order relation on R if it is consistent, linear, and co-transitive, i.e. if the following holds

(a) $\forall x, y \in R)(x \nleqslant y \implies x \neq y)$ (consistency),

(b) $(\forall x, y \in R)(x \neq y \implies (x \nleqslant y \vee y \nleqslant x))$ (linearity) and

(c) $(\forall x, y, z \in R)(x \nleqslant z \implies (x \nleqslant y \vee y \nleqslant z))$ (co-transitivity).

If R is an algebraic structure with respect to the internal binary operation '$\cdot$' in R, then the co-order relation $\nleqslant$ should satisfy the additional condition

(d) $(\forall x, y, u, v \in R)(x \cdot u \nleqslant y \cdot v \implies (x \nleqslant y \vee u \nleqslant v))$.

From (a) immediately follows

(a') $(\forall x \in R)\neg(x \nleqslant x)$.

More about 'co-order relation' in sets and in algebraic structures, an interested reader can find in our articles [12, 13, 14, 15, 16, 18].

3.1. **Co-ordered Γ-semiring.** This subsection we will begin by introducing the concept of Γ-semirings with the apartness ordered under a co-order relation.

Definition 3.1. A Γ-semiring R with apartness is called Γ-semiring ordered under co-order (or R is a co-ordered Γ-semiring, for short) if it admits a co-order relation $\nleqslant$ compatible with the operations in R, i.e., $\nleqslant$ is a co-order on R satisfies the following conditions:
(9) $(\forall x, y, z \in R)((x + z \nleqslant y + z \lor z + x \nleqslant z + y) \implies x \nleqslant y)$; and
(10) $(\forall x, y, z \in R)(\forall a \in \Gamma)((xay \nleqslant yaz \lor zax \nleqslant zay) \implies x \nleqslant y)$.

Remark 3.1. *Speaking in the language of classical algebra, the co-order relation* $'\nleqslant'$ *on* Γ*-semigroup* R *is compatible with the internal operations in* R *if the operations ares right cancellative and left cancellative with respect to the co-order relation.*

Lemma 3.2. *The condition (9) is equivalent to the condition*
(11) $(\forall x, y, u, v \in R)(x + u \nleqslant y + v \implies (x \nleqslant y \lor u \nleqslant v))$.

Proof. (9) $\implies$ (11). Let $x, y, u, v \in R$ be elements such that $x + u \nleqslant y + v$. Then $x + u \nleqslant y + u$ or $+u \nleqslant y+v$ by co-transitivity of the co-order relation, Thus $x \nleqslant y$ or $u \nleqslant v$ by (9). The second implication can be proved by analogy to previous.

(11) $\implies$ (9). Obviously, this implication is valid since co-order is a consistent relation. For example, for v = u, we have $x + u \nleqslant y + u \implies z \nleqslant y$. □

Lemma 3.3. *The condition (10) is equivalent to the condition*
(12) $(\forall x, y, u, v \in R)(\forall a \in \Gamma)((xau \nleqslant yav \lor uax \nleqslant yay) \implies (x \nleqslant y \lor u \nleqslant v))$.

Proof. (12) $\implies$ (10). Obviously, this implication is valid since co-order is a consistent relation.

(10) $\implies$ (12). Let $x, y, u, v \in R$ and $a \in \Gamma$ be arbitrary elements such that $xau \nleqslant yav$ or $uax \nleqslant yay$. Then $xau \nleqslant yau \lor yau \nleqslant yav$ by co-transitivity of the co-order relation. Thus $x \nleqslant y \lor u \nleqslant v$ by (10). The second implication can be proved by analogy. □

The correctness of this determination of the co-order relation $'\nleqslant'$ on Γ-semiring R with the apartness relation is shown by proving that the $\nleqslant^{\lhd} =: \{(u, v) \in R \times R : (u, v) \lhd \nleqslant\}$ is a partial order relation on R compatible with the operations in R. More specifically:

Theorem 3.4. *Let* $((R, =, \neq), +)$ *be a* Γ*-semiring with apartness ordered under a co-order* $'\nleqslant'$*. Then the relation* $\nleqslant^{\lhd}$ *is a partial order in* $((R, \neq^{\lhd}, \neq), +)$ *compatible with the internal operations in* $((R, \neq^{\lhd}, \neq), +)$.

Proof. (1) Let $x, u, v \in R$ be arbitrary elements such that $u \nleqslant v$. Then $u \nleqslant x$ or $x \nleqslant v$ by co-transitivity of $\nleqslant$. Thus $u \neq x \lor x \neq v$ by consistency of $\nleqslant$. So, we have $(x, x) \neq (u, v) \in \nleqslant$ and $x \nleqslant^{\lhd} x$. Therefore, the relation $\nleqslant^{\lhd}$ is a reflexive relation in R.

Let $x, y, z, u, v \in X$ be arbitrary elements of R such that $x \nleqslant^{\lhd} y$, $y \nleqslant^{\lhd} z$ and $u \nleqslant v$. Then $u \nleqslant x \lor x \nleqslant y \lor y \nleqslant z \lor z \nleqslant v$ by co-transitivity of $\nleqslant$. Thus $u \neq x$ or $z \neq v$ because the options $x \nleqslant y$ and $y \nleqslant z$ ate impossible by hypothesis. So, $(x, z) \neq (u, v) \in \nleqslant$ and $x \nleqslant^{\lhd} z$. Therefore, the relation $\nleqslant^{\lhd}$ is a transitive relation in R.

Let $x, y, u, v \in R$ be arbitrary elements such that $x \nleqslant^{\lhd} y$ and $y \nleqslant^{\lhd} x$ and $u \neq v$. Suppose $x \neq y$. Then we would have $x \nleqslant x \lor y \nleqslant x$, which contradicts the hypothesis. So $\neg(x \neq y)$. On the other hand, from $u \neq v$ follows $u \neq x \lor x \neq y \lor y \neq v$ and $u \neq x \lor (x \nleqslant y \lor y \nleqslant c) \lor y \neq v$. Thus $(x, y) \neq (u, v) \in \neq$ by hypothesis $x \nleqslant^{\lhd} y$ and $y \nleqslant^{\lhd} x$. Finally, we have $x \neq^{\lhd} y$.

We have shown by this the relation $\nleqslant^{\lhd}$ is a partial order $((R, \neq^{\lhd}, \neq), +)$.

(2) Let $x, y, u, v, s, t \in R$ be arbitrary elements such that $x \nleqslant^{\lhd} y$, $u \nleqslant^{\lhd} v$ and $s \nleqslant t$. From $s \nleqslant t$ follows $s \nleqslant x+u \lor x+u \nleqslant y+v \lor y+v \nleqslant t$. From the second option $x+u \nleqslant y+v$ follows $x \nleqslant y \lor u \nleqslant v$

by (11). This contradicts the hypothesis. So, have to be $s \nleq x+u \vee y+v \nleq t$ and $s \neq x+u \vee y+v \neq t$ by consistency of the co-order relation. So, we have $(x+u, t+v) \neq (s,t) \in \nleq$. This means $x+u \nleq^{\lhd} y+v$.

Therefore, we have proved by this that $\nleq^{\lhd}$ is compatible with the addition in $((R, \neq^{\lhd}, \neq), +)$.

(3) Let $x, y, u, v, s, t \in R$, $a \in \Gamma$ be arbitrary elements such that $x \nleq^{\lhd} y$, $u \nleq^{\lhd} v$ and $s \nleq t$. From $s \nleq t$ follows $s \nleq xau \vee xau \nleq yav \vee yav \nleq t$. From the second option $x \nleq y \vee u \nleq v$ by (12). It is a contradiction. So, we have $s \neq xau \vee yav \neq t$ by (a). Thus $(xau, yav) \neq (s,t) \in \nleq$. So, we have $xau \nleq^{\lhd} yav$. Implication $((x \nleq^{\lhd} y \wedge u \nleq^{\lhd} v) \Longrightarrow xau \nleq yav)$ can be proved analogous to the previous proof.

Therefore, we have proved by this that $\nleq^{\lhd}$ is compatible with the multiplication in Γ-semiring R. $\square$

If the apartness relation is tight, i.e. if

$$(\forall x, y \in R)(\neg(x \neq y) \implies x = y)$$

holds, then $\nleq^{\lhd}$ is literally a partial order in R compatible with the operations in R. However, in general case, the apartness relation does not have to be tight. In accepting this commitment, we are entering into a much broader and more specific field that contains far more specific phenomena than is the case with classical theory.

3.2. **Ordered co-ideals.** In the following definition, we introduce the concept of ordered co-ideals in a Γ-semiring ordered under a co-order relation.

Definition 3.2. A non-empty subset B of a co-ordered Γ-semiring R is called a (left, right) ordered co-ideal of the co-ordered Γ-semiring R if B is a (left, right) co-ideal in R and the following holds

(13) $(\forall x, y \in R)(x \in B \Longrightarrow (x \nleq y \vee y \in B))$.

As is usual in the **Bish** framework, this concept should be associated with the concept of ordered ideals in such algebraic structures ([11], Definition 3.1) in the following since:

Theorem 3.5. *Let B be a (left, right) ordered co-ideal of a co-ordered Γ-semiring R. Then the set $B^{\lhd} =: \{x \in R : x \lhd B\}$ is a (lest, right) ideal of R.*

Proof. According to Proposition 2.2, $B^{\lhd}$ is a (left, right) ideal in R. Now, suppose that $x, y, u \in R$ be such $x \nleq^{\lhd} y$, $u \in B$ and $y \in B^{\lhd}$. Then $u \nleq y \vee y \in B$ by (13). Since the option $y \in B$ is impossible by hypothesis, it should be $u \nleq y$ and $y \neq u \in B$ by (a). So, the set $B^{\lhd}$ meets the requirements of Definition 2.19 in the article [11] and, therefore, it is an ideal in R. $\square$

Theorem 3.6. *The family $\mathfrak{K}(R)$ of all ordered co-ideals in co-ordered Γ-semiring R forms a complete lattice.*

Proof. Let $\{B_i\}_{i \in I}$ be an arbitrary subfamily of the family $\mathfrak{K}(R)$. If we look at the this family as a family of co-ideals, then $\bigcup_{i \in I} B_i$ is a co-ideal in R, according to Corollary 2.9 in the article [20]. It remains to prove the property (13) for union $\bigcup_{i \in I} B_i$. Suppose $x \in \bigcup_{i \in I} B_i$ and $y \in R$. Then there exists an index $j \in I$ such that $x \in B_j$. Thus $x \nleq y$ or $y \in B_j$. So, $x \nleq y \vee y \in \bigcup_{i \in I} B_i$. Therefore, $\bigcup_{i \in I} B_i$ is an ordered co-ideal in R.

Let $\mathfrak{X}$ be a family of all co-ideals in R contained in $\bigcap_{i \in I} B_i$. Then $\cup\mathfrak{X}$ is the maximal co-ideal contained in $\bigcap_{i \in I} B_i$.

If we pit $\sqcup_{i \in I} B_i = \bigcup_{i \in I} B_i$ and $\sqcap_{i \in I} B_i = \cup\mathfrak{X}$, then $(\mathfrak{K}(R), \sqcup, \sqcap)$ is a complete lattice. $\square$

Corollary 3.7. *If X is a subset in a co-ordered Γ-semiring R, then there exists the maximal ordered co-ideal of R contained in X.*

Proof. The claim of this Corollary follows from the second part of the proof in the previous theorem. $\square$

Corollary 3.8. *If x is an arbitrary element in a co-ordered Γ-semiring R, then there exists the maximal ordered co-ideal K_a such that $a \lhd K_x$.*

Proof. The claim of this Corollary follows from the previous Corollary if we put $X = R \backslash \{a\}$. $\square$

3.3. **Co-filters.** The concept of co-filters in a co-ordered semigroup with apartness is discussed in [21], while the concept of co-filters in Γ-semigroup with apartness ordered under a co-order relation is discussed in [22].

Definition 3.3. A non-empty subset G of a co-ordered Γ-semiring R is called co-filter of the co-ordered Γ-semiring R is G is an additive cosub-semigroup of R and the following holds

(14) $(\forall x, y \in R)(y \in G \implies (x \in G \vee x \nleqslant y))$.

Theorem 3.9. *If G is a co-filter in co-ordered Γ-semiring R with apartness, then the set $G^{\triangleleft}$ is a filter in R.*

Proof. Since G is an additive cosub-semiring in co-ordered Γ-semiring R, then $G^{\triangleleft}$ is a sub-semiring in R. Now, let $x, y, u \in R$ be such that $x \nleqslant^{\triangleleft} y$, $u \in G$ and $x \in G^{\triangleleft}$. Then $x \nleqslant u \vee x \in G$ by (14). The second option is impossible by hypothesis $x \triangleleft G$. Thus must be $x \nleqslant u \in G$. So, we have $y \neq u \in G$ by (a). With this we have shown that the set $G^{\triangleleft}$ satisfies the requirements of Definition 3.1 in article [11]. Therefore, $G^{\triangleleft}$ is a filter in R. □

The proof of the following theorem can be demonstrated analogously to the proof of Theorem 3.6, so we will omit it.

Theorem 3.10. *The family $\mathfrak{G}(R)$ of all co-filters in a co-ordered Γ-semiring R forms a complete lattice.*

We end this subsection with the following definition and theorem

Definition 3.4. Let $((R, =_R, \neq_R), +_R, \nleqslant_R)$ be a co-ordered Γ-semiring and $((T, =_T, \neq_T), +_T, \nleqslant_T)$ a co-ordered Λ-semiring. Then:

- $(f, \varphi) : (R, \Gamma) \longrightarrow (T, \Lambda)$ is called a *isotone homomorphism* if $f : R \longrightarrow T$ is an isotone mapping; and

- $(f, \varphi) : (R, \Gamma) \longrightarrow (T, \Lambda)$ is called a *reverse isotone homomorphism* if $f : R \longrightarrow T$ is a reverse isotone mapping.

Theorem 3.11. *Let $(f, \varphi) : (R, \Gamma) \longrightarrow (T, \Lambda)$ be a reverse isotone homomorphism. Then if K is a ordered co-ideal (co-filter) of T, then $(f, \varphi)^{-1}(K)$ is a ordered co-ideal (co-filter) of R.*

Proof. (i) Let $x, y \in R$ and $a \in \Gamma$ be arbitrary elements such that $x +_R y \in (f, \varphi)^{-1}(K)$ and $xay \in (f, \varphi)^{-1}(K)$. Then:

$f(x) +_T f(y) = (f, \varphi)(x +_R y) \in K$ and $f(x) \in K \vee f(y) \in K$ since K is an additive cosub-semigroup in T. Thus $x \in (f, \varphi)^{-1}(K) \vee y \in (f, \varphi)^{-1}(K)$. So, the set $(f, \varphi)^{-1}(K)$ ia an additive cosub-semigroup in R.

$f(x)\varphi(a)f(y) = (f, \varphi)(xay) \in K$. Thus $f(x) \in K$ or $f(y) \in K$ by (5) because K is a cosub-semiring in ordered Λ-semiring T. So, we have $x \in (f, \varphi)^{-1}(K) \vee y \in (f, \varphi)^{-1}(K)$.

We have shown that $(f, \varphi)^{-1}(K)$ is a cosub-semiring in R.

(ii) Suppose K is a co-filter in T. From $y \in (f, \varphi)^{-1}(K)$, i.e. from $f(x) \in K$ follows $f(x) \in K \vee f(x) \nleqslant_T f(y)$ since K is a co-filter in T. Thus $x \in (f, \varphi)^{-1}(K) \vee x \nleqslant_R y$ because the mapping f is reverse isotone. Therefore, the set $(f, \varphi)^{-1}(K)$ is a co-filter in R.

(iii) Suppose K is an ordered co-ideal in T. Let $x, y \in R$ and $a \in \Gamma$ be arbitrary elements such that $xay \in (f, \varphi)^{-1}(K)$. Then $f(x)\varphi(a)f(y) = (f, \varphi)(xay) \in K$. Thus $f(x) \in K \wedge f(y) \in K$ by (8). So, this means $x \in (f, \varphi)^{-1}(K) \wedge y \in (f, \varphi)^{-1}(K)$. Therefore, $\in (f, \varphi)^{-1}(K)$ is a co-ideal in T.

Now, from $x \in (f, \varphi)^{-1}(K)$, i.e. from $f(x) \in K$ follows $f(x) \nleqslant_T f(y) \vee f(y) \in K$ since K is a ordered co-ideal in T. Hence $x \nleqslant_T y \vee y \in (f, \varphi)^{-1}(K)$ since (f, φ) is a reverse isotone homomorphism. So, the set $(f, \varphi)^{-1}(K)$ is an ordered co-ideal in R. □

4. Conclusion

The sets and algebraic structures with the apartness relation within the Bishop's constructive framework have long been in the focus of this author's interest. In articles [14, 15, 16], the author deals with various algebraic structures determined on sets with apartness relation. The articles [18, 22] introduces and analyzes the structure of Γ-semigroup with apartness. The article [19] is a recapitulation of algebraic structures with apartness relation. The concept of co-filters in a co-ordered semigroup with apartness is discussed in [21], while the concept of co-filters in Γ-semigroup with apartness ordered under a co-order relation is discussed in [22].

In the article [20], the concept of Γ-semirings with apartness was introduced and analyzed. In this report, the concept of a Γ-semirings with apartness ordered under a co-order relation is presented. In addition, the concepts of ordered co-ideals and co-filters in such structures are introduced and analyzed. In doing so, the specificities of this chosen research framework are highlighted.

The possibility of further research of this algebraic structure may include, for example, designing isomorphism theorems for a co-ordered Γ-semiring, or determining of ordered interior, ordered weak-interior and ordered quasi-interior co-ideals in co-ordered Γ-semirings.

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INEQUALITIES FOR (α, m_1, m_2)-ARITHMETIC GEOMETRICALLY CONVEX FUNCTIONS VIA BY GAMMA AND INCOMPLETE GAMMA FUNCTIONS

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Abstract. In this study, we introduce concepts of (α, m_1, m_2)-arithmetic geometrically (AG) convex functions and establish some Hermite-Hadamard type inequalities of these classes of functions via gamma and incomplete gamma function.

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1. Preliminaries

Definition 1.1. A function $f : I \to \mathbb{R}$ is said to be convex if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t) f(y)$$

is valid for all $x, y \in I$ and $t \in [0, 1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$.

One of the most important inequalities in convex theory is the Hermite-Hadamard integral inequality. This inequality is given below. Let $f : I \to \mathbb{R}$ be a convex function. Then the following inequalities hold

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

for all $a, b \in I$ with $a < b$. The above inequalities was firstly discovered by the famous scientist Charles Hermite. In recent years, readers can find more information in [2, 5, 6, 7, 8, 11, 10, 12, 13, 14, 15, 16, 17, 19] for different convex classes and related Hermite-Hadamard integral inequalities.

Definition 1.2 ([7])**.** The function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be (m_1, m_2)-convex, if

$$f(m_1tx + m_2(1-t)y) \leq m_1tf(x) + m_2(1-t)f(y)$$

for all $x, y \in I$, $t \in [0, 1]$ and $(m_1, m_2) \in (0, 1]^2$.

Definition 1.3 ([1, 20])**.** If a function $f : I \subseteq \mathbb{R} \to (0, \infty)$ satisfies

$$f(\lambda x + m(1-\lambda)y) \leq [f(x)]^{\lambda} [f(y)]^{1-\lambda}$$

for all $x, y \in I$, $\lambda \in [0, 1]$, the function f is called logarithmically convex on I. If this inequality reverses, the function f is called logarithmically concave on I.

Definition 1.4 ([9])**.** A function $f : [0, b] \to (0, \infty)$ is said to be (m_1, m_2)-arithmetic geometrically convex (or logarithmically convex) if the inequality

$$f(m_1tx + m_2(1-t)y) \leq [f(x)]^{m_1t}[f(y)]^{m_2(1-t)}$$

holds for all $x, y \in [0, b]$, $(m_1, m_2) \in (0, 1]^2$, and $t \in [0, 1]$.

Definition 1.5 ([8])**.** $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be (α, m_1, m_2)-convex function, if

$$f(m_1tx + m_2(1-t)y) \leq m_1t^\alpha f(x) + m_2(1-t^\alpha)f(y)$$

for all $x, y \in I$, $t \in [0, 1]$ and $(\alpha, m_1, m_2) \in (0, 1]^3$.

Definition 1.6 ([3])**.** A function $f : [0, b] \to (0, \infty)$ is said to be m-logarithmically convex if the inequality

$$f(tx + m(1-t)y) \leq [f(x)]^t [f(y)]^{m(1-t)}$$

holds for all $x, y \in [0, b]$, $m \in (0, 1]$, and $t \in [0, 1]$.

Definition 1.7 ([18])**.** Let $0 < s \leq 1$. A function $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ where $\mathbb{R}_+ = [0, \infty)$ is said to be s-convex in the first sense if

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y)$$

for all $x, y \in I$ and $\alpha, \beta \geq 0$ with $\alpha^s + \beta^s = 1$.

Definition 1.8 (Gamma function)**.** The classic gamma function is usually defined as

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt.$$

Definition 1.9 (Upper incomplete gamma function)**.** The upper incomplete gamma function is defined as

$$\Gamma(s, x) = \int_x^\infty t^{s-1}e^{-t}dt.$$

Definition 1.10 (Lower incomplete gamma function)**.** The lower incomplete gamma function is defined as

$$\gamma(s, x) = \int_0^x t^{s-1}e^{-t}dt.$$

Throughout this paper, for shortness we will use the following notations for special means of two nonnegative numbers a, b with $b > a$:

1. The arithmetic mean

$$A := A(a, b) = \frac{a+b}{2}, \quad a, b > 0,$$

2. The geometric mean

$$G := G(a, b) = \sqrt{ab}, \quad a, b \geq 0,$$

3. The logarithmic mean

$$L := L(a, b) = \begin{cases} \frac{b-a}{\ln b - \ln a}, & a \neq b \\ a, & a = b \end{cases} ; \quad a, b > 0.$$

The main purpose of this paper is to introduce the concept of (α, m_1, m_2)-AG or (α, m_1, m_2)-logarithmically convex functions and then establish some results connected with new inequalities similar to the Hermite-hadamard integral inequality for these classes of functions.

2. Main results for (α, m_1, m_2)-AG convex functions

In this section, we introduce a new concept, which is called (α, m_1, m_2)-AG convex functions and we give by setting some algebraic properties for the (α, m_1, m_2)-AG convex functions, as follows:

Definition 2.1. A function $f : [0, b] \to (0, \infty)$ is said to be (α, m_1, m_2)-arithmetic geometrically convex (AG) if the inequality

$$f(m_1tx + m_2(1-t)y) \leq [f(x)]^{m_1t^\alpha} [f(y)]^{m_2(1-t^\alpha)} \tag{2.1}$$

holds for all $x, y \in [0, b]$, $(\alpha, m_1, m_2) \in (0, 1]^3$, and $t \in [0, 1]$.

Remark 2.2. i) If we take $m_1 = m_2 = 1$ and $\alpha = 1$ in (2.1), then the (α, m_1, m_2)-AG function becomes a arithmetic geometrically convex function in defined [1, 20].

ii) If we take $m_1 = 1$, $m_2 = m$ in (2.1), then the (α, m_1, m_2)-AG function becomes the (α, m)-arithmetic geometrically convex function defined in [3].

iii) If we take $\alpha = 1$ in (2.1), then the (α, m_1, m_2)-AG function becomes the (m_1, m_2)-arithmetic geometrically convex function defined in [9].

iv) If we take $m_1 = m_2 = 1$ and $\alpha = s$ in (2.1), then the (α, m_1, m_2)-AG convex function becomes s-convex in the first sense defined in [18].

The following Theorems can be proved similar to those in [9].

Theorem 2.3. *The function $f : I \subset (0, \infty) \to \mathbb{R}$ is (α, m_1, m_2)-AG convex function on I if and only if $\ln \circ f : (0, \infty) \to \mathbb{R}$ is (α, m_1, m_2)-convex function on $(0, \infty)$.*

Proof. $(\Rightarrow)$ Let $f : I \subset (0, \infty) \to \mathbb{R}$ (α, m_1, m_2)-AG convex function. Then, we have

$$(\ln \circ f)(m_1ta + m_2(1-t)b) \quad \leq \quad \ln\left([f(a)]^{m_1t^\alpha} [f(b)]^{m_2(1-t^\alpha)}\right) = m_1t^\alpha f(\ln a) + m_2(1-t^\alpha)f(\ln b).$$

Therefore, $\ln \circ f$ is (α, m_1, m_2)-convex function on $(0, \infty)$.

$(\Leftarrow)$ Let $\ln \circ f : (0, \infty) \to \mathbb{R}$, (α, m_1, m_2)-convex function on $(0, \infty)$. Then, we get

$$\begin{aligned} (\ln \circ f)(m_1ta + m_2(1-t)b) &\leq m_1t^\alpha f(\ln a) + m_2(1-t^\alpha)f(\ln b) \\ e^{(\ln \circ f)(m_1ta + m_2(1-t)b)} &\leq e^{m_1t^\alpha f(\ln a) + m_2(1-t^\alpha)f(\ln b)}. \end{aligned}$$

So, the function $f(x)$ (α, m_1, m_2)-AG convex function on I. □

Theorem 2.4. *If $f : I \to J$ is a (α, m_1, m_2)-convex and $g : J \to \mathbb{R}$ is a (α, m_1, m_2)-AG convex function and nondecreasing, then $g \circ f : I \to \mathbb{R}$ is a (m_1, m_2)-AG convex function.*

Proof. For $a, b \in I$ and $t \in [0, 1]$, we get

$$\begin{aligned} (g \circ f)(m_1ta + m_2(1-t)b) &= g(f(m_1ta + m_2(1-t)b)) \\ &\leq g(m_1t^\alpha f(a) + m_2(1-t^\alpha)f(b)) \\ &\leq [g(f(a))]^{m_1t^\alpha} [g(f(a))]^{m_2(1-t^\alpha)} \\ &\leq [(g \circ f)(a)]^{m_1t^\alpha} [(g \circ f)(b)]^{m_2(1-t^\alpha)}. \end{aligned}$$

This completes the proof of theorem. □

Theorem 2.5. *Let $b > 0$ and $f_\alpha : [a, b] \to \mathbb{R}$ be an arbitrary family of (α, m_1, m_2)-AG convex functions and let $f(x) = \sup_\alpha f_\alpha(x)$. If $J = \{u \in [a, b] : f(u) < \infty\}$ is nonempty, then J is an interval and f is an (α, m_1, m_2)-AG convex function on J.*

Proof. Let $t \in [0,1]$ and $a, b \in J$ be arbitrary. Then

$$\begin{aligned} f(ta+(1-t)b) &= \sup_\alpha f_\alpha\left(m_1 t\frac{a}{m_1}+m_2(1-t)\frac{b}{m_2}\right) \\ &\le \sup_\alpha\left(\left[f_\alpha\left(\frac{a}{m_1}\right)\right]^{m_1 t^\alpha}\left[f_\alpha\left(\frac{b}{m_2}\right)\right]^{m_2(1-t^\alpha)}\right) \\ &\le \left[\sup_\alpha f_\alpha\left(\frac{a}{m_1}\right)\right]^{m_1 t^\alpha}\left[\sup_\alpha f_\alpha\left(\frac{b}{m_2}\right)\right]^{m_2(1-t^\alpha)} \\ &= \left[f\left(\frac{a}{m_1}\right)\right]^{m_1 t^\alpha}\left[f\left(\frac{b}{m_2}\right)\right]^{m_2(1-t^\alpha)} < \infty. \end{aligned}$$

So, this shows simultaneously that J is an interval, since it contains every point between any two of its points, and that f is an (α, m_1, m_2)-AG convex function on J. This completes the proof of theorem. ☐

Theorem 2.6. *Let $f : [0, b^*] \to \mathbb{R}$ a finite function on $\frac{a}{m_1}, \frac{b}{m_2} \in [0, b^*]$, (α, m_1, m_2)-AG convex function with $\alpha, m_1, m_2 \in (0,1]$. Then the function f is bounded on any closed interval $[a, b]$.*

Proof. Let $M = \max\left\{f\left(\frac{a}{m_1}\right), f\left(\frac{b}{m_2}\right)\right\}$ and $x \in [a,b]$ is an arbitrary point. Then there exist a $t \in [0,1]$ such that $x = ta + (1-t)b$. In this case, since $m_1 t^\alpha + m_2(1-t^\alpha) \le 1$ we have

$$f(x) = f(ta+(1-t)b) = f\left(m_1 t\frac{a}{m_1}+m_2(1-t)\frac{b}{m_2}\right) \le \left[f\left(\frac{a}{m_1}\right)\right]^{m_1 t^\alpha}\left[f\left(\frac{b}{m_2}\right)\right]^{m_2(1-t^\alpha)} \le M.$$

Therefore, f is upper bounded in $[a,b]$. Now we notice that any $z \in [a,b]$ can be written as $\frac{a+b}{2} + t$ for $|t| \le \frac{b-a}{2}$, hence

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{1}{2}\left(\frac{a+b}{2}+t\right)+\frac{1}{2}\left(\frac{a+b}{2}-t\right)\right) \\ &= f\left(\frac{m_1}{2}\left(\frac{\frac{a+b}{2}+t}{m_1}\right)+\frac{m_2}{2}\left(\frac{\frac{a+b}{2}-t}{m_2}\right)\right) \\ &\le \left[f\left(\frac{\frac{a+b}{2}+t}{m_1}\right)\right]^{m_1\left(\frac{1}{2}\right)^\alpha}\left[f\left(\frac{\frac{a+b}{2}-t}{m_2}\right)\right]^{m_2\left[1-\left(\frac{1}{2}\right)^\alpha\right]}. \end{aligned}$$

So, we get

$$f\left(\frac{\frac{a+b}{2}+t}{m_1}\right) \ge \left\{\frac{f\left(\frac{a+b}{2}\right)}{\left[f\left(\frac{\frac{a+b}{2}-t}{m_2}\right)\right]^{m_2\left[1-\left(\frac{1}{2}\right)^\alpha\right]}}\right\}^{\frac{2^\alpha}{m_1}} \ge \frac{\left[f\left(\frac{a+b}{2}\right)\right]^{\frac{2^\alpha}{m_1}}}{M^{\frac{m_2(2^\alpha-1)}{m_1}}}$$

and similarly

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{1}{2}\left(\frac{a+b}{2}+t\right)+\frac{1}{2}\left(\frac{a+b}{2}-t\right)\right) \\ &= f\left(\frac{m_2}{2}\left(\frac{\frac{a+b}{2}+t}{m_2}\right)+\frac{m_1}{2}\left(\frac{\frac{a+b}{2}-t}{m_1}\right)\right) \\ &\le \left[f\left(\frac{\frac{a+b}{2}+t}{m_2}\right)\right]^{m_2\left(\frac{1}{2}\right)^\alpha}\left[f\left(\frac{\frac{a+b}{2}-t}{m_1}\right)\right]^{m_1\left[1-\left(\frac{1}{2}\right)^\alpha\right]}, \end{aligned}$$

hence, we get

$$f\left(\frac{\frac{a+b}{2}+t}{m_2}\right) \geq \left\{\frac{f\left(\frac{a+b}{2}\right)}{\left[f\left(\frac{\frac{a+b}{2}-t}{m_1}\right)\right]^{m_1\left[1-\left(\frac{1}{2}\right)^{\alpha}\right]}}\right\}^{\frac{2^{\alpha}}{m_2}} \geq \frac{\left[f\left(\frac{a+b}{2}\right)\right]^{\frac{2^{\alpha}}{m_2}}}{M^{\frac{m_1(2^{\alpha}-1)}{m_2}}}$$

and since $\frac{a+b}{2}+t$ is arbitrary in $[a,b]$, the function f is also bounded below in $[a,b]$. This completes the proof of theorem. □

3. Hermite-Hadamard inequality for (α, m_1, m_2)-AG convex function

The goal of this section is to establish some inequalities of Hermite-Hadamard type integral inequalities for (α, m_1, m_2)-AG convex functions. In this section, we will denote by $L[a,b]$ the space of (Lebesgue) integrable functions on the interval $[a,b]$.

Theorem 3.1. *Let $f:[a,b]\to\mathbb{R}$ be an (α, m_1, m_2)-AG convex function and let $[f(b)]^{m_2} > [f(a)]^{m_1}$. If $a<b$ and $f\in L[a,b]$, then the following Hermite-Hadamard type inequalities hold:*

$$f\left(\frac{a+b}{2}\right) \leq \exp\left\{\frac{m_1^2}{2^{\alpha}(b-a)}\int_a^b \ln f(m_1x)dx + \frac{(2^{\alpha}-1)\,m_2^2}{2^{\alpha}(b-a)}\int_a^b \ln f(m_2y)dy\right\}$$

and

$$\frac{1}{m_2b-m_1a}\int_{m_1a}^{m_2b} f(x)dx = [f(b)]^{m_2}\,\frac{\Gamma\left(\frac{1}{\alpha}\right)-\Gamma\left(\frac{1}{\alpha}, \ln[f(b)]^{m_2}-\ln[f(a)]^{m_1}\right)}{\alpha\left(\ln[f(b)]^{m_2}-\ln[f(a)]^{m_1}\right)^{\frac{1}{\alpha}}} \leq \frac{m_1f(a)+\alpha m_2f(b)}{\alpha+1}.$$

Proof. Firstly, from the property of the (α, m_1, m_2)-AG convex function of f, we can write

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) &= f\left(\frac{\left[m_1t\frac{a}{m_1}+m_2(1-t)\frac{b}{m_2}\right]+\left[m_1(1-t)\frac{a}{m_1}+m_2t\frac{b}{m_2}\right]}{2}\right)\\
&= f\left(\frac{m_1}{2}\left[t\frac{a}{m_1}+\frac{m_2}{m_1}(1-t)\frac{b}{m_2}\right]+\frac{m_2}{2}\left[t\frac{b}{m_2}+\frac{m_1}{m_2}(1-t)\frac{a}{m_1}\right]\right)\\
&\leq \left[f\left(t\frac{a}{m_1}+\frac{m_2}{m_1}(1-t)\frac{b}{m_2}\right)\right]^{m_1\left(\frac{1}{2}\right)^{\alpha}}\left[f\left(t\frac{b}{m_2}+\frac{m_1}{m_2}(1-t)\frac{a}{m_1}\right)\right]^{m_2\left[1-\left(\frac{1}{2}\right)^{\alpha}\right]}.
\end{aligned}$$

By taking the logarithm on the both sides of the above inequality we obtain

$$\begin{aligned}
&\ln f\left(\frac{a+b}{2}\right)\\
\leq\ & \ln\left\{\left[f\left(t\frac{a}{m_1}+\frac{m_2}{m_1}(1-t)\frac{b}{m_2}\right)\right]^{m_1\left(\frac{1}{2}\right)^{\alpha}}\left[f\left(t\frac{b}{m_2}+\frac{m_1}{m_2}(1-t)\frac{a}{m_1}\right)\right]^{m_2\left[1-\left(\frac{1}{2}\right)^{\alpha}\right]}\right\}\\
=\ & m_1\left(\frac{1}{2}\right)^{\alpha}\ln f\left(t\frac{a}{m_1}+\frac{m_2}{m_1}(1-t)\frac{b}{m_2}\right)+m_2\left[1-\left(\frac{1}{2}\right)^{\alpha}\right]\ln f\left(t\frac{b}{m_2}+\frac{m_1}{m_2}(1-t)\frac{a}{m_1}\right).
\end{aligned}$$

Now, if we take integral in the last inequality with respect to $t \in [0,1]$ and choose $m_1 x = ta + (1-t)b$ and $m_2 y = tb + (1-t)a$, we get

$$\begin{aligned}
\ln f\left(\frac{a+b}{2}\right) &\leq m_1 \left(\frac{1}{2}\right)^{\alpha} \int_0^1 \ln f\left(t\frac{a}{m_1} + \frac{m_2}{m_1}(1-t)\frac{b}{m_2}\right) dt \\
&\quad + m_2 \left[1 - \left(\frac{1}{2}\right)^{\alpha}\right] \int_0^1 \ln f\left(t\frac{b}{m_2} + \frac{m_1}{m_2}(1-t)\frac{a}{m_1}\right) dt \\
&= \frac{m_1^2}{2^{\alpha}(b-a)} \int_a^b \ln f(m_1 x) dx + \frac{(2^{\alpha}-1)\, m_2^2}{2^{\alpha}(b-a)} \int_a^b \ln f(m_2 y) dy \\
f\left(\frac{a+b}{2}\right) &\leq \exp\left\{\frac{m_1^2}{2^{\alpha}(b-a)} \int_a^b \ln f(m_1 x) dx + \frac{(2^{\alpha}-1)\, m_2^2}{2^{\alpha}(b-a)} \int_a^b \ln f(m_2 y) dy\right\}.
\end{aligned}$$

Secondly, by using the property of the (α, m_1, m_2)-AG convex function of f, we can write

$$f\left(m_1 t x + m_2(1-t)y\right) \leq [f(x)]^{m_1 t^{\alpha}} [f(y)]^{m_2(1-t^{\alpha})} \leq m_1 t^{\alpha} f(a) + m_2 \left(1-t^{\alpha}\right) f(b).$$

If the variable is changed as $u = [f(a)]^{m_1 t^{\alpha}} [f(b)]^{m_2(1-t^{\alpha})}$ in the above inequality, then

$$\begin{aligned}
\frac{1}{m_2 b - m_1 a} \int_{m_1 a}^{m_2 b} f(x) dx &= \int_0^1 [f(a)]^{m_1 t^{\alpha}} [f(b)]^{m_2(1-t^{\alpha})} dt \\
&= [f(b)]^{m_2} \frac{\Gamma\left(\frac{1}{\alpha}\right) - \Gamma\left(\frac{1}{\alpha}, \ln [f(b)]^{m_2} - \ln [f(a)]^{m_1}\right)}{\alpha \left(\ln [f(b)]^{m_2} - \ln [f(a)]^{m_1}\right)^{\frac{1}{\alpha}}} \\
&\leq \frac{m_1 f(a) + \alpha m_2 f(b)}{\alpha + 1}.
\end{aligned}$$

This completes the proof of theorem. □

Corollary 3.2. *Under the conditions of Theorem 3.1, If we take $\alpha = 1$, then we get*

$$f\left(\frac{a+b}{2}\right) \leq \exp\left\{\frac{m_1^2}{2(b-a)} \int_a^b \ln f(m_1 x) dx + \frac{m_2^2}{2(b-a)} \int_a^b \ln f(m_2 y) dy\right\}$$

and

$$\frac{1}{m_2 b - m_1 a} \int_{m_1 a}^{m_2 b} f(x) dx \leq \frac{m_1 f(a) + m_2 f(b)}{2}.$$

This inequality coincides with the inequality in [9].

Corollary 3.3. *Under the conditions of Theorem 3.1, If we take $m_1 = m_2 = 1$ and $\alpha = 1$, then we get*

$$\ln\ f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b \ln f(x) dx \leq L\left([f(a)], [f(b)]\right) \leq A\left([f(a)], [f(b)]\right).$$

This inequality coincides with the inequality in [4].

4. Some new inequalities for (α, m_1, m_2)-AG convex functions

The main purpose of this section is to establish new estimates that refine Hermite-Hadamard integral inequality for functions whose first derivative in absolute value is (α, m_1, m_2)-AG convex function. Will use the following lemma to obtain our main results.

Lemma 4.1 ([8])**.** *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $m_1a, m_2b \in I^\circ$ with $m_1a < m_2b$. If $f' \in L[m_1a, m_2b]$, then the following equality*

$$\begin{aligned}
&\frac{1}{m_2b-m_1a}\int_{m_1a}^{m_2b} f(x)dx - f\left(\frac{m_1a+m_2b}{2}\right) \\
= \;& (m_2b-m_1a)\left[\int_0^{\frac{1}{2}} tf'\left(m_1ta+m_2(1-t)b\right)dt + \int_{\frac{1}{2}}^1 (t-1)f'\left(m_1ta+m_2(1-t)b\right)dt\right]
\end{aligned}$$

holds for $t \in [0,1]$ and $m_1, m_2 \in (0,1]^2$.

Theorem 4.2. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $m_1a, m_2b \in I^\circ$ with $m_1a < m_2b$, $f' \in L[m_1a, m_2b]$ and $|f'(b)|^{m_2} > |f'(a)|^{m_1}$. If $|f'|$ is (α, m_1, m_2)-AG convex on $[m_1a, m_2b]$, then the following inequality*

$$\begin{aligned}
&\left|\frac{1}{m_2b-m_1a}\int_{m_1a}^{m_2b} f(x)dx - f\left(\frac{m_1a+m_2b}{2}\right)\right| \\
\leq \;& \frac{(m_2b-m_1a)\,|f'(b)|^{m_2}}{\alpha\left(\ln|f'(b)|^{m_2}-\ln|f'(a)|^{m_1}\right)^{\frac{2}{\alpha}}}\left\{\Gamma\left(\frac{2}{\alpha}\right) - \Gamma\left(\frac{2}{\alpha}, \ln|f'(b)|^{m_2}-\ln|f'(a)|^{m_1}\right)\right. \\
&+ \left(\ln|f'(b)|^{m_2}-\ln|f'(a)|^{m_1}\right)^{\frac{1}{\alpha}} \\
&\times\left[\Gamma\left(\frac{1}{\alpha}, \frac{\ln|f'(b)|^{m_2}-\ln|f'(a)|^{m_1}}{2^\alpha}\right) - \Gamma\left(\frac{1}{\alpha}, \ln|f'(b)|^{m_2}-\ln|f'(a)|^{m_1}\right)\right] \\
&\left.-\Gamma\left(\frac{2}{\alpha}, \frac{\ln|f'(b)|^{m_2}-\ln|f'(a)|^{m_1}}{2^\alpha}\right) + \Gamma\left(\frac{2}{\alpha}, \ln|f'(b)|^{m_2}-\ln|f'(a)|^{m_1}\right)\right\},
\end{aligned}$$

holds for $t \in [0,1]$ and $\alpha, m_1, m_2 \in (0,1]^3$.

Proof. From Lemma 4.1 and the following inequality

$$\left|f'\left(m_1ta+m_2(1-t)b\right)\right| \leq |f'(a)|^{m_1t^\alpha}\,|f'(b)|^{m_2(1-t^\alpha)},$$

we get

$$\begin{aligned}
&\left|\frac{1}{m_2b-m_1a}\int_{m_1a}^{m_2b} f(x)dx - f\left(\frac{m_1a+m_2b}{2}\right)\right| \\
\leq \;& \left|(m_2b-m_1a)\left[\int_0^{\frac{1}{2}} tf'\left(m_1ta+m_2(1-t)b\right)dt + \int_{\frac{1}{2}}^1 (t-1)f'\left(m_1ta+m_2(1-t)b\right)dt\right]\right| \\
\leq \;& (m_2b-m_1a)\left[\int_0^{\frac{1}{2}} t\left|f'\left(m_1ta+m_2(1-t)b\right)\right|dt + \int_{\frac{1}{2}}^1 |t-1|\left|f'\left(m_1ta+m_2(1-t)b\right)\right|dt\right] \\
\leq \;& (m_2b-m_1a)\left[\int_0^{\frac{1}{2}} t\,|f'(a)|^{m_1t^\alpha}\,|f'(b)|^{m_2(1-t^\alpha)}dt + \int_{\frac{1}{2}}^1 |t-1|\,|f'(a)|^{m_1t^\alpha}\,|f'(b)|^{m_2(1-t^\alpha)}dt\right] \\
= \;& \frac{(m_2b-m_1a)\,|f'(b)|^{m_2}}{\alpha\left(\ln|f'(b)|^{m_2}-\ln|f'(a)|^{m_1}\right)^{\frac{2}{\alpha}}}\left\{\Gamma\left(\frac{2}{\alpha}\right) - \Gamma\left(\frac{2}{\alpha}, \ln|f'(b)|^{m_2}-\ln|f'(a)|^{m_1}\right)\right. \\
&+ \left(\ln|f'(b)|^{m_2}-\ln|f'(a)|^{m_1}\right)^{\frac{1}{\alpha}} \\
&\times\left[\Gamma\left(\frac{1}{\alpha}, \frac{\ln|f'(b)|^{m_2}-\ln|f'(a)|^{m_1}}{2^\alpha}\right) - \Gamma\left(\frac{1}{\alpha}, \ln|f'(b)|^{m_2}-\ln|f'(a)|^{m_1}\right)\right] \\
&\left.-\Gamma\left(\frac{2}{\alpha}, \frac{\ln|f'(b)|^{m_2}-\ln|f'(a)|^{m_1}}{2^\alpha}\right) + \Gamma\left(\frac{2}{\alpha}, \ln|f'(b)|^{m_2}-\ln|f'(a)|^{m_1}\right)\right\}.
\end{aligned}$$

This completes the proof of theorem. □

Corollary 4.3. *Under the conditions of Theorem 4.2, If we take* $m_1 = m_2 = 1$ *and* $\alpha = 1$ *then we get the following inequality:*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \\
\leq\ & \frac{(b-a)\left|f'(b)\right|}{\ln\left|f'(b)\right| - \ln\left|f'(a)\right|}\left\{1 - \Gamma\left(2, \ln\left|f'(b)\right| - \ln\left|f'(a)\right|\right)\right. \\
& + \left(\ln\left|f'(b)\right| - \ln\left|f'(a)\right|\right)\left[\Gamma\left(1, \frac{\ln\left|f'(b)\right| - \ln\left|f'(a)\right|}{2}\right) - \Gamma\left(1, \ln\left|f'(b)\right| - \ln\left|f'(a)\right|\right)\right] \\
& \left. -\Gamma\left(2, \frac{\ln\left|f'(b)\right| - \ln\left|f'(a)\right|}{2}\right) + \Gamma\left(2, \ln\left|f'(b)\right| - \ln\left|f'(a)\right|\right)\right\},
\end{aligned}$$

Theorem 4.4. *Let* $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* I°, $m_1a, m_2b \in I^\circ$ *with* $m_1a < m_2b$ *and* $f' \in L\left[m_1a, m_2b\right]$, $\left|f'(b)\right|^{m_2} > \left|f'(a)\right|^{m_1}$, *and let* $q > 1$. *If* $|f'|$ *is* (α, m_1, m_2)*-AG convex on* $[m_1a, m_2b]$, *then the following inequality*

$$\begin{aligned}
&\left|\frac{1}{m_2b - m_1a}\int_{m_1a}^{m_2b} f(x)dx - f\left(\frac{m_1a + m_2b}{2}\right)\right| \\
\leq\ & (m_2b - m_1a)\left(\frac{1}{(p+1)2^{p+1}}\right)^{\frac{1}{p}}\left[\left|f'(b)\right|^{qm_2}\frac{\Gamma\left(\frac{1}{\alpha}\right) - \Gamma\left(\frac{1}{\alpha}, \frac{\ln\left|f'(b)\right|^{qm_2} - \ln\left|f'(a)\right|^{qm_1}}{2^\alpha}\right)}{\alpha\left(\ln\left|f'(b)\right|^{qm_2} - \ln\left|f'(a)\right|^{qm_1}\right)^{\frac{1}{\alpha}}}\right. \\
& \left. + \left|f'(b)\right|^{qm_2}\frac{\Gamma\left(\frac{1}{\alpha}, \frac{\ln\left|f'(b)\right|^{qm_2} - \ln\left|f'(a)\right|^{qm_1}}{2^\alpha}\right) - \Gamma\left(\frac{1}{\alpha}, \ln\left|f'(b)\right|^{qm_2} - \ln\left|f'(a)\right|^{qm_1}\right)}{\alpha\left(\ln\left|f'(b)\right|^{qm_2} - \ln\left|f'(a)\right|^{qm_1}\right)^{\frac{1}{\alpha}}}\right],
\end{aligned}$$

holds for $t \in [0,1]$ *and* $\alpha, m_1, m_2 \in (0,1]^3$, *where* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 4.1, Hölder's integral inequality and the inequality

$$\left|f'\left(m_1ta + m_2(1-t)b\right)\right|^q \leq \left|\left[f(x)\right]^{m_1t^\alpha}\left[f(y)\right]^{m_2(1-t^\alpha)}\right|^q \leq \left|f'(a)\right|^{qm_1t^\alpha}\left|f'(b)\right|^{qm_2(1-t^\alpha)},$$

we obtain

$$\begin{aligned}
&\left|\frac{1}{m_2b-m_1a}\int_{m_1a}^{m_2b} f(x)dx - f\left(\frac{m_1a+m_2b}{2}\right)\right| \\
\leq\ & (m_2b-m_1a)\left[\int_0^{\frac{1}{2}} |t|\left|f'\left(m_1ta+m_2(1-t)b\right)\right| dt + \int_{\frac{1}{2}}^1 |t-1|\left|f'\left(m_1ta+m_2(1-t)b\right)\right| dt\right] \\
\leq\ & (m_2b-m_1a)\left[\left(\int_0^{\frac{1}{2}} t^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|[f(a)]^{m_1t^\alpha}[f(b)]^{m_2(1-t^\alpha)}\right|^q dt\right)^{\frac{1}{q}}\right. \\
& \left.+\left(\int_{\frac{1}{2}}^1 |t-1|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|[f(a)]^{m_1t^\alpha}[f(b)]^{m_2(1-t^\alpha)}\right|^q dt\right)^{\frac{1}{q}}\right] \\
\leq\ & (m_2b-m_1a)\left[\left(\int_0^{\frac{1}{2}} t^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}|f'(a)|^{qm_1t^\alpha}|f'(b)|^{qm_2(1-t^\alpha)} dt\right)^{\frac{1}{q}}\right. \\
& \left.+\left(\int_{\frac{1}{2}}^1 |t-1|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1|f'(a)|^{qm_1t^\alpha}|f'(b)|^{qm_2(1-t^\alpha)} dt\right)^{\frac{1}{q}}\right] \\
=\ & (m_2b-m_1a)\left(\frac{1}{(p+1)2^{p+1}}\right)^{\frac{1}{p}}\left[|f'(b)|^{qm_2}\frac{\Gamma\left(\frac{1}{\alpha}\right)-\Gamma\left(\frac{1}{\alpha},\frac{\ln|f'(b)|^{qm_2}-\ln|f'(a)|^{qm_1}}{2^\alpha}\right)}{\alpha\left(\ln|f'(b)|^{qm_2}-\ln|f'(a)|^{qm_1}\right)^{\frac{1}{\alpha}}}\right. \\
& \left.+|f'(b)|^{qm_2}\frac{\Gamma\left(\frac{1}{\alpha},\frac{\ln|f'(b)|^{qm_2}-\ln|f'(a)|^{qm_1}}{2^\alpha}\right)-\Gamma\left(\frac{1}{\alpha},\ln|f'(b)|^{qm_2}-\ln|f'(a)|^{qm_1}\right)}{\alpha\left(\ln|f'(b)|^{qm_2}-\ln|f'(a)|^{qm_1}\right)^{\frac{1}{\alpha}}}\right],
\end{aligned}$$

where

$$\int_0^{\frac{1}{2}} t^p dt = \int_{\frac{1}{2}}^1 |t-1|^p dt = \frac{1}{(p+1)2^{p+1}}.$$

This completes the proof of theorem. □

Corollary 4.5. *Under the conditions of Theorem 4.4, if we take $\alpha = 1$ then we get the following inequality:*

$$\begin{aligned}
&\left|\frac{1}{m_2b-m_1a}\int_{m_1a}^{m_2b} f(x)dx - f\left(\frac{m_1a+m_2b}{2}\right)\right| \\
\leq\ & (m_2b-m_1a)\left(\frac{1}{(p+1)2^{p+1}}\right)^{\frac{1}{p}}\left[|f'(b)|^{qm_2}\frac{1-\Gamma\left(1,\frac{\ln|f'(b)|^{qm_2}-\ln|f'(a)|^{qm_1}}{2}\right)}{\ln|f'(b)|^{qm_2}-\ln|f'(a)|^{qm_1}}\right. \\
& \left.+|f'(b)|^{qm_2}\frac{\Gamma\left(1,\frac{\ln|f'(b)|^{qm_2}-\ln|f'(a)|^{qm_1}}{2}\right)-\Gamma\left(1,\ln|f'(b)|^{qm_2}-\ln|f'(a)|^{qm_1}\right)}{\ln|f'(b)|^{qm_2}-\ln|f'(a)|^{qm_1}}\right],
\end{aligned}$$

where $\Gamma(1) = 1$.

Corollary 4.6. *Under the conditions of Theorem 4.4, if we take $m_1 = m_2 = 1$ and $\alpha = 1$ then we get the following inequality:*

$$\begin{aligned}\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq\ & (b-a)\left(\frac{1}{(p+1)2^{p+1}}\right)^{\frac{1}{p}}\left[|f'(b)|^q \frac{1-\Gamma\left(1, \frac{\ln|f'(b)|^q - \ln|f'(a)|^q}{2}\right)}{\ln|f'(b)|^q - \ln|f'(a)|^q}\right.\\ & \left. + |f'(b)|^q \frac{\Gamma\left(1, \frac{\ln|f'(b)|^q - \ln|f'(a)|^q}{2}\right) - \Gamma\left(1, \ln|f'(b)|^q - \ln|f'(a)|^q\right)}{\ln|f'(b)|^q - \ln|f'(a)|^q}\right],\end{aligned}$$

where $\Gamma(1) = 1$.

Theorem 4.7. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $m_1a, m_2b \in I^\circ$ with $m_1a < m_2b$ and $f' \in L[m_1a, m_2b]$, $|f'(b)|^{m_2} > |f'(a)|^{m_1}$ and let $q \geq 1$. If $|f'|$ is (α, m_1, m_2)-AG convex on $[m_1a, m_2b]$, then the following inequality*

$$\begin{aligned}&\left|\frac{1}{m_2b-m_1a}\int_{m_1a}^{m_2b} f(x)dx - f\left(\frac{m_1a+m_2b}{2}\right)\right|\\ \leq\ & (m_2b-m_1a)\left(\frac{1}{8}\right)^{1-\frac{1}{q}}\left[K_1^{\frac{1}{q}}(a,b,q,f;m_1,m_2,\alpha) + K_2^{\frac{1}{q}}(a,b,q,f;m_1,m_2,\alpha)\right]\end{aligned}$$

holds for $t \in [0,1]$ and $\alpha, m_1, m_2 \in (0,1]^3$, where $\frac{1}{p}+\frac{1}{q}=1$ and

$$\begin{aligned}K_1(a,b,q,f;m_1,m_2,\alpha) &:= \int_0^{\frac{1}{2}} t^q |f'(a)|^{qm_1t^\alpha} |f'(b)|^{qm_2(1-t^\alpha)} dt,\\ K_2(a,b,q,f;m_1,m_2,\alpha) &:= \int_{\frac{1}{2}}^1 |t-1|^q |f'(a)|^{qm_1t^\alpha} |f'(b)|^{qm_2(1-t^\alpha)} dt.\end{aligned}$$

Proof. Using Lemma 4.1, power mean inequality and the inequality

$$|f'(m_1ta + m_2(1-t)b)|^q \leq |f'(a)|^{qm_1t^\alpha} |f'(b)|^{qm_2(1-t^\alpha)},$$

we get

$$\begin{aligned}&\left|\frac{1}{m_2b-m_1a}\int_{m_1a}^{m_2b} f(x)dx - f\left(\frac{m_1a+m_2b}{2}\right)\right| \qquad (4.1)\\ \leq\ & (m_2b-m_1a)\left[\int_0^{\frac{1}{2}} |t|\,|f'(m_1ta+m_2(1-t)b)|\,dt + \int_{\frac{1}{2}}^1 |t-1|\,|f'(m_1ta+m_2(1-t)b)|\,dt\right]\\ \leq\ & (m_2b-m_1a)\left[\int_0^{\frac{1}{2}} t^q \left|[f(a)]^{m_1t^\alpha}[f(b)]^{m_2(1-t^\alpha)}\right|^q dt + \int_{\frac{1}{2}}^1 |t-1|\left|[f(a)]^{m_1t^\alpha}[f(b)]^{m_2(1-t^\alpha)}\right|^q dt\right]\\ \leq\ & (m_2b-m_1a)\left[\left(\int_0^{\frac{1}{2}} tdt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}} t^q |f'(a)|^{qm_1t^\alpha}|f'(b)|^{qm_2(1-t^\alpha)}dt\right)^{\frac{1}{q}}\right.\\ & \left. + \left(\int_{\frac{1}{2}}^1 |t-1|\,dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1 |t-1|^q |f'(a)|^{qm_1t^\alpha}|f'(b)|^{qm_2(1-t^\alpha)}dt\right)^{\frac{1}{q}}\right]\\ =\ & (m_2b-m_1a)\left(\frac{1}{8}\right)^{1-\frac{1}{q}}\left[K_1^{\frac{1}{q}}(a,b,q,f;m_1,m_2,\alpha) + K_2^{\frac{1}{q}}(a,b,q,f;m_1,m_2,\alpha)\right].\end{aligned}$$

□

Corollary 4.8. *Under the conditions of Theorem 4.7, if we take $q = 1$, then we get the following inequality:*

$$\begin{aligned} & \left| \frac{1}{m_2 b - m_1 a} \int_{m_1 a}^{m_2 b} f(x)dx - f\left(\frac{m_1 a + m_2 b}{2}\right) \right| \\ \leq \quad & (m_2 b - m_1 a) \left[K_1(a, b, 1, f; m_1, m_2, \alpha) + K_2(a, b, 1, f; m_1, m_2, \alpha) \right]. \end{aligned}$$

Corollary 4.9. *Under the conditions of Theorem 4.7, if we take $m_1 = m_2 = 1$ and $\alpha = 1$ then we get the following inequality:*

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right| \quad \leq \quad (b-a) \left(\frac{1}{8}\right)^{1-\frac{1}{q}} \left[K_1^{\frac{1}{q}}(a, b, q, f; 1, 1, 1) + K_2^{\frac{1}{q}}(a, b, q, f; 1, 1, 1) \right].$$

Corollary 4.10. *Under the conditions of Theorem 4.7, if we take $m_1 = m_2 = 1$ and $\alpha = 1, q = 1$ then we get the following inequality:*

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right| \quad \leq \quad 2L(a,b) \left[\frac{A(|f'(a)|, |f'(b)|) - 2G^2(|f'(a)|, |f'(b)|)}{\ln b - \ln a} \right].$$

Corollary 4.11. *Under the conditions of Theorem 4.7, we can also write the following inequality:*

$$\begin{aligned} & \left| \frac{1}{m_2 b - m_1 a} \int_{m_1 a}^{m_2 b} f(x)dx - f\left(\frac{m_1 a + m_2 b}{2}\right) \right| \\ \leq \quad & (m_2 b - m_1 a) \left[|f'(b)|^{q m_2} \frac{\Gamma\left(\frac{1}{\alpha}\right) - \Gamma\left(\frac{1}{\alpha}, \frac{\ln|f'(b)|^{q m_2} - \ln|f'(a)|^{q m_1}}{2^\alpha}\right)}{\alpha\left(\ln|f'(b)|^{q m_2} - \ln|f'(a)|^{q m_1}\right)^{\frac{1}{\alpha}}} \right. \\ & \left. + |f'(b)|^{q m_2} \frac{\Gamma\left(\frac{1}{\alpha}, \frac{\ln|f'(b)|^{q m_2} - \ln|f'(a)|^{q m_1}}{2^\alpha}\right) - \Gamma\left(\frac{1}{\alpha}, \ln|f'(b)|^{q m_2} - \ln|f'(a)|^{q m_1}\right)}{\alpha\left(\ln|f'(b)|^{q m_2} - \ln|f'(a)|^{q m_1}\right)^{\frac{1}{\alpha}}} \right]. \end{aligned}$$

Proof. By using $t \leq 1$, $|t-1| \leq 1$, $|t-1|^q \leq 1$ and $t^q \leq 1$ in the the inequality (4.1), we obtain the desired result. □

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TOPOLOGICAL PROPERTIES OF GENERALIZED CLOSURE SPACES

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ABSTRACT. This paper deals with some points on generalized closure spaces and its related spaces. This paper also interrelates between interior points, closure points, boarder points, exterior points and limit points. These points will characterize continuity and homeomorphism. This paper also gives a brief discussion on category sets of generalized closure spaces.

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1. Introduction and Preliminaries

Renowned topologist Hausdorff [11] had started the study of closure spaces after that the mathematicians Day [1], Hammer [9, 10], Gnilka [3, 4, 5], Stadler [15, 16] and Habil and Elzenati [7, 8, 6] have studied this space and introduced various types of generalization spaces. The study of these concepts are the comparative study with topological spaces. The study of closure spaces and its related spaces like generalized closure spaces, isotonic closure spaces [14, 16, 8] are going on. The following are the formal definitions of closures and its related spaces:

Let X be a set, $\wp(X)$ its power set and $cl : \wp(X) \to \wp(X)$ be an arbitrary set-valued set-function. We call $cl(A),\ A \subseteq X$, the closure of A, and we call the pair (X, cl)(or simply X) a generalized closure space. Consider the following axioms of the closure function for all $A,\ B,\ A_\lambda \in \wp(X)$:

K0) $cl(\emptyset) = \emptyset$.
K1) $A \subseteq B$ implies $cl(A) \subseteq cl(B)$ (isotonic).
K2) $A \subseteq cl(A)$ (expanding).
K3) $cl(A \cup B) \subseteq cl(A) \cup cl(B)$ (subadditive).
K4) $cl(cl(A)) = cl(A)$ (idempotent).
K5) $\bigcup_{\lambda\in\Lambda} cl(A_\lambda) = cl(\bigcup_{\lambda\in\Lambda}(A_\lambda))$ (additive).
K6) $cl(A \cup B) = cl(A) \cup cl(B)$.

The dual of a closure function is called the interior function $int : \wp(X) \to \wp(X)$ which is defined by

$$int(A) = X \setminus cl(X \setminus A).$$

A set $A \in \wp(X)$ is said to be closed in the generalized closure space (X, cl) if $cl(A) = A$ holds. It is said to be open if its complement $X \setminus A$ is closed or equivalently $A = int(A)$.

Definition 1.1. [15, 16, 12] *An isotonic space is a pair (X, cl) (or simply X), where X is a set and $cl : \wp(X) \to \wp(X)$ satisfies the axioms (K0) and (K1). An isotonic space (X, cl) that satisfies the axiom*

(K2) is called a neighborhood space. A closure space is a neighborhood space. A closure space is a neighborhood space that satisfies (K4). A topological space is a closure space satisfies (K3).

In this paper, we consider some points of generalized closure spaces and its related spaces. We characterize continuity in terms of boundary, border, exterior points. We discuss about the topological property of these points. We also give a brief discussion on category sets on generalized closure spaces.

2. Limit points

Definition 2.1. [15] *Let cl and int be a closure function and its dual interior function on X, respectively. Then the neighborhood function $\mathcal{N} : X \to \wp(\wp(X))$ (resp. the convergent function $\mathcal{N}^* : X \to \wp(\wp(X))$) assigns to each $x \in X$ the collection*

$$\mathcal{N}(x) = \{N \in \wp(X) : \ x \in int(N)\}.$$
$$(resp.\ \mathcal{N}^*(x) = \{Q \in \wp(X) : \ x \in cl(Q)\}).$$

Definition 2.2. [15] *Let A be a subset of a generalized closure space X. A point p is a limit point of A if each neighborhood $N \in \mathcal{N}(p)$ satisfies $N \cap (A \setminus \{p\}) \neq \emptyset$. The set*

$$A^{\blacktriangledown} = \{p \in X : \ N \cap (A \setminus \{p\}) \neq \emptyset, \ for\ each\ N \in \mathcal{N}(p)\}$$

of all limit points of A is called the derived set of A.

Theorem 2.3. *For subsets A, B of a generalized closure space X, the following statements hold:*
1. *If $A \subseteq B$, then $A^{\blacktriangledown} \subseteq B^{\blacktriangledown}$* [15].
2. *$A^{\blacktriangledown} \cup B^{\blacktriangledown} = (A \cup B)^{\blacktriangledown}$ and $(A \cap B)^{\blacktriangledown} \subseteq A^{\blacktriangledown} \cap B^{\blacktriangledown}$.*
3. *$(A^{\blacktriangledown})^{\blacktriangledown} \setminus A \subseteq A^{\blacktriangledown}$.*
4. *$A^{\blacktriangledown} \subseteq cl(A)$ and $A^{\blacktriangledown} \setminus A = cl(A) \setminus A$* [15].
5. *$(A \cup A^{\blacktriangledown})^{\blacktriangledown} \subseteq A \cup A^{\blacktriangledown}$.*
6. *If A is closed, then $A^{\blacktriangledown} \subseteq A$.*

Proof. (2) From (1), we have $A^{\blacktriangledown} \cup B^{\blacktriangledown} \subseteq (A \cup B)^{\blacktriangledown}$. For the reverse inclusion, suppose $x \in (A \cup B)^{\blacktriangledown}$. Then each $N \in \mathcal{N}(x)$, $N \cap ((A \cup B) \setminus \{x\}) \neq \emptyset$. Hence $N \cap [(A \setminus \{x\}) \cup (B \setminus \{x\})] \neq \emptyset$ and $N \cap (A \setminus \{x\}) \neq \emptyset$ or $N \cap (B \setminus \{x\}) \neq \emptyset$. Therefore, $x \in A^{\blacktriangledown} \cup B^{\blacktriangledown}$.

(3) If $x \in (A^{\blacktriangledown})^{\blacktriangledown} \setminus A$ and $N \in \mathcal{N}(x)$, then $N \cap (A^{\blacktriangledown} \setminus \{x\}) \neq \emptyset$. Let $y \in N \cap (A^{\blacktriangledown} \setminus \{x\})$. Then, since $y \in A^{\blacktriangledown}$ and $y \in N$, $N \cap (A \setminus \{y\}) \neq \emptyset$. Let $z \in N \cap (A \setminus \{y\})$. Then $z \neq x$ for $z \in A$ and $x \notin A$. Hence $N \cap (A \setminus \{x\}) \neq \emptyset$. Therefore $x \in A^{\blacktriangledown}$.

(5) Let $x \in (A \cup A^{\blacktriangledown})^{\blacktriangledown}$. If $x \in A$, the result is obvious. So let $x \in (A \cup A^{\blacktriangledown})^{\blacktriangledown} \setminus A$, then for each $N \in \mathcal{N}(x)$, $N \cap [(A \cup A^{\blacktriangledown}) \setminus \{x\}] \neq \emptyset$. Thus $N \cap (A \setminus \{x\}) \neq \emptyset$ or $N \cap (A^{\blacktriangledown} \setminus \{x\}) \neq \emptyset$. Now it follows similarly from (3) that $N \cap (A \setminus \{x\}) \neq \emptyset$. Hence $x \in A^{\blacktriangledown}$. Therefore, in any case $(A \cup A^{\blacktriangledown})^{\blacktriangledown} \subseteq A \cup A^{\blacktriangledown}$. □

Definition 2.4. *Let A be a subset of a generalized closure space X, then $b(A) = A \setminus int(A)$ called the border of A.*

Before discussing the properties of the border of a set, we recall following:

Lemma 2.5. [8] *A generalized space X is isotonic if and only if the interior function $int : \wp(X) \to \wp(X)$ satisfies:*
1. *$int(X) = X$;*
2. *$int(A) \subseteq int(B)$ for every $A \subseteq B \subseteq X$.*

Theorem 2.6. *For a subset A of a generalized closure space X, the following statements hold:*
1. *$A = int(A) \cup b(A)$.*
2. *$int(A) \cap b(A) = \emptyset$.*
3. *A is open if and only if $b(A) = \emptyset$, when the closure function cl is expanding.*
4. *$int(b(A)) = \emptyset$, if the space X is a neighborhood space.*
5. *$b(b(A)) = b(A)$, when the closure function cl is isotonic and expanding.*
6. *$b(A) = A \cap cl(X \setminus A)$.*

Proof. (1), (2) The proofs are obvious.

(3) Suppose A is open. Then $A = int(A)$, so $b(A) = \emptyset$. Conversely suppose that $b(A) = \emptyset$, then $A \subseteq int(A)$. Since cl is expanding, $int(A) \subseteq A$ and $A = int(A)$. Hence A is open.

(4) If possible suppose that $int(b(A)) \neq \emptyset$. Let $x \in int(b(A))$, then $x \in b(A)$, since the closure function is expanding. Since $b(A) \subseteq A$, then $x \in int(A)$ (since the closure function is isotonic that is, $x \in int(b(A)) \subseteq int(A)$). Thus we have, $x \in int(A) \cap b(A)$ which is contrary to (2). Thus $int(b(A)) = \emptyset$.

(5) $b(b(A)) = b(A) \setminus int(b(A)) = b(A)$ (since $int(b(A) = \emptyset)$).

(6) $b(A) = A \setminus int(A) = A \cap (X \setminus int(A)) = A \cap cl(X \setminus A)$.

□

Definition 2.7. *A subset D of a generalized closure space X is said to be dense in X (or simply dense) if $cl(D) = X$.*

Chattopadhyay [2] defined this idea by the name of sgc-dense and showed that a dense set D in the generalized closure space X dose not imply that $U \cap D \neq \emptyset$ for all nonempty open set U in X.

If the closure function is isotonic of a generalized closure space X, then every super set of a dense set is always a dense set.

Theorem 2.8. *Let A be a subset of a generalized closure space X. Then following hold:*

(1) If $int(A) = \emptyset$, Then $X \setminus A$ is dense in X.

(2) $X \setminus b(A)$ is dense in X, if the space X is a neighborhood space.

(3) If $int(X \setminus A) = \emptyset$, then A is dense in X.

Proof. (1), (3) The proofs are obvious.

(2) Since $int(b(A)) = \emptyset$ (from Theorem 2.6), then $cl(X \setminus b(A)) = (X \setminus int(b(A))) = X$.

□

3. Topological property

The following are the relevant definitions of this section.

Definition 3.1. [15] *Let (X, cl_X) and (Y, cl_Y) be two generalized closure spaces. A function $f : X \to Y$ is continuous if $cl_X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B))$ for each $B \in \wp(X)$.*

Definition 3.2. [15] *Let (X, cl_X) and (Y, cl_Y) be two generalized closure spaces. Then $f : X \to Y$ is closure-preserving if for all $A \in \wp(X)$, $f(cl_X(A)) \subseteq cl_Y(f(A))$.*

Lemma 3.3. *Let (X, cl_X) and (Y, cl_Y) be two generalized closure spaces and $f : X \to Y$ be a bijective closure-preserving function. Then $f(int_X(A)) \supseteq int_Y f(A))$ for each $A \in \wp(X)$.*

Proof. Since f is closure preserving, then $f(cl_X(X \setminus A) \subseteq cl_Y(f(X \setminus A))$. Then $f(X) \setminus f(int_X(A)) \subseteq Y \setminus int_Y f(A)$, Thus $int_Y f(A) \subseteq f(int_X(A))$. □

Definition 3.4. [8] *Let (X, cl_X) and (Y, cl_Y) be two generalized closure spaces. A function $f : X \to Y$ is called a homeomorphism if f is a continuous bijection and f^{-1} is also continuous. In this case, we say that X and Y are homeomorphic.*

Theorem 3.5. [15] *Let (X, cl_X) and (Y, cl_Y) be isotonic spaces. Then, for a function $f : X \to Y$, the following properties are equivalent:*

(1) $f : X \to Y$ is continuous.

(2) $f : X \to Y$ is closure-preserving.

(3) $f(A) \subseteq B$ implies $f(cl_X(A)) \subseteq cl_Y(B)$ for all $A \in \wp(X)$ and $B \in \wp(Y)$.

Theorem 3.6. [8] *Let (X, cl_X) and (Y, cl_Y) be isotonic spaces and $f : X \to Y$ be a bijection. Then f is a homeomorphism if and only if $f(cl_X(A)) = cl_Y(f(A))$ for each $A \subseteq X$.*

Remark 3.7. [8] *It should be noted that in Theorem 3.6 we can replace* $f(cl_X(A)) = cl_Y(f(A))$ *by* $f(int_X(A)) = int_Y(f(A))$, *and hence if* $U \in \mathcal{N}(x)$ *then* $f(U) \in \mathcal{N}(f(x))$ *for all* $x \in X$.

Definition 3.8. [8] *A topological property is a property which, if possible by a space* X, *then it is possessed by all spaces homeomorphic to* X.

Definition 3.9. [13] *Let* A *be a subset of a generalized closure space* X, $bd(A) = cl(A) \cap cl(X \setminus A)$ *is said to be boundary of* A.

Theorem 3.10. *Let* (X, cl_X) *and* (Y, cl_Y) *be two generalized closure spaces and* $f : X \to Y$ *be a closure-preserving, injective function. Then* $f(bd_X(A)) \subseteq bd_Y(f(A))$ *for each* $A \in \wp(X)$.

Proof. $f(bd_X(A)) = f[cl_X(A) \cap cl_X(X \setminus A)] \subseteq cl_Y f(A) \cap cl_Y f(X \setminus A) = bd_Y f(A)$. □

Theorem 3.11. *Let* (X, cl_X) *and* (Y, cl_Y) *be two isotonic spaces and* $f : X \to Y$ *be a homeomorphism. Then* $f(bd_X(A)) = bd_Y(f(A))$ *for each* $A \in \wp(X)$.

Proof. The proof is obvious from the following facts:
(1) $f(cl_X(A)) = cl_Y(f(A))$ for each $A \in \wp(X)$.
(2) If f is injective, then $f(A \cap B) = f(A) \cap f(B)$ for each $A,\ B \in \wp(X)$. □

Now we recall the following theorem from [15].

Theorem 3.12. *Let* (X, cl_X) *and* (Y, cl_Y) *be generalized closure spaces and* $f : X \to Y$ *be a function. Then the following conditions (for continuity) are equivalent:*
(1) $cl_X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B))$ *for all* $B \in \wp(Y)$.
(2) $f^{-1}(int_Y(B)) \subseteq int_X(f^{-1}(B))$ *for all* $B \in \wp(Y)$.
(3) $B \in \mathcal{N}(f(x))$ *implies* $f^{-1}(B) \in \mathcal{N}(x)$ *for all* $B \in \wp(Y)$ *and all* $x \in X$.
(4) $f^{-1}(B) \in \mathcal{N}^*(x)$ *implies* $B \in \mathcal{N}^*(f(x))$ *for all* $B \in \wp(Y)$ *and all* $x \in X$.
Conditions (3) and (4) are equivalent for each individual $x \in X$ *as well.*

Theorem 3.13. *Let* (X, cl_X) *and* (Y, cl_Y) *be generalized closure spaces. Then* $f : X \to Y$ *is continuous if and only if* $b_X(f^{-1}(B)) \subseteq f^{-1}(b_Y(B))$ *for each* $B \in \wp(Y)$.

Proof. Let $f : X \to Y$ be continuous. Then $f^{-1}(int_Y(B)) \subseteq int_X(f^{-1}(B))$. Now $f^{-1}(b_Y(B)) = f^{-1}[B \setminus int_Y(B)] = f^{-1}(B) \setminus f^{-1}(int_Y(B)) \supseteq f^{-1}(B) \setminus int_X(f^{-1}(B)) = b_X(f^{-1}(B))$.

Conversely suppose that $b_X(f^{-1}(B)) \subseteq f^{-1}(b_Y(B))$. Then $f^{-1}(B) \setminus int_X f^{-1}(B) \subseteq f^{-1}(B \setminus int_Y(B)) = f^{-1}(B) \setminus f^{-1}(int_Y(B))$. This implies that $f^{-1}(int_Y(B)) \subseteq int_X f^{-1}(B)$.
□

Theorem 3.12 can be restated as follows:

Corollary 3.14. *Let* (X, cl_X) *and* (Y, cl_Y) *be generalized closure spaces and* $f : X \to Y$ *be a function. Then the following conditions (for continuity) are equivalent:*
(1) $cl_X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B))$ *for all* $B \in \wp(Y)$.
(2) $b_X(f^{-1}(B)) \subseteq f^{-1}(b_Y(B))$ *for each* $B \in \wp(Y)$.
(3) $f^{-1}(int_Y(B)) \subseteq int_X(f^{-1}(B))$ *for all* $B \in \wp(Y)$.
(4) $B \in \mathcal{N}(f(x))$ *implies* $f^{-1}(B) \in \mathcal{N}(x)$ *for all* $B \in \wp(Y)$ *and all* $x \in X$.
(5) $f^{-1}(B) \in \mathcal{N}^*(x)$ *implies* $B \in \mathcal{N}^*(f(x))$ *for all* $B \in \wp(Y)$ *and all* $x \in X$.
Conditions (4) and (5) are equivalent for each individual $x \in X$ *as well.*

Theorem 3.15. *Let* (X, cl_X) *and* (Y, cl_Y) *be two generalized closure spaces and* $f : X \to Y$ *be a closure-preserving bijective function. Then* $f(b_X(A)) \subseteq b_Y(f(A))$ *for each* $A \in \wp(X)$.

Proof. $f(b_X(A)) = f[A \setminus int_X(A)] = f(A) \setminus f(int_X(A)) \subseteq f(A) \setminus int_Y f(A)$ (from Lemma 3.3) $= b_Y(f(A))$.
□

Theorem 3.16. *Let (X, cl_X) and (Y, cl_Y) be two isotonic spaces and $f : X \to Y$ be a homeomorphism. Then $f(b_X(A)) = b_Y(f(A))$ for each $A \in \wp(X)$.*

Proof. Since f is injective, then $f(A \setminus int_X(A)) = f(A) \setminus f(int_X(A)) = f(A) \setminus int_Y f(A)$ (from Remark 3.7). □

Theorem 3.17. *Let (X, cl_X) and (Y, cl_Y) be two isotonic spaces and $f : X \to Y$ be a bijective function such that $f(b_X(A)) = b_Y(f(A))$ for each $A \in \wp(X)$. Then f is a homeomorphism.*

Proof. Given that $f(b_X(A)) = b_Y(f(A))$. Then $f(A \setminus int_X(A)) = f(A) \setminus int_Y f(A)$ and hence $f(int_X(A)) = int_Y(f(A))$. Thus from Remark 3.7, f is a homeomorphism. □

Corollary 3.18. *Let (X, cl_X) and (Y, cl_Y) be two isotonic spaces and $f : X \to Y$ be a bijective function. Then f is a homeomorphism if and only if $f(b_X(A)) = b_Y(f(A))$ for each $A \in \wp(X)$.*

Definition 3.19. *Let A be a subset of a generalized closure space X. Then $ext(A) = int(X \setminus A)$ is called the exterior of A.*

Theorem 3.20. *Let (X, cl_X) and (Y, cl_Y) be two generalized closure spaces. If $f : X \to Y$ is bijective and closure-preserving, then $f(ext_X(A)) \supseteq ext_Y(f(A))$ for each $A \in \wp(X)$.*

Proof. $f(ext_X(A)) = f(int_X(X \setminus A)) = f(X \setminus cl_X(A)) = f(X) \setminus f(cl_X(A)) \supseteq Y \setminus cl_Y f(A) = int_Y(Y \setminus f(A)) = ext_Y(f(A))$. □

Theorem 3.21. *Let (X, cl_X) and (Y, cl_Y) be generalized closure spaces and $f : X \to Y$ be a function. Then f is continuous if and only if $f^{-1}(ext_Y(B)) \subseteq ext_X f^{-1}(B)$ for each $B \in \wp(Y)$.*

Proof. Suppose f is continuous. Then for all $B \in \wp(Y)$, $f^{-1}(int_Y(B)) \subseteq int_X f^{-1}(B)$ for all $B \in \wp(Y)$. Now for $B \in \wp(Y)$, $f^{-1}(ext_Y(B)) = f^{-1}(int_Y(Y \setminus B)) \subseteq int_X f^{-1}(Y \setminus B) = int_Y(X \setminus f^{-1}(B)) = ext_Y f^{-1}(B)$.

Conversely suppose that $f^{-1}(ext_Y(B)) \subseteq ext_X(f^{-1}(B))$ for each $B \in \wp(Y)$. Then $f^{-1}(int_Y(Y \setminus B)) \subseteq int_X(X \setminus f^{-1}(B))$ implies $f^{-1}(Y \setminus cl_Y(B)) \subseteq X \setminus cl_X f^{-1}(B)$. Thus, $X \setminus f^{-1}(cl_Y(B)) \subseteq X \setminus cl_X f^{-1}(B)$ and hence $cl_X f^{-1}(B) \subseteq f^{-1}(cl_X(B))$. Thus from Theorem 3.12, f is continuous. □

Theorem 3.12 can be restated as follows:

Corollary 3.22. *Let (X, cl_X) and (Y, cl_Y) be generalized closure spaces and $f : X \to Y$. Then following conditions (for continuity) are equivalent:*
(1) $cl_X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B))$ for all $B \in \wp(Y)$.
(2) $f^{-1}(int_X(B)) \subseteq int_Y(f^{-1}(B))$ for all $B \in \wp(Y)$.
(3) $b_X(f^{-1}(B)) \subseteq f^{-1}(b_Y(B))$ for all $B \in \wp(Y)$.
(4) $f^{-1}(ext_Y(B)) \subseteq ext_X(f^{-1}(B))$ for all $B \in \wp(Y)$.
(5) $B \in \mathcal{N}(f(x))$ implies $f^{-1}(B) \in \mathcal{N}(x)$ for all $B \in \wp(Y)$ and all $x \in X$.
(6) $f^{-1}(B) \in \mathcal{N}^*(x)$ implies $B \in \mathcal{N}^*(f(x))$ for all $B \in \wp(Y)$ and all $x \in X$.
Conditions (5) and (6) are equivalent for each individual $x \in X$ as well.

Theorem 3.23. *Let (X, cl_X) and (Y, cl_Y) be two isotonic spaces and $f : X \to Y$ be a homeomorphism. Then $f(ext_X(A)) = ext_Y(f(A))$ for each $A \in \wp(X)$.*

Proof. Since f is bijective, then $f(ext_X(A)) = f(int_X(X \setminus A)) = f(X \setminus cl_X(A)) = Y \setminus f(cl_X(A)) = Y \setminus cl_Y f(A) = int_Y(Y \setminus f(A)) = ext_Y f(A)$. □

The converse of Theorem 3.23 is stated as follows:

Theorem 3.24. *Let (X, cl_X) and (Y, cl_Y) be two isotonic spaces and $f : X \to Y$ be a bijective function such that $f(ext_X(A)) = ext_Y f(A)$ for each $A \in \wp(X)$. Then f is a homeomorphism.*

Proof. Given that $f(X \setminus cl_X(A)) = Y \setminus cl_Y(f(A))$. Then $f(cl_X(A)) = cl_Y f(A)$. Thus, by Theorem 3.6 f is a homeomorphism. □

Corollary 3.25. *Let (X, cl_X) and (Y, cl_Y) be two isotonic spaces and $f : X \to Y$ be a bijective function. Then f is a homeomorphism if and only if $f(ext_X(A)) = ext_Y(f(A))$ for each $A \in \wp(X)$.*

Theorem 3.26. *Let (X, cl_X) and (Y, cl_Y) be two isotonic spaces and $f : X \to Y$ be a homeomorphism. Then p is a limit point of A if and only if $f(p)$ is a limit point of $f(A)$ for each $A \in \wp(X)$.*

Proof. Let p be a limit point of A in (X, cl_X). Then for each $N \in \mathcal{N}(f(p))$, $f^{-1}(N) \in N(p)$ and $f^{-1}(N) \cap (A \setminus \{p\}) \neq \emptyset$. This implies that $\emptyset \neq f(f^{-1}(N) \cap (A \setminus \{p\})) = N \cap f(A \setminus \{p\}) = N \cap (f(A) \setminus \{f(p)\})$. Therefore, $f(p)$ is a limit point of $f(A)$ (from Remark 3.7).

Conversely suppose that $f(p)$ is a limit point of $f(A)$. Then for each $N \in \mathcal{N}(p)$, $f(N) \in \mathcal{N}(f(p))$ and $f(N) \cap (f(A) \setminus \{f(p)\}) \neq \emptyset$. Now $\emptyset \neq f^{-1}[f(N) \cap (f(A) \setminus \{f(p)\})] = N \cap (A \setminus \{p\})$. Hence, p is a limit point of A. □

Theorem 3.27. *Let (X, cl_X) and (Y, cl_Y) be two generalized closure spaces and $f : X \to Y$ be a continuous surjection. If $f^{-1}(D)$ is dense in X, then D is dense in Y.*

Proof. The proof follows easily from the fact that

$$cl_X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B)) \text{ for all } B \in \wp(Y).$$

□

Theorem 3.28. *Let (X, cl_X) and (Y, cl_Y) be two generalized closure spaces and $f : X \to Y$ be a closure-preserving surjective mapping. If D is dense in (X, cl_X), then $f(D)$ is dense in (Y, cl_Y).*

Proof. Let D be dense in (X, cl_X). Then from closure-preserving, $f(cl_X D) \subseteq cl_Y f(D)$. This implies that $f(X) \subseteq cl_X f(D)$ and hence $Y \subseteq cl_X f(D)$, since f is surjective. □

Theorem 3.29. *Let (X, cl_X) and (Y, cl_Y) be two isotonic space and $f : X \to Y$ be a homeomorphism. Then D is dense in (X, cl_X) if and only if $f(D)$ is dense in (Y, cl_Y).*

Proof. Suppose D is dense in (X, cl_X). Then, by Theorem 3.6, $f(cl_X(D)) = cl_Y(f(D))$, $Y = cl_Y(f(D))$.

Conversely suppose that $f(D)$ is dense in (Y, cl_Y). Then for $f(cl_X(D)) = cl_Y(f(D))$, $f(cl_X(D)) = Y$ and $cl_X(D) = f^{-1}(Y) = X$. Hence D is dense in (X, cl_X). □

Definition 3.30. [2] *A subset A of a generalized closure space X is said to be sgc-nwdense if $int(cl(A)) = \emptyset$.*

Theorem 3.31. *Let (X, cl_X) and (Y, cl_Y) be two generalized closure spaces, $f : X \to Y$ be a closure-preserving surjective function and $f(int_X(A)) = int_Y(f(A))$ for each subset A of X. If B is sgc-nwdense in (Y, cl_Y) then $f^{-1}(B)$ is sgc-nwdense in (X, cl_X).*

Proof. $f(int_X(cl_X(f^{-1}(B))) = int_Y(f(cl_X(f^{-1}(B))) \subseteq int_Y(cl_Y(f(f^{-1}(B))) = int_Y(cl_Y(B)) = \emptyset$. Hence $f^{-1}(B)$ is sgc-nwdense in (X, cl_X).

□

Theorem 3.32. *Let (X, cl_X) and (Y, cl_Y) be isotonic generalized closure spaces and $f : X \to Y$ be a homeomorphism. A is sgc-nwdense in (X, cl_X) if and only if $f(A)$ is sgc-nwdense in (Y, cl_Y).*

Proof. The proof follows from the fact that $f(int_X(cl_X(A)) = int_Y(cl_Y(f(A)))$. □

Definition 3.33. *A generalized closure space X is said to be first category if it is the union of a countable family of sgc-nwdense sets. A generalized closure space that is not of first category is said to be of second category.*

Theorem 3.34. *Let (X, cl_X) and (Y, cl_Y) be two generalized closure spaces, $f : X \to Y$ be a closure-preserving surjective function and $f(int_X(A)) = int_Y(f(A))$ for each subset A of X. If Y is first category, then X first category.*

Proof. The proof follows from Theorem 3.31. □

Theorem 3.35. *Let (X, cl_X) and (Y, cl_Y) be isotonic spaces and $f : X \to Y$ be a homeomorphism. Then X is first category if and only if Y is first category.*

Proof. The proof follows from Theorem 3.32. □

Conclusion:

We have reached the following points through this paper:

(1) Role of limit points in generalized closure spaces and its related spaces.

(2) Role of continuity, closure-preserving and homeomorphism on border, exterior and boundary points on generalized closure spaces.

(3) Role of Category sets under continuity, closure-preserving and homeomorphism on generalized closure spaces.

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DOUBLE LACUNARY STATISTICAL BOUNDEDNESS OF ORDER α

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ABSTRACT. The main goal of this paper is to introduce the concepts of double statistical boundedness and double lacunary statistical boundedness. Additionally, we shall present the relations between double statistical boundedness of α and double lacunary statistical boundedness of α.

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1. Introduction

The idea of convergence of a real sequence had been extended to statistical convergence by Fast [4] (see also Schoenberg [21] as follows : If $\mathbb{N}$ denotes the set of natural numbers and $K \subset \mathbb{N}$ then $K(m,n)$ denotes the cardinality of the set $K \cap [m,n]$. The upper and lower natural density of the subset K is defined by

$$\overline{d}(K) = \lim_{n\to\infty} \sup \frac{K(1,n)}{n} \text{ and } \underline{d}(K) = \lim_{n\to\infty} \inf \frac{K(1,n)}{n}.$$

If $\overline{d}(K) = \underline{d}(K)$ then we say that the natural density of K exists and it is denoted simply by $d(K)$. Clearly $d(K) = \lim_{n\to\infty} \frac{K(1,n)}{n}$.

A sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers is said to be statistically convergent to L if for arbitrary $\epsilon > 0$, the set $K(\epsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \epsilon\}$ has natural density zero. Statistical convergence turned out to be one of the most active areas of research in summability theory after the work of Fridy [6] and Šalát [15].

Over the years and under different names statistical convergence was discussed in the theory of Banach spaces, Trigonometric series, Number theory, Measure theory and Fourier analysis. Subsequently a lot of interesting investigations have been done by various authors on this convergence. For the single case, the concept of statistical boundedness and lacunary statistical boundedness were given by Bhardwaj and Gupta [1] and Bhardwaj et al. [2] respectively. In many branches of science and engineering we often come across double sequences, i.e. sequences of matrices and certainly there are situations where either the idea of ordinary convergence does not work or the underlying space does not serve our purpose. Therefore to deal with such situations we have to introduce some new type of measures which can provide a better tool and suitable frame work. Mursaleen et al. [10] studied generalized statistical convergence and statistical core of double sequences. Also the double lacunary statistical convergence was introduced by Patterson and Savas [13].

The idea of statistical convergence depends upon the density of subsets of the set $\mathbb{N}$. The density of a subset E of $\mathbb{N}$ is defined by

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k)$$

provided that the limit exists. A sequence $x = (x_k)$ is said to be statistically convergent to L if for every $\varepsilon > 0$,

$$\delta(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}) = 0.$$

We shall be particularly concerned with those subsets of $\mathbb{N}$ which have natural density zero. For this, Fridy [6] introduced the following notation: if $x = (x_k)$ is a sequence such that x_k satisfies property P for all k except a set of natural density zero, then we say that (x_k) satisfies P for 'almost all k' and we abbreviate this by '$a.a.k$'.

Further, the concept of statistical boundedness was given by Bhardwaj and Gupta [1] as follows:

The sequence $x = (x_k)$ of reals is said to be statistically bounded if there is a number M such that $\delta(\{k : |x_k| > M\}) = 0$.

It is well known that every bounded sequence is statistically bounded, but the converse is not true.

Following Freedman et al.[5], by a lacunary sequence $\theta = \{k_r\}_{r=0}^{\infty}$ where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and we let $h_r = k_r - k_{r-1}$. Sums of the form $\sum\limits_{i=k_{r-1}+1}^{k_r} |x_i| = \sum\limits_{i\in I_r} |x_i|$ will be written for convenience as $\sum\limits_{i\in I_r} |x_i|$ and the ration $\frac{k_r}{k_{r-1}}$ will be denoted by q_r. Recently a lot of interesting developments have occurred in lacunary sequence and related topics, (see, [20]).

In another direction, in [7], a new type of convergence called lacunary statistical convergence was introduced a follows: A sequence (x_k) of real numbers is said to be lacunary statistically convergent to L (or, S_θ-convergent to L) if for any $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \epsilon\}| = 0$$

where $|A|$ denotes the cardinality of $A \subset \mathbb{N}$ and the relation between lacunary statistical convergence and statistical convergence was established among other things.

We now begin with some basic definitions that will be needed in the result of this paper.

Definition 1.1. *(see, [14]) A double sequence $x = (x_{k,l})$ is said to be convergent in the Pringsheim's sense if for every $\varepsilon > 0$ there exists $N \in \mathbf{N}$ such that $|x_{k,l} - L| < \epsilon$ whenever*

$k, l \geq N$, L is called the Pringsheim limit of x. We shall describe such an x more briefly as "P-convergent".

Let $K \subseteq \mathbf{N} \times \mathbf{N}$ be a two dimensional set of positive integers and let $K_{m,n}$ be the numbers of (i, j) in K such that $i \leq n$ and $j \leq m$. Then the two-dimensional analogue of natural case density can be defined as follows: The lower asymptotic density of K is defined as

$$P - \liminf_{m,n} \frac{K_{m,n}}{mn} = \delta_2(K).$$

In the case when the sequence $\{\frac{K_{m,n}}{mn}\}_{m,n=1,1}^{\infty,\infty}$ has a limit then we say that K has a natural density and is defined as

$$P - \lim_{m,n} \frac{K_{m,n}}{mn} = \delta_2(K).$$

For example, let $K = \{(i^2, j^2) : i, j \in \mathbf{N} \times \mathbf{N}\}$. Then

$$\delta_2(K) = P - \lim_{m,n} \frac{K_{m,n}}{mn} \leq P - \lim_{m,n} \frac{\sqrt{m}\sqrt{n}}{mn} = 0$$

(i.e. the set K has double natural density zero), while the set $\{(i, 2j) : i, j \in \mathbf{N} \times \mathbf{N}\}$ has double natural density $1/2$.

Recently the studies of double sequences has a rapid growth. Mursaleen and Edely [11] extended the idea of statistical convergence of single sequences to double sequences of scalars and established relations between statistical convergence and strongly Cesáro summable double sequences. R. F. Patterson [12] studied the analogues of some fundamental theorems of summability theory. Also, the double lacunary statistical convergence was introduced by Patterson and Savas [13]. Mursaleen and Edely [11] introduced main definition as follows:

Definition 1.2. *A double sequences $x = (x_{k,l})$ is said to be P-statistically convergent to L provided that for each $\varepsilon > 0$*

$$P - \lim_{m,n} \frac{1}{mn} \{number\ of\ (k, l) : k < m\ and\ l < n, |x_{k,l} - L| \geq \varepsilon\} = 0,$$

In this case we write $S^2 - \lim_{k,l} x_{k,l} = L$ and we denote the set of all statistical convergent double sequences by S^2.

Definition 1.3. *The double sequence x is bounded if there exists a positive number M such that $|x_{k,l}| < M$ for all k and l,*

$$||x||_{(\infty,2)} = \sup_{k,l} |x_{k,l}| < \infty.$$

We will denote the set of all bounded double sequences by l^2_∞.

Note that in contrast to the case for single sequence, a P-convergent double sequence need to be bounded but every P-convergent real (or complex) double sequences is Cauchy and convergent. We are ready to present the definition of double statistically bounded.

Definition 1.4. *The sequence $x = (x_{k,l})$ is double statistically bounded if there is a number $M > 0$ such that $\delta(\{(k,l) : |x_{k,l}| > M\}) = 0$, i.e., $|x_{k,l}| \leq M$ almost all (k,l). We denote the set of all double statistically bounded sequences by $S^2(B)$.*

We now introduce the following notation: The double sequence $\theta = \{(k_r, l_s)\}$ is called **double lacunary** if there exist two increasing of integers such that

$$k_0 = 0, h_r = k_r - k_{k-1} \to \infty \text{ as } r \to \infty$$

and

$$l_0 = 0, \bar{h}_s = l_s - l_{s-1} \to \infty \text{ as } s \to \infty.$$

Notations: $k_{r,s} = k_r l_s$, $h_{r,s} = h_r \bar{h}_s$, θ is determine by $I_r = \{(k) : k_{r-1} < k \leq k_r\}$, $I_s = \{(l) : l_{s-1} < l \leq l_s\}$, $I_{r,s} = \{(k,l) : k_{r-1} < k \leq k_r \,\&\, l_{s-1} < l \leq l_s\}$, $q_r = \frac{k_r}{k_{r-1}}$, $\bar{q}_s = \frac{l_s}{l_{s-1}}$, and $q_{r,s} = q_r \bar{q}_s$. We will denote the set of all double lacunary sequences by $\mathbf{N}_{\theta_{r,s}}$ (see,[13]).

Using this notation, we have the following.

Let $K \subseteq \mathbf{N} \times \mathbf{N}$ has double lacunary density $\delta_2^{\theta}(K)$ if

$$P - lim_{r,s} \frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s} : (k,l) \in K\}|$$

exists.

In 2005, R. F. Patterson and E. Savas [13] studied double lacunary statistically convergence by giving the definition as follows:

Definition 1.5. *Let $\theta = \{(k_r, l_s)\}$ be a double lacunary sequence; the double number sequence $x = (x_{k,l})$ is st_θ^2-convergent to L provided that for every $\varepsilon > 0$,*

$$P - \lim_{r,s} \frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s} : |x_{k,l} - L| \geq \varepsilon\}| = 0.$$

In this case write $st_\theta^2 - \lim x = L$ or $x_{k,l} \to L(S_\theta^2)$.

More investigation in this direction and more applications of double lacunary and double sequences can be found in ([16], [17], [18],[19]).

2. Main Results

The main object of this part is to introduce and study the new concept of double lacunary statistical boundedness.

Definition 2.1. *Let $\theta = \{(k_r, l_s)\}$ be a double lacunary sequence. The number sequence $x = (x_{k,l})$ is said to be double lacunary statistical bounded or S_θ^2- bounded if there exists $M > 0$, such that*

$$\lim_{r,s \to \infty} \frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s} : |x_{k,l}| > M\}| = 0$$

i.e.,

$$\delta_\theta^2(\{(k,l) \in \mathbb{N} \times \mathbb{N} : |x_{k,l}| > M\}) = 0.$$

For a given lacunary sequence $\theta = \{(k_r, l_s)\}$, by $S_\theta^2(B)$ we denote the set of all S_θ^2-bounded sequences.

We begin by establishing elementary connections between boundedness, lacunary statistical boundedness and lacunary statistical convergence. We state following result without proof in view of the fact that the empty set has zero lacunary density for every lacunary sequence $\theta = \{(k_r, l_s)\}$.

Theorem 2.2. *Every double bounded sequence is double lacunary statistical bounded, i.e, $\ell^2_\infty \subset S^2_\theta(B)$ for every double lacunary sequence θ.*

Theorem 2.3. *Every double lacunary statistical convergent sequence is double lacunary statistical bounded, but not conversely.*

Proof. $x \in S^2_\theta$ and $\varepsilon > 0$ be given. Then there exists $L \in \mathbb{C}$ such that

$$\lim_{r,s\to\infty} \frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s} : |x_{k,l} - L| \geq \varepsilon\}| = 0$$

by using the following, we have

$$\lim_{r,s\to\infty} \frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s} : |x_{k,l}| > |L| + \varepsilon\}| \leq \lim_{r,s\to\infty} \frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s} : |x_{k,l} - L| \geq \varepsilon\}|$$

The show the strictness of the inclusion, the sequence $x = (x_{k,l})$ defined by

$$x_{k,l} = \begin{cases} 1 & k = 2r, l = 2s \\ -1 & k \neq 2r, l \neq 2s \end{cases} r, s = 1, 2, 3, ...$$

Then $x \in S^2_\theta(B)$, but $x \notin S^2_\theta$. ☐

Theorem 2.4. *For a given double lacunary sequence $\theta = \{(k_r, l_s)\}$, a sequence $x = (x_{k,l})$ is double lacunary statistical bounded if and only if there exists a bounded double sequence $y = (y_{k,l})$ such that $x_{k,l} = y_{k,l}$ almost all (k, l) with respect to θ.*

Proof. Assume that $x = (x_{k,l})$ is double lacunary statistical bounded sequence. Then there exists $M \geq 0$ such that $\delta^\theta(K) = 0$ where $K = \{(k,l) \in \mathbb{N} \times \mathbb{N} : |x_{k,l}| > M\}$. Let

$$y_{k,l} = \begin{cases} x_{k,l} & if \ (k,l) \notin K \times K \\ 0 & otherwise \end{cases}$$

Then $y = (y_{k,l}) \in \ell^2_\infty$ and $x_{k,l} = y_{k,l}$ almost all (k, l) with respect to θ.

Conversely, $y = (y_{k,l}) \in \ell^2_\infty$ so there exists $L > 0$ such that $|y_{k,l}| \leq L$ for all $(k, l) \in \mathbb{N} \times \mathbb{N}$. Let $E = \{(k,l) \in \mathbb{N} \times \mathbb{N} \ , x_{k,l} \neq y_{k,l}\}$, Since $\delta_\theta(E) = 0$, $|x_{k,l}| \leq L$ almost all (k, l) with respect to θ. ☐

Theorem 2.5. *For any lacunary sequence $\theta = \{(k_r, l_s)\}$, $S^2(B) \subset S^2_\theta(B)$ if and only if $\liminf_r q_r > 1$ and $\liminf_r \bar{q}_s > 1$.*

Proof. (Sufficiency). Suppose that $\liminf_r q_r > 1$ and $\liminf_s \bar{q}_s > 1$, $\liminf_r q_r = \alpha_1$ and $\liminf_s \bar{q}_s = \alpha_2$, $\beta_1 = (\alpha_1 - 1) \setminus 2$ and $\beta_2 = (\alpha_2 - 1) \setminus 2$ then there exists two positive integers r_0 and s_0 such that $q_r \geq 1 + \beta_1$ for $r \geq r_0$ and $\bar{q}_s \geq 1 + \beta_2$ for $s \geq s_0$. Hence, for

$r \geq r_0$ and $s \geq s_0$.

$$\begin{aligned} \frac{h_{r,s}}{k_r l_s} &= \frac{1-k_{r-1}}{k_r} \cdot \frac{1-l_{s-1}}{l_s} \\ &= \left(1-\frac{1}{q_r}\right)\left(1-\frac{1}{\overline{q_s}}\right) \\ &\geq \left(1-\frac{1}{1+\beta_1}\right)\left(1-\frac{1}{1+\beta_2}\right) \\ &= \beta_1 \frac{1}{1+\beta_1} \cdot \beta_2 \frac{1}{1+\beta_2} \end{aligned}$$

For $(x_{k,l}) \in S^2_\theta(B)$ have exists $M > 0$ such that

$$\lim_{r,s\to\infty} \frac{1}{h_{r,s}} \left|\{(k,l) \in I_{r,s} : |x_{k,l}| > M\}\right| = 0$$

For $r \geq r_0$ and $s \geq s_0$ we write

$$\begin{aligned} \frac{1}{k_r l_s} \left|\{k \leq k_r \;\&\; l \leq l_s : |x_{k,l}| > M\}\right| &\geq \frac{h_{r,s}}{k_r l_s} \frac{1}{h_{r,s}} \left|\{(k,l) \in I_{r,s} : |x_{k,l}| > M\}\right| \\ &\geq \frac{\beta_1}{1+\beta_1} \frac{\beta_2}{1+\beta_2} \frac{1}{h_{r,s}} \left|\{(k,l) \in I_{r,s} : |x_{k,l}| > M\}\right| \end{aligned}$$

This proves the sufficiency.

(Necessity). Suppose that $\liminf_r q_r = 1$ and $\liminf_s \bar{q}_s = 1$. We can select a subsequence of the lacunary sequence $\theta_{r,s} = \{k_r, l_s\}$ such that

$$\frac{k_{r_j}}{k_{r_j}-1} < 1 + \frac{1}{j} \text{ and } \frac{k_{r_j}-1}{k_{r_{j-1}}} > 1 \ .$$

and

$$\frac{l_{s_i}}{l_{s_i}-1} < 1 + \frac{1}{i} \text{ and } \frac{l_{s_i}-1}{l_{s_{i-1}}} > i$$

where $r_j \geq r_{j-1} + 2$ and $s_i > s_{i-1} + 2$.

Define $x = (x_{k,l})$ by

$$x_{k,l} = \begin{cases} kl & \text{if } (k,l) \in I_{r_j s_i}, \text{ for some } i,j = 1,2,3,...; \\ 0 & otherwise \end{cases}$$

Then for any $M > 0$, there exists $j_0, i_0 \in \mathbb{N} \times \mathbb{N}$ such that $k_{r_j-1} > M$ and $l_{s_i-1} > M$ we write

$$\frac{1}{h_{r_{j_0} s_{i_0}}} \left|\left\{(k,l) \in I_{r_{j_0} s_{i_0}} : |x_{k,l}| > M\right\}\right| \geq \frac{1}{h_{r_{j_0} s_{i_0}}} \left|\left\{(k,l) \in I_{r_{j_0} s_{i_0}} : |x_{k,l}| > k_{r_{j_0-1}, s_{i_0-1}}\right\}\right| = 1$$

that is,

$$\frac{1}{h_{r_{j_0} s_{i_0}}} \left|\left\{(k,l) \in I_{r_{j_0} s_{i_0}} : |x_{k,l}| > M\right\}\right| = 1$$

for all $j \geq j_0$, $i \geq i_0$. But, for $r \neq r_j$, $s \neq s_i$

$$\frac{1}{h_{r,s}} \left|\{(k,l) \in I_{r,s} : |x_{k,l}| > M\}\right| = 0$$

Hence, $x = (x_{k,l}) \notin S^2_\theta(B)$.

On the other hand, for each m and n we find two positive number j_m and i_n such that $k_{r_{jm}} < m \leq k_{r_{jm}} + 1$ and $l_{s_{in}} < n \leq l_{s_{in}} + 1$. Then, we write,

$$\frac{1}{mn} \left|\left\{k \leq m, l \leq n : |x_{k,l}| > \frac{1}{2}\right\}\right| < \frac{2}{j} + \frac{2}{i}.$$

Since $m, n \to \infty$, it follows that i, $j \to \infty$. Hence, $(x_{k,l}) \in S^2(B)$. Consequently, we are granted the proof of theorem. □

Theorem 2.6. *For any double lacunary sequence* $\theta = \{(k_r, l_s)\}$ $S^2_\theta(B) \subset S^2(B)$ *if and only if* $\limsup_r q_r < \infty$ *and* $\limsup_s \bar{q}_s < \infty$.

Proof. The proof of sufficiency part follows on the same lines as adopted by Patterson and Savas [13], so we omit it. For necessity, we suppose that $\limsup_r q_r = \infty$ and $\limsup_s \bar{q}_s = \infty$. In [13] we can select two subsequences (k_{r_j}) and (l_{s_i}) of the lacunary sequence $\theta_{r,s} = (k_r, l_s)$ such that $k_{r_j} + > j$ and $l_{s_i} > i$, $k_{r_j} > j + 3$ and $l_{s_i} > i + 3$ and define a sequence $x = (x_{k,l})$ by

$$x_{i,j} = \begin{cases} ij & \text{if } k_{r_j-1} < j < 2k_{r_j-1}, l_{s_i-1} < i < 2l_{s_i-1}, i,j = 1,2,.. \\ 0 & otherwise \end{cases}$$

Then for $i, j > 1$

$$\tau_{r_j,s_i} = \frac{1}{h_{r_js_i}} \left|\left\{(i,j) \in I_{r_j,s_i} : |x_{k,l}| > \frac{1}{2}\right\}\right| < \frac{k_{r_j-1} l_{s_i-1}}{h_{r_js_i}} < \frac{1}{j-1}\frac{1}{i-1}$$

and if $r \neq r_j$, $s \neq s_i, \tau_{r_js_i} = 0$. Hence, $(x_{k,l}) \in S^2_\theta(B)$. However, for any real $M > 0$, there exists some j_0, i_0 such that $k_{r_j-1} > M$, $l_{s_i-1} > M$ for all $j \geq j_0, i \geq i_0$

$$\begin{aligned} &\frac{1}{2k_{r_j-1}} \frac{1}{2l_{s_i-1}} \left|\left\{i \leq 2k_{r_j-1},\ j \leq 2l_{s_i-1} : |x_{k,l}| > M\right\}\right| \\ = \; &\frac{1}{2k_{r_j-1}} \frac{1}{2l_{s_i-1}} \left[k_{r_1} - 1 + k_{r_2} - 1 + ... + k_{r_j} - 1\right] \left[l_{s_1} - 1 + l_{s_2} - 1 + ... l_{s_i} - 1\right] \geq \frac{1}{4} \end{aligned}$$

for $j \geq j_0$ and $i \geq i_0$. Thus $(x_{k,l}) \in S^2(B)$ □

Theorem 2.7. *Let* $\theta = \{(k_r, l_s)\}$ *be a double lacunary sequence, then* $S^2(B) = S^2_\theta(B)$ *if and only if*

$$1 < \liminf_r q_r \leq \limsup_r q_r < \infty.$$

and

$$1 < \liminf_s \overline{q_s} \leq \limsup_s \overline{q_s} < \infty.$$

Theorem 2.8. *For any lacunary sequence $\theta = \{(k_r, l_s)\}$, if*

$$\lim_{r,s} \inf \frac{h_{r,s}}{k_{r,s}} > 0 \tag{2.1}$$

then $S^2(B) \subset S^2_\theta(B)$.

Proof. For $M > 0$ we have

$$\{k \leq k_r,\ l \leq l_s : |x_{k,l}| > M\} \supset \{(k,l) \in I_{r,s} : |x_{k,l}| > M\}$$

Then

$$\begin{aligned} \frac{1}{k_r l_s} |\{k \leq k_r,\ l \leq l_s : |x_{kl}| > M\}| &\geq \frac{1}{k_r l_s} \{(k,l) \in I_{r,s} : |x_{k,l}| > M\} \\ &= \frac{h_{r,s}}{k_{r,s}} \frac{1}{h_{r,s}} \{(k,l) \in I_{r,s} : |x_{k,l}| > M\} \end{aligned}$$

by taking limit an $r, s \to \infty$ and using (2.1) we have $S^2(B) \subset S^2_\theta(B)$. ☐

Theorem 2.9. *Let $\theta_1 = \{(k_r, l_s)\}$ and $\theta_2 = \{(\overline{k}_r, \overline{l}_s)\}$ be two lacunary sequences such that $I_{r,s} \subset \overline{J}_{r,s}$ for all $(r,s) \in \mathbb{N} \times \mathbb{N}$. If*

$$\lim_{r,s\to\infty} \inf \frac{h_{r,s}}{l_{r,s}} > 0 \tag{2.2}$$

then $S^2_{\theta_2}(B) \subset S^2_{\theta_1}(B)$.

Proof. Suppose that $I_{r,s} \subset \overline{J}_{r,s}$ for all $(r,s) \in \mathbb{N} \times \mathbb{N}$ and let (2.2) be satisfied. For $M > 0$ we write,

$$\left\{(k,l) \in \bar{j}_{r,s} : |x_{k,l}| > M\right\} \supseteq \{(k,l) \in I_{r,s} : |x_{k,l}| > M\}$$

and

$$\frac{1}{l_{rs}} \left|\left\{(k,l) \in \overline{J}_{r,s} : |x_{k,l}| > M\right\}\right| \geq \frac{h_{rs}}{l_{rs} h_{rs}} |\{(k,l) \in I_{r,s} : |x_{k,l}| > M\}|$$

for all $(r,s) \in \mathbb{N} \times \mathbb{N}$ when

$$\overline{J}_{r,s} = \left\{(k,l) : \overline{k}_{r-1} < k \leq \overline{k}_r \text{ and } \overline{l}_{s-1} < l < \overline{l}_s\right\}$$

If we take limit an $r \to \infty$ in the last inequality and using (2.2) we have $S^2_{\theta_2}(B) \subset S^2_{\theta_1}(B)$. ☐

3. Double Lacunary Refinement

In [13] R.F. Patterson and E. Savas defined double lacunary refinement and established an multidimensional analog of Theorem 2.2 of Fridy and Orhan [7].

Definition 3.1. *The double index sequence $\rho = \left\{\overline{k_r}, \overline{l_s}\right\}$ is called a double lacunary refinement of the double lacunary sequence $\theta = \{k_r, l_s\}$ if $\{k_r, l_s\} \subseteq \left\{\overline{k_r}, \overline{l_s}\right\}$.*

Theorem 3.2. *Suppose* $\rho = \left\{\overline{k_r}, \overline{l_s}\right\}$ *is a lacunary refinement of the double lacunary sequence* $\theta = \{k_r, l_s\}$. *Let* $I_{r,s}$ *and* $\overline{I}_{r,s}$; $r,s = 1,2,3,...$ *be defined as above. If there exists* $\delta > 0$ *such that*

$$\frac{\left|\overline{I}_{\alpha,\beta}\right|}{I_{r,s}} \geq \delta \text{ for every } \overline{I}_{\alpha,\beta} \subseteq I_{r,s} \tag{3.1}$$

Then $x_{k,l} \to L\left(S_\theta^2\right)$ *implies* $x_{k,l} \to L\left(S_\rho^2\right)$ $\left(i.e. S_\theta^2 \subseteq S_\rho^2\right)$.

Proof. Given any $\varepsilon > 0$ and $\overline{I}_{\alpha,\beta}$ we can find $I_{r,s}$ such that $\overline{I}_{\alpha,\beta} \subseteq I_{r,s}$, and we obtain the following

$$\begin{aligned}
& \left(\frac{1}{|I_{r,s}|}\right)|\{(k,l) \in I_{r,s} : |x_{k,l} - L| \geq \varepsilon\}| \\
= & \left(\frac{\left|\overline{I}_{\alpha,\beta}\right|}{|I_{r,s}|}\right)\left(\frac{1}{\left|\overline{I}_{\alpha,\beta}\right|}\right)|\{(k,l) \in I_{r,s} : |x_{k,l} - L| \geq \varepsilon\}| \\
\leq & \left(\frac{\left|\overline{I}_{\alpha,\beta}\right|}{|I_{r,s}|}\right)\left(\frac{1}{\left|\overline{I}_{\alpha,\beta}\right|}\right)\left|\left\{(k,l) \in \overline{I}_{\alpha,\beta} : |x_{k,l} - L| \geq \varepsilon\right\}\right| \\
\leq & \left(\frac{1}{\delta}\right)\left(\frac{1}{\left|\overline{I}_{\alpha,\beta}\right|}\right)\left|\left\{(k,l) \in \overline{I}_{\alpha,\beta} : |x_{k,l} - L| \geq \varepsilon\right\}\right|
\end{aligned}$$

This grants us the proof of theorem. □

4. Double Lacunary Statistical Boundedness of Order α

In this section we introduce the concept of double lacunary boundedness of order α. The order of statistical convergence of a sequence of numbers was given by Gadjiev and Orhan [9] and after then statistical convergence of order α was studied by Çolak [3]. For single case, the concept of lacunary statistical convergence of order α was given by Sengül and Et [22] as follows: Let $\theta = (k_r)$ be a lacunary sequence and $0 < \alpha \leq 1$ be given. The sequence $x = (x_k)$ is said to be lacunary statistically convergent of order α, if there is a real number L such that

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} |\{k \in I_r : |x_k - L| \geq \varepsilon\}| = 0$$

The set of all lacunary statistically convergent sequences of order α will be denoted by S_θ^α.

Now we present some new definitions and main results of the paper.

Definition 4.1. *The sequence* $x = (x_{k,l})$ *is said to be double statistically bounded sequences of order* α, *if there is a number* $M > 0$ *such that*

$$\lim_{m,n\to\infty} \frac{1}{(mn)^\alpha} |\{k \leq m,\ l \leq n : |x_{k,l}| > M\}| = 0$$

i.e

$$\delta^2(\{(k,l) : |x_{k,l}| > M\}) = 0$$

we denote the set of all double statistically bounded sequences of order α by $S^{\alpha}(B)^2$.

Definition 4.2. *Let $\theta = \{(k_r, l_s)\}$ be a double lacunary sequence and $0 < \alpha \le 1$ be given. We define double lacunary $\delta-$density of a subset E of $\mathbb{N} \times \mathbb{N}$ by*

$$\delta_{\theta}^{\alpha}(E) = \lim_{r,s\to\infty} \frac{1}{h_{rs}^{\alpha}} \left|\{k_{r-1} < k \le k_r \ \&\ l_{s-1} < l \le l_s : (k,l) \in E\}\right|$$

provided the limit exists. We remark that double lacunary $\delta-$density $\delta_{\theta}^{\alpha}(E)$ reduces to the natural double density $\delta^2(E)$ in the special cases $\alpha = 1$ and $\theta = (2^{rs})$

Proposition 4.3. *Let $\theta = \{(k_r, l_s)\}$ be a double lacunary sequence and $\alpha, \beta \in (0,1]$ such that $\alpha \le \beta$, then $\delta_{\theta}^{\beta}(E) \le \delta_{\theta}^{\alpha}(E)$.*

Proof. Proof follows from the following inequality

$$\begin{aligned} & \frac{1}{h_{rs}^{\alpha}} \left|\{k_{r-1} < k \le k_r \ \&\ l_{s-1} < l \le l_s : (k,l) \in E\}\right| \\ \le\ & \frac{1}{h_{rs}^{\beta}} \left|\{k_{r-1} < k \le k_r \ \&\ l_{s-1} < l \le l_s : (k,l) \in E\}\right|. \end{aligned}$$

□

Definition 4.4. *Let $\theta = \{(k_r, l_s)\}$ be a double lacunary sequence and $0 < \alpha \le 1$ be given. The sequence $x = (x_{k,l})$ is said to be double lacunary statistically bounded of order α, if there is a $M \ge 0$ such that*

$$(1) \quad \lim_{r,s\to\infty} \frac{1}{h_{rs}^{\alpha}} \left|\{(k,l) \in I_{r,s} : |x_{k,l}| > M\}\right| = 0,$$

where $I_{r,s} = \{(k,l) : k_{r-1} < k \le k_r \ \&\ l_{s-1} < l \le l_s\}$. The set of all double lacunary statistically bounded sequences of order α will be denoted by $S_{\theta}^{\alpha}(B)^2$.

For $\theta = (2^{rs})$, double lacunary statistically bounded sequences of order α reduces to double statistically bounded sequences of order α; for $\alpha = 1$, double lacunary statistically bounded sequences of order α reduces to double lacunary statistically bounded sequences. For $\alpha = 1$ and $\theta = (2^{rs})$, double lacunary statistically bounded sequences of order α reduces to double statistically bounded sequences. The set of all double statistically bounded sequences of order α and the set of all lacunary statistically bounded sequences of order α, $S^{\alpha}(B)^2$ and $S_{\theta}^{\alpha}(B)^2$, respectively.

Proposition 4.5. *Every double lacunary statistically convergent sequence of order α is double lacunary statistically bounded of order α, but the converse is not true.*

Proof. Let $x = (x_{k,l})$ double lacunary convergence to L statistically, $x \in S_{\theta}^{\alpha}$ and $\varepsilon > 0$ be given, then there exists $L \in \mathbb{C}$ such that

$$\lim_{r,s\to\infty} \frac{1}{h_{r,s}^{\alpha}} \left|\{(k,l) \in I_{r,s} : |x_{k,l} - L| \ge \varepsilon\}\right| = 0$$

by using the following, we have the result

$$\lim_{r,s\to\infty}\frac{1}{h_{r,s}^{\alpha}}\left|\{(k,l)\in I_{r,s}:|x_{k,l}|>|L|+\varepsilon\}\right|$$
$$\leq \lim_{r,s\to\infty}\frac{1}{h_{r,s}^{\alpha}}\left|\{(k,l)\in I_{r,s}:|x_{k,l}-L|\geq\varepsilon\}\right|.$$

To show the strictness of the inclusion, the sequence $x=(x_{k,l})$ defined by

$$x_{k,l}=\begin{cases}1 & k=2r, l=2s\\ -1 & k\neq 2r, l\neq 2s\end{cases} \quad r,s=1,2,3$$

Then $x\in S_{\theta}^{\alpha}(B)^2$, but $x\notin S_{\theta}^{\alpha}$. □

Proposition 4.6. *Every double bounded sequence is double lacunary statistical bounded of order α, but the converse is not true.*

Theorem 4.7. *If $0<\alpha\leq\beta\leq 1$ then $S_{\theta}^{\alpha}(B)^2\subseteq S_{\theta}^{\beta}(B)^2$ and the inclusion is strict.*

Proof. The inclusion part of proof is obvious. To show the strictness of the inclusion, let $\theta=\{(k_r,l_s)\}$ be given and the sequence $x=(x_{k,l})$ defined by

$$x_{k,l}=\begin{pmatrix} 1 & 2 & 3 & \cdots & [\sqrt[3]{h_{r,s}}] & 0 & \cdots\\ 2 & 2 & 3 & \cdots & [\sqrt[3]{h_{r,s}}] & 0 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 2 & [\sqrt[3]{h_{r,s}}] & \cdots & \cdots & [\sqrt[3]{h_{r,s}}] & 0 & \cdots\\ 0 & 0 & 0 & 0 & 0 & 0 & \vdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Then $x\in S_{\theta}^{\beta}(B)^2$ for $\frac{1}{2}<\beta\leq 1$ but $x\notin S_{\theta}^{\alpha}(B)^2$ for $0<\alpha\leq\frac{1}{2}$. □

Corollary 4.8. *If a double sequence is lacunary statistically bounded of order α, then it is double lacunary statistically bounded.*

The proofs of the following theorems are straightforward and therefore omitted.

Theorem 4.9. *Let $0<\alpha\leq 1$ and $\theta=\{(k_r,l_s)\}$ be a double lacunary sequence. If $\liminf_r q_r>1$ and $\liminf_s \overline{q_s}>1$ then $S^{\alpha}(B)^2\subset S_{\theta}^{\alpha}(B)^2$.*

Theorem 4.10. *Let $0<\alpha\leq 1$ and $\theta=\{(k_r,l_s)\}$ be a double lacunary sequence. If $\limsup_r q_r<\infty$ and $\limsup_s \overline{q_s}<\infty$ then $S_{\theta}^{\alpha}(B)^2\subset S^{\alpha}(B)^2$.*

Theorem 4.11. *Let $\theta=\{(k_r,l_s)\}$ and $\theta'=\{(u_r,v_s)\}$ be two lacunary sequences such that $I_{rs}\subseteq J_{rs}$ for all $r,s\in\mathbb{N}\times\mathbb{N}$. Suppose also that the parameters α and β are fixed real numbers such that $0<\alpha\leq\beta\leq 1$. We have the following:*

(1) If

$$\lim_{r,s\to\infty}\inf\frac{h_{rs}^{\alpha}}{\ell_{rs}^{\beta}}>0$$

then $S_{\theta'}^{\beta}(B)^2\subseteq S_{\theta}^{\alpha}(B)^2$.

(2) If

$$\lim_{r,s\to\infty} \inf \frac{\ell_{rs}}{h_{rs}^{\beta}} = 1$$

then $S_{\theta}^{\alpha}(B)^2 \subseteq S_{\theta'}^{\beta}(B)^2$

5. Conclusion

In this paper, we presented the concepts of double statistical boundedness of α and double lacunary statistical boundedness of α, and also we examined the relationship between these two new notions. Hence, we filled the big gap in the literature of Summability Theory.

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A NOTE ON THE HYPERBOLIC CURVATURE OF EUCLIDEAN PLANE CURVES

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Dedicated to the memory of Academician Radu Miron

Abstract. We introduce and study a new curvature function for plane curves inspired by the weighted mean curvature of M. Gromov. We call it *hyperbolic* being the difference between the usual curvature and the inner product of the normal vector field and the hyperbolic vector field. But, since the problem of vanishing of this curvature involves complicated expressions, we computed it for several examples.

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1. Preliminaries

The last forty years known an intensive research in the area of geometric flows. The most simple of them is the *curve shortening flow* and already the excellent survey [1] is almost twenty years old. Recall that the main geometric tool in this last flow is the well-known curvature of plane curves. Hence, to give a re-start to this problem seams to search for variants of the curvature or in terms of [4], deformations of the usual curvature. The goal of this short note is to propose such a deformation.

Fix $I \subseteq \mathbb{R}$ an open interval and $C \subset \mathbb{R}^2$ a regular parametrized curve of equation:

$$C : r(t) = (x(t), y(t)), \quad \|r'(t)\| > 0, \quad t \in I. \tag{1}$$

The ambient setting, namely $\mathbb{R}^2$, is an Euclidean vector space with respect to the canonical inner product:

$$\langle u, v\rangle = u^1v^1 + u^2v^2, \quad u = (u^1, u^2), \quad v = (v^1, v^2) \in \mathbb{R}^2, \quad 0 \leq \|u\|^2 = \langle u, u\rangle. \tag{2}$$

The infinitesimal generator of the rotations in $\mathbb{R}^2$ is the linear vector field, called *angular*:

$$\xi(u) := -u^2\frac{\partial}{\partial u^1} + u^1\frac{\partial}{\partial u^2}, \quad \xi(u) = i \cdot u = i \cdot (u^1 + iu^2). \tag{3}$$

It is a complete vector field with integral curves the circles $\mathcal{C}(O, R)$:

$$\begin{cases} \gamma^{\xi}_{u_0}(t) = (u^1_0 \cos t - u^2_0 \sin t, u^1_0 \sin t + u^2_0 \cos t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \cdot \begin{pmatrix} u^1_0 \\ u^2_0 \end{pmatrix} = SO(2) \cdot u_0, \\ R = \|u_0\| = \|(u^1_0, u^2_0)\|, \quad t \in \mathbb{R}, \end{cases} \tag{4}$$

and since the rotations are isometries of the Riemannian metric $g_{can} = dx^2 + dy^2$ it follows that ξ is a Killing vector field of the Riemannian manifold $(\mathbb{R}^2(x, y), g_{can})$. The first integrals of ξ are the Gaussian functions i.e. multiples of the square norm: $f_C(x,y) = C(x^2+y^2)$, $C \in \mathbb{R}$. For an arbitrary vector field $X = A(x,y)\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y}$ its Lie bracket with ξ is:

$$[X,\xi] = (yA_x - xA_y - B)\frac{\partial}{\partial x} + (A + yB_x - xB_y)\frac{\partial}{\partial y}$$

where the subscript denotes the variable corresponding to the partial derivative.

In the following we fix the complete vector field:

$$\Gamma_h = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$$

which we call *hyperbolic* since its integral curves are the equilateral hyperbolas $(x(t),y(t)) = (e^t x_0, e^{-t}y_0)$. Its Lie bracket with ξ is:

$$[\Gamma_h,\xi] = 2y\frac{\partial}{\partial x} + 2x\frac{\partial}{\partial y}.$$

The Frenet apparatus of the curve C is provided by:

$$\begin{cases} T(t) = \frac{r'(t)}{\|r'(t)\|}, \quad N(t) = i\cdot T(t) = \frac{1}{\|r'(t)\|}(-y'(t), x'(t)) \\ k(t) = \frac{1}{\|r'(t)\|}\langle T'(t), N(t)\rangle = \frac{1}{\|r'(t)\|^3}\langle r''(t), ir'(t)\rangle = \frac{1}{\|r'(t)\|^3}[x'(t)y''(t) - y'(t)x''(t)]. \end{cases} \tag{5}$$

Hence, if C is naturally parametrized (or parametrized by arc-length) i.e. $\|r'(t)\| = 1$ for all $t \in I$ then $r''(t) = k(t)ir'(t)$. In a complex approach based on $z(t) = x(t) + iy(t) \in \mathbb{C} = \mathbb{R}^2$ we have $2\lambda = Im(\bar{z}dz)$ and:

$$\begin{cases} k(t) = \frac{1}{|z'(t)|^3}Im(\bar{z}'(t)\cdot z''(t)) = \frac{1}{|z'(t)|}Im\left(\frac{z''(t)}{z'(t)}\right) = \frac{1}{|z'(t)|}Im\left[\frac{d}{dt}\left(\ln z'(t)\right)\right], \\ Re(\bar{z}'(t)\cdot z''(t)) = \frac{1}{2}\frac{d}{dt}\|r'(t)\|^2, \quad f_C(z) = C|z|^2. \end{cases}$$

2. Results and Examples

This short note defines a new curvature function for C inspired by a notion introduced by M. Gromov in [3, p. 213] and concerning with hypersurfaces M^n in a weighted Riemannian manifold $(\tilde{M}, g, f \in C^\infty_+(\tilde{M}))$. More precisely, the *weighted mean curvature* of M is the difference:

$$H^f := H - \langle \tilde{N}, \tilde{\nabla} f\rangle_g \tag{6}$$

where H is the usual mean curvature of M and $\tilde{N}$ is the unit normal to M. This curvature was studied in several papers and we point out that the curve shortening problem associated to a density is studied in the paper [5].

The rotational field ξ is not a g_{can}-gradient vector field but Γ_h is the gradient of the function $f_h(x,y) = \frac{1}{2}(x^2-y^2)$. Hence we follow this path and we introduce:

Definition 1 The *hyperbolic curvature* of C is the smooth function $k_h : I \to \mathbb{R}$ given by:

$$k_h(t) := k(t) - \langle N(t), \Gamma_h(r(t))\rangle. \tag{7}$$

Before starting its study we point out that this work is dedicated to the memory of Academician Radu Miron (1927-2022). He was always interested in the geometry of curves and besides its theory of *Myller configuration* ([7]) he generalized also a type of curvature for space curves in [6]. Returning to our subject we note:

Proposition 2 *The expression of the hyperbolic curvature is*:

$$k_h(t) = k(t) + \frac{1}{\|r'(t)\|}\frac{d}{dt}[x(t)y(t)]. \tag{8}$$

Proof We have directly:

$$\langle N(t), \Gamma_h(r(t))\rangle = \frac{1}{\|r'(t)\|}\langle(-y'(t), x'(t)), (x(t), -y(t))\rangle = -\frac{1}{\|r'(t)\|}[x(t)y(t)]' \tag{9}$$

and the conclusion (8) follows. $\square$

Example 3 i) If C is the line $r_0 + tu, t \in \mathbb{R}$ with the vector $u = (u^1, u^2) \neq \bar{0} = (0, 0)$ and the point $r_0 = (x_0, y_0) \in \mathbb{R}^2$ then k_h is the affine map:

$$k_h(t) = \frac{1}{\|u\|}[u^1(y_0 + tu^2) + u^2(x_0 + tu^1)], \quad t \in \mathbb{R}. \tag{10}$$

In particular, if $O \in C$ then $k_h(t) = \frac{2u^1u^2t}{\|u\|}$.
ii) If C is the circle $\mathcal{C}(O, R) : r(t) = Re^{it}$ the k_h is again a non-constant function:

$$k_h(t) = \frac{1}{R}\left(1 + R^2\cos 2t\right) \in \left[\frac{1 - R^2}{R}, \frac{1 + R^2}{R} \geq 2\right]. \tag{11}$$

Hence, the unit circle S^1 has the hyperbolic curvature $k_h(t) = 2\cos^2 t \in [0, 2]$.
iii) The equilateral hyperbola $H_e(R) : xy = R^2$ has the hyperbolic curvature equal to the usual curvature:

$$h_k(t) = k(t) = \frac{2R^2}{t\sqrt{t^4 + R^4}} \geq 0, \quad t \in (0, +\infty). \tag{12}$$

We note that in the paper [2] is studied a product on the set of equilateral hyperbolas. If the equilateral hyperbola is expressed as $H^e(R) : x^2 - y^2 = R^2$ then its curvatures are:

$$k(t) = -\frac{1}{R(\cosh 2t)^{\frac{3}{2}}} < 0, \quad k_h(t) = \frac{R^2(\cosh 2t)^2 - 1}{R(\cosh 2t)^{\frac{3}{2}}}.$$

In particular for $H^e(1)$ its hyperbolic curvature is positive:

$$k_h(t) = \frac{(\sinh 2t)^2}{R(\cosh 2t)^{\frac{3}{2}}} \geq 0.$$

iv) Suppose that C is positively oriented in the terms of Definition 1.14 from [8, p. 17]. Suppose also that C is convex; then applying the Theorem 1.18 of page 19 from the same book it results for the usual curvature the inequality $k \geq 0$; it results that $k_k(t) \geq \frac{1}{\|r'(t)\|}\frac{d}{dt}[x(t)y(t)]$, for all $t \in I$. $\square$

An important problem is the class of curves with prescribed hyperbolic curvature. Using the formalism of [9, p. 2] if $r : S^1 \simeq [0, 2\pi) \to \mathbb{R}^2$ is naturally parametrized then there exists the smooth function $\theta : S^1 \to \mathbb{R}$, called *normal angle*, such that:

$$N(t) = e^{i\theta(t)} = (\cos\theta(t), \sin\theta(t)), \quad T(t) = -iN(t) = -ie^{i\theta(t)} = e^{i(\theta(t) - \frac{\pi}{2})} \tag{13}$$

and then the Frenet equations yields:

$$\frac{d\theta}{dt}(t) = k(t). \tag{14}$$

It follows that the hyperbolic curvature is a derivative:

$$k_h(t) = \frac{d}{dt}[\theta(t) + x(t)y(t)]. \tag{15}$$

Proposition 4 *Suppose that t is a natural parameter on the curve C. Then C is hyperbolic-flat i.e. $k_h \equiv 0$ if and only if $\theta + x \cdot y$ is a constant.*

In the following we present other two examples in order to remark the computational aspects of our approach.

Example 5 Recall that for $R > 0$ the cycloid of radius R has the equation:

$$C : r(t) = R(t - \sin t, 1 - \cos t) = R[(t, 1) - e^{i(\frac{\pi}{2} - t)}], \quad t \in \mathbb{R}. \tag{16}$$

We have immediately:

$$r'(t) = R(1 - \cos t, \sin t) = R[(1, 0) - e^{it}], \quad k(t) = -\frac{1}{4R \sin \frac{t}{2}}, \quad \|r'(t)\| = 2R|\sin \frac{t}{2}| \tag{17}$$

and then we restrict our definition domain to $(0, \pi)$. It follows:

$$k_h(t) = -\frac{1}{4R \sin \frac{t}{2}} + \frac{R[2(\cos^2 t - \cos t) + t \sin t]}{2 \sin \frac{t}{2}}. \tag{18}$$

A natural parameter s for C is provided by: $t = 2 \arccos \left(1 - \frac{s}{4R}\right)$. $\square$

Example 6 Fix a graph $C : y = f(x)$, $x \in I$ with the second derivative f'' strictly positive. With the usual parametrization $C : r(t) = (t, f(t))$ we have:

$$r'(t) = (1, f'(t)), \quad k(t) = \frac{f''(t)}{[1 + (f'(t))^2]^{\frac{3}{2}}} > 0, \quad \|r(t)\|^2 = t^2 + f^2(t) \tag{19}$$

which gives that C is convex and:

$$k_k(t) = \frac{f''(t)}{[1 + (f'(t))^2]^{\frac{3}{2}}} + \frac{f(t) + tf'(t)}{[1 + (f'(t))^2]^{\frac{1}{2}}}. \tag{20}$$

It follows that the function f making k_h constant zero is a solution of the non-autonomous differential equation:

$$f''(t) = -[f(t) + tf'(t)][1 + (f'(t))^2]. \tag{21}$$

$\square$

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