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SUBFAMILIES OF BI-UNIVALENT FUNCTIONS DEFINED BY THE (p,q)-DERIVATIVE OPERATOR CONNECTED TO LUCAS-BALANCING POLYNOMIALS

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Abstract. This paper introduces and investigates two subclasses of bi-univalent functions defined via the (p,q)-derivative operator subordinate to Lucas-balancing polynomials, considering their generating function in the corrected domain $\varkappa \in (-\frac{1}{3}, \frac{1}{3})$. Coefficient estimates for $|d_2|$ and $|d_3|$ are obtained, along with bounds for the Fekete–Szegö functional $|d_3 - \mu d_2^2|$, where $\mu \in \mathbb{R}$. It is also shown that the findings of the present research have relevant connections with earlier published findings.

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1. Preliminaries

The q-analysis is a generalization of the ordinary analysis that does not employ limit notation. Jackson described the use and application of the q-calculus in [24, 25]. The extension of the q-calculus to the (p,q)-calculus was examined by the researchers. Around the same time, in 1991, Arik [5], Brod [10], Chakrabarti [11], Wach [45], and others conducted the first analysis of the (p,q)-number. The (p,q)-number was introduced in [5] to investigate Fibonacci oscillators. The (p,q)-number investigation in [10] allows for the construction of a (p,q)-Harmonic oscillator. The (p,q)-number was utilized in [11] to unify various q-oscillator algebra types, and in [45], the (p,q)-numbers are analyzed to determine the (p,q)-Stirling numbers. Since 1991, many scholars have investigated the (p,q)-calculus in a range of scientific domains, building on the publications previously mentioned. The findings in [26] gave a syntax for embedding q-series into a (p,q)-series. Additionally, they found a few results that matched (p,q)-extensions of the known q-identities. The (p,q)-series is a corresponding extension of the q-identities (see, for example, [4]). We explain some of the fundamental ideas in (p,q)-calculus. The (p,q)-bracket number is given by $[j]_{p,q} = p^{j-1} + p^{j-2}q + p^{j-3}q^2 + ... + pq^{j-2} + q^{j-1} = \frac{p^j - q^j}{p-q}$ $(p \neq q)$, which is an extension of q-number (see [25]), that is $[j]_q = \frac{1-q^j}{1-q}$ $(q \neq 1)$. If $p = 1$, then $[j]_{p,q}$=$[j]_q$ and note that $[j]_{p,q}$ is symmetric.

Let $\mathbb{C}$ be the set of complex numbers, and the open unit disk be represented by $\mathfrak{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$. Let $\mathbb{R} := (-\infty, \infty)$ and $\mathbb{N} := \{1, 2, 3, \cdots\}$.

Definition 1.1. [37] *Let $1 \geq p > q > 0$ and consider a function φ defined on $\mathbb{C}$. Then, the definition of (p, q)-derivative of φ is*

$$D_{p,q}\varphi(\zeta) = \frac{\varphi(p\zeta) - \varphi(q\zeta)}{(p - q)\zeta}$$

where $\zeta \neq 0$, and $D_{p,q}\varphi(0) = \varphi'(0)$, if $\varphi'(0)$ exists.

We note that $D_{p,q}\zeta^j = [j]_{p,q}\zeta^{j-1}, D_{p,q}\ln(\zeta) = \frac{\ln(p/q)}{(p-q)\zeta}$. $If p = 1$ and $q \to 1^-$, then i). $[j]_{p,q} \to j$, and ii). $D_{p,q}\varphi(\zeta) \to \varphi'(\zeta)$. According to [33], the exponential functions are used to define the (p, q)-analogues of trigonometric functions. If κ and δ are constants, it is clear that $D_{p,q}(\kappa\varphi_1(\zeta) + \delta\varphi_2(\zeta)) = \kappa D_{p,q}\varphi_1(\zeta) + \delta D_{p,q}\varphi_2(\zeta)$.

Assume that $\mathcal{A}$ denotes the set of functions Φ that have the following form and are analytic in $\mathfrak{D}$.

$$\Phi(\zeta) = \zeta + \sum_{j=2}^{\infty} d_j\zeta^j, \quad (\zeta \in \mathfrak{D}), \tag{1.1}$$

and if $\Phi \in \mathcal{A}$ is of the form (1.1), then

$$D_{p,q}\Phi(\zeta) = 1 + \sum_{j=2}^{\infty} [j]_{p,q} d_j\zeta^{j-1}, \quad (\zeta \in \mathfrak{D}). \tag{1.2}$$

We define $\mathcal{S}$ as the subset of $\mathcal{A}$ consisting of functions that are univalent in $\mathfrak{D}$; that is,

$$S = \{\Phi \in \mathcal{A} : \Phi \textit{ is univalent in } \mathfrak{D}\}.$$

The Koebe theorem ([15]) states that for any function Φ in $\mathcal{S}$, the image of $\Phi(\mathfrak{D})$ contains the disk with radius 1/4 and center at 0. Thus, $\Phi(\mathfrak{D})$ retains an inverse $\Phi^{-1} : \Phi(\mathfrak{D}) \to \mathfrak{D}$ satisfying $\Phi^{-1}(\Phi(\zeta)) = \zeta$, $\zeta \in \mathfrak{D}$ and $\Phi(\Phi^{-1}(\varpi)) = \varpi$, $(r_0(\Phi) > |\varpi|; r_0(\Phi) \geq 1/4), \varpi \in \mathfrak{D}$. In fact, Φ^{-1} has the expression

$$\Phi^{-1}(\varpi) = \varpi - d_2\varpi^2 + (2d_2^2 - d_3)\varpi^3 - (5d_2^3 - 5d_2d_3 + d_4)\varpi^4 + \cdots = \Psi(\varpi). \tag{1.3}$$

If $\Phi \in \mathcal{S}$ and $\Phi^{-1} \in \mathcal{S}$, then $\Phi \in \mathcal{A}$ is bi-univalent in $\mathfrak{D}$. The set of all bi-univalent functions in $\mathfrak{D}$ is represented by the symbol σ; that is,

$$\sigma = \{\Phi \in \mathcal{A} : \Phi \textit{ and } \Phi^{-1} \textit{ are both univalent in } \mathfrak{D}\}.$$

Examples of functions belonging to the class σ include $\frac{1}{2}\log\left(\frac{1+\zeta}{1-\zeta}\right)$, $-\log(1 - \zeta)$ and $\frac{\zeta}{1-\zeta}$. However, $\zeta - \frac{\zeta^2}{2}$, $\frac{\zeta}{1-\zeta^2}$, and the Koebe function do not belong to σ family, even though they are in $\mathcal{S}$. For a brief overview and to discover the fascinating characteristics of the family σ, see [8, 9, 27, 36] and the citation provided in these papers. Comparable to the established subclasses of the $\mathcal{S}$ family, Srivastava et al. [34] have introduced a number of subclasses of the family σ. In reality, many authors have since investigated a variety of alternative subclasses of σ as follow-ups to the aforementioned subfamilies (see, for example, [14, 17, 18, 35]). The majority of these publications focus on the analysis of the Fekete-Szegó problem of functions in distinct σ subclasses.

Researches studied several subclasses of the class σ using the (p,q)-calculus. The (p,q)-derivative operator and the subordination principle, for instance, were used in [38] to introduce the new generalized classes of (p,q)-convex and (p,q)-starlike functions. The (p,q)-Bernardi integral operator for analytic functions is defined and the Fekete-Szegö inequalities are also studied. In ([1, 2, 12, 29, 43, 44]), new subclasses of the class σ related to the (p,q)-differential operator have also been presented and examined.

The current investigation in GFT is largely motivated by the rich structural and applied properties of special polynomials. Examples include Bernoulli, Fibonacci, Gegenbauer, Horadam, and Lucas-Lehmer polynomials, all of which have demonstrated utility in diverse areas such as combinatorics, number theory, numerical analysis, physics, and computer science. Owing to their versatility, several generalizations have been proposed in the literature. Currently, researchers are focusing on a specific class of functions within the σ family that are subordinate to well-known polynomials (see [3, 19, 39, 41, 42]), with the Lucas-balancing polynomials emerging as a particularly compelling subject of study.

The balancing numbers, denoted by $\mathcal{C}_j, j \geq 0$, satisfy the recurrence relation $6\mathcal{C}_j - \mathcal{C}_{j-1} = \mathcal{C}_{j+1}$, $(j \geq 1)$, with $\mathcal{C}_0 = 1$, and $\mathcal{C}_1 = 1)$ (see [6]). The sequence $\mathcal{B}_j = \sqrt{8\mathcal{C}_j^2 + 1}, j \geq 1$ is called a Lucas-balancing numbers. It satisfies the recurrence relation

$$\mathcal{B}_{j+1} = 6\mathcal{B}_j - \mathcal{B}_{j-1}, (j \geq 1, \mathcal{B}_0 = 1, \mathcal{B}_1 = 3).$$

These numbers have been thoroughly examined in the articles [13, 20, 30, 31, 32]. We now discuss the natural extensions of balancing numbers and Lucas-balancing numbers. The recursive definition of balancing polynomials, represented by $\mathcal{C}_j(\varkappa), j \geq 0$, is

$$6\varkappa\mathcal{C}_{j-1}(\varkappa) - \mathcal{C}_{j-2}(\varkappa) = \mathcal{C}_j(\varkappa), \quad (j \geq 2, \mathcal{C}_0(\varkappa) = 0, \mathcal{C}_1(\varkappa) = 1),$$

where $\varkappa \in \mathbb{C}$. It is evident that $\mathcal{C}_2(\varkappa) = 6\varkappa$, $\mathcal{C}_3(\varkappa) = 36\varkappa^2 - 1$, and $\mathcal{C}_4(\varkappa) = 216\varkappa^3 - 12\varkappa$, and so forth. Lucas-balancing polynomials, denoted by $\mathcal{B}_j(\varkappa), \varkappa \in \mathbb{C}$, are recursively defined as

$$\mathcal{B}_j(\varkappa) = 6\varkappa\mathcal{B}_{j-1}(\varkappa) - \mathcal{B}_{j-2}(\varkappa), \quad (j \in \mathbb{N}\backslash\{1\}, \mathcal{B}_0(\varkappa) = 1, \ \mathcal{B}_1(\varkappa) = 3\varkappa). \tag{1.4}$$

$\mathcal{B}_2(\varkappa) = 18\varkappa^2 - 1, \mathcal{B}_3(\varkappa) = 108\varkappa^3 - 9\varkappa, \cdots$ are evident from (1.4). To learn more about this field, researchers can visit [7, 28, 32]. According to [21], the generating function (GF) of the Lucas-balancing polynomials is represented by the following $\mathsf{B}(\varkappa, \zeta)$.

$$\mathsf{B}(\varkappa, \zeta) := \sum_{j=0}^{\infty} \mathcal{B}_j(\varkappa)\zeta^j = \frac{1 - 3\varkappa\zeta}{1 - 6\varkappa\zeta + \zeta^2}, \tag{1.5}$$

where $\varkappa \in (-\frac{1}{3}, \frac{1}{3})$ and $\zeta \in \mathfrak{D}$.

Remark 1.2. *The generating function under consideration is analytic in ζ inside the unit disk $\mathfrak{D}$ if and only if $\varkappa \in (-\frac{1}{3}, \frac{1}{3})$. This analyticity condition is crucial for the convergence and validity of the associated expansions. However, in* [22, 23]*, the range of $\varkappa$ is incorrectly stated as $(-\frac{1}{2}, 1]$, which does not ensure analyticity within $\mathfrak{D}$. The correct range $(-\frac{1}{3}, \frac{1}{3})$ follows directly from the requirement that the singularities of the generating function lie outside the unit disk.*

For functions θ_1, θ_2 analytic in the unit disk $\mathfrak{D}$, we say that θ_1 is subordinate to θ_2, if there is a Schwarz function $\kappa(\zeta)$ that is regular in $\mathfrak{D}$ with $\kappa(0) = 0$, and $|\kappa(\zeta)| < 1$, such that $\theta_1(\zeta) = \theta_2(\kappa(\zeta)), \zeta \in \mathfrak{D}$. This is indicated as $\theta_1 \prec \theta_2$ or $\theta_1(\zeta) \prec \theta_2(\zeta)$. In particular, if $\theta_2 \in \mathcal{S}$, then

$$\theta_1(\zeta) \prec \theta_2(\zeta) \Leftrightarrow \ \theta_1(0) = \theta_2(0) \quad \textit{and} \quad \theta_1(\mathfrak{D}) \subset \theta_2(\mathfrak{D}).$$

Fekete and Szegö [16] discovered an inequality for the coefficients of univalent analytic functions, which is known as the Fekete-Szegö inequality in mathematics. It states that if a function Φ of the form (1.1) $\in \mathcal{S}$, $\mu \in \mathbb{C}$, and $0 \leq \kappa < 1$, then $|d_3 - \mu d_2^2| \leq 1 + 2\exp(-2\mu/(1-\mu))$, $0 \leq \mu < 1$. Finding comparable estimates for different subclasses of $\mathcal{S}$ is the Fekete-Szegö problem.

Motivated by the articles [22, 40], we present two new subfamilies $\mathfrak{A}_{\sigma}^{\tau,\delta}(\nu,\varkappa)$ and $\mathfrak{V}_{\sigma}^{\tau,\delta}(\gamma,\varkappa)$ of σ subordinate to $\mathsf{B}_j(\varkappa), j \geq 0$ as in (1.4) with the GF (1.5). $\mathsf{B}(\varkappa,\zeta)$ as in (1.5), $\Phi^{-1}(\varpi) = \Psi(\varpi)$ an inverse as in (1.3), $\zeta \in \mathfrak{D}$ and $\varpi \in \mathfrak{D}$ are taken for granted throughout this work, unless specified.

Definition 1.3. *A function Φ in σ that has the series (1.1) is said to be in the set $\mathfrak{A}_{\sigma,p,q}^{\tau,\delta}(\nu,\varkappa)$, if*

$$\frac{1}{2}\left\{\frac{1-\nu+\nu[D_{p,q}(\zeta D_{p,q}\Phi(\zeta))]^{\tau}}{D_{p,q}\Phi(\zeta)} + \left(\frac{1-\nu+\nu[D_{p,q}(\zeta D_{p,q}\Phi(\zeta))]^{\tau}}{D_{p,q}\Phi(\zeta)}\right)^{\frac{1}{\delta}}\right\} \prec \mathsf{B}(\varkappa,\zeta),$$

and

$$\frac{1}{2}\left\{\frac{1-\nu+\nu[D_{p,q}(\varpi D_{p,q}\Psi(\varpi))]^{\tau}}{D_{p,q}\Psi(\varpi)} + \left(\frac{1-\nu+\nu[D_{p,q}(\varpi D_{p,q}\Psi(\varpi))]^{\tau}}{D_{p,q}\Psi(\varpi)}\right)^{\frac{1}{\delta}}\right\} \prec \mathsf{B}(\varkappa,\varpi),$$

where $\nu \geq 1$, $0 < \delta \leq 1$, $\tau \geq 1$, and $\varkappa \in (-\frac{1}{3}, \frac{1}{3})$.

The family $\mathfrak{A}_{\sigma,p,q}^{\tau,\delta}(\nu,\varkappa)$ contains numerous subfamilies of σ for particular choices of p, q, ν, and τ, as illustrated below:

1. Let $\tau = 1$. Then, for $\nu \geq 1$, $0 < \delta \leq 1$, and $\varkappa \in (-\frac{1}{3}, \frac{1}{3})$, the class $\mathfrak{A}_{\sigma,p,q}^{1,\delta}(\nu,\varkappa)$ consists of all functions $\Phi \in \sigma$ satisfying

$$\frac{1}{2}\left\{\frac{1-\nu+\nu[D_{p,q}(\zeta D_{p,q}\Phi(\zeta))]}{D_{p,q}\Phi(\zeta)} + \left(\frac{1-\nu+\nu[D_{p,q}(\zeta D_{p,q}\Phi(\zeta))]}{D_{p,q}\Phi(\zeta)}\right)^{\frac{1}{\delta}}\right\} \prec \mathsf{B}(\varkappa,\zeta),$$

and

$$\frac{1}{2}\left\{\frac{1-\nu+\nu[D_{p,q}(\varpi D_{p,q}\Psi(\varpi))]}{D_{p,q}\Psi(\varpi)} + \left(\frac{1-\nu+\nu[D_{p,q}(\varpi D_{p,q}\Psi(\varpi))]}{D_{p,q}\Psi(\varpi)}\right)^{\frac{1}{\delta}}\right\} \prec \mathsf{B}(\varkappa,\varpi).$$

2. Suppose $\nu = 1$. Then, for $0 < \delta \leq 1$, $\tau \geq 1$, and $\varkappa \in (-\frac{1}{3}, \frac{1}{3})$, the class $\mathfrak{A}_{\sigma,p,q}^{\tau,\delta}(1,\varkappa)$ consists of all elements $\Phi \in \sigma$ satisfying

$$\frac{1}{2}\left\{\frac{[D_{p,q}(\zeta D_{p,q}\Phi(\zeta))]^{\tau}}{D_{p,q}\Phi(\zeta)} + \left(\frac{[D_{p,q}(\zeta D_{p,q}\Phi(\zeta))]^{\tau}}{D_{p,q}\Phi(\zeta)}\right)^{\frac{1}{\delta}}\right\} \prec \mathsf{B}(\varkappa,\zeta),$$

and

$$\frac{1}{2}\left\{\frac{[D_{p,q}(\varpi D_{p,q}\Psi(\varpi))]^{\tau}}{D_{p,q}\Psi(\varpi)} + \left(\frac{[D_{p,q}(\varpi D_{p,q}\Psi(\varpi))]^{\tau}}{D_{p,q}\Psi(\varpi)}\right)^{\frac{1}{\delta}}\right\} \prec \mathsf{B}(\varkappa,\varpi).$$

3. If $p = 1$ and $q \to 1^-$. Then, for $0 < \delta \leq 1$, $\tau \geq 1$, $\nu \geq 1$, and $\varkappa \in (-\frac{1}{3}, \frac{1}{3})$, the class $\mathfrak{A}_{\sigma,p=1,q\to 1^-}^{\tau,\delta}(\nu,\varkappa)$ consists of all functions $\Phi \in \sigma$ satisfying

$$\frac{1}{2}\left\{\frac{1-\nu+\nu[(\zeta\Phi'(\zeta))']^{\tau}}{\Phi'(\zeta)} + \left(\frac{1-\nu+\nu[(\zeta\Phi'(\zeta))']^{\tau}}{\Phi'(\zeta)}\right)^{\frac{1}{\delta}}\right\} \prec \mathsf{B}(\varkappa,\zeta),$$

and

$$\frac{1}{2}\left\{\frac{1-\nu+\nu[(\varpi\Psi'(\varpi))']^{\tau}}{\Psi'(\varpi)}+\left(\frac{1-\nu+\nu[(\varpi\Psi'(\varpi))']^{\tau}}{\Psi'(\varpi)}\right)^{\frac{1}{\delta}}\right\}\prec \mathsf{B}(\varkappa,\varpi).$$

4. Let $\delta=1$. Then, for $\tau\geq 1$, $\nu\geq 1$, and $\varkappa\in(-\frac{1}{3},\frac{1}{3})$, the class $\mathfrak{A}^{\tau,1}_{\sigma,p,q}(\nu,\varkappa)$ consists of elements $\Phi\in\sigma$ satisfying

$$\left\{\frac{1-\nu+\nu[D_{p,q}(\zeta D_{p,q}\Phi(\zeta))]^{\tau}}{D_{p,q}\Phi(\zeta)}\right\}\prec \mathsf{B}(\varkappa,\zeta),$$

and

$$\left\{\frac{1-\nu+\nu[D_{p,q}(\varpi D_{p,q}\Psi(\varpi))]^{\tau}}{D_{p,q}\Psi(\varpi)}\right\}\prec \mathsf{B}(\varkappa,\varpi).$$

Definition 1.4. *A function Φ in σ that has the series (1.1) is said to be in the set $\mathfrak{V}^{\tau,\delta}_{\sigma,p,q}(\gamma,\varkappa)$, if*

$$\frac{1}{2}\left\{\frac{\zeta(D_{p,q}\Phi(\zeta))^{\tau}}{(1-\gamma)\zeta+\gamma\Phi(\zeta)}+\left(\frac{\zeta(D_{p,q}\Phi(\zeta))^{\tau}}{(1-\gamma)\zeta+\gamma\Phi(\zeta)}\right)^{\frac{1}{\delta}}\right\}\prec \mathsf{B}(\varkappa,\zeta),$$

and

$$\frac{1}{2}\left\{\frac{w(D_{p,q}\Psi(\varpi))^{\tau}}{(1-\gamma)\varpi+\gamma\Psi(\varpi)}+\left(\frac{\varpi(D_{p,q}\Psi(\varpi))^{\tau}}{(1-\gamma)\varpi+\gamma\Psi(\varpi)}\right)^{\frac{1}{\delta}}\right\}\prec \mathsf{B}(\varkappa,\varpi),$$

where $0\leq\gamma\leq 1$, $0<\delta\leq 1$, $\tau\geq 1$, and $\varkappa\in(-\frac{1}{3},\frac{1}{3})$.

For particular selections of γ and τ, the family $\mathfrak{V}^{\tau,\delta}_{\sigma,p,q}(\gamma,\varkappa)$ includes many many subfamilies of σ, as shown below:

1. Let $\gamma=0$. Then, for $\tau\geq 1$, $0<\delta\leq 1$, and $\varkappa\in(-\frac{1}{3},\frac{1}{3})$, the class $\mathfrak{V}^{\tau,\delta}_{\sigma,p,q}(0,\varkappa)$ consists of elements $\Phi\in\sigma$ satisfying

$$\frac{1}{2}\left((D_{p,q}\Phi(\zeta))^{\tau}+(D_{p,q}\Phi(\zeta))^{\frac{\tau}{\delta}}\right)\prec \mathsf{B}(\varkappa,\zeta),$$

and

$$\frac{1}{2}\left((D_{p,q}\Psi(\varpi))^{\tau}+D_{p,q}\Psi(\varpi)^{\frac{\tau}{\delta}}\right)\prec \mathsf{B}(\varkappa,\varpi).$$

2. Let $\gamma=1$. Then, for $\tau\geq 1$, $0<\delta\leq 1$, and $\varkappa\in(-\frac{1}{3},\frac{1}{3})$, the class $\mathfrak{V}^{\tau,\delta}_{\sigma,p,q}(1,\varkappa)$ consists of elements $\Phi\in\sigma$ satisfying

$$\frac{1}{2}\left\{\frac{\zeta(D_{p,q}\Phi(\zeta))^{\tau}}{\Phi(\zeta)}+\left(\frac{\zeta(D_{p,q}\Phi(\zeta))^{\tau}}{\Phi(\zeta)}\right)^{\frac{1}{\delta}}\right\}\prec \mathsf{B}(\varkappa,\zeta),$$

and

$$\frac{1}{2}\left\{\frac{\varpi(D_{p,q}\Psi(\varpi))^{\tau}}{\Psi(\varpi)}+\left(\frac{\varpi(D_{p,q}\Psi(\varpi))^{\tau}}{\Psi(\varpi)}\right)^{\frac{1}{\delta}}\right\}\prec \mathsf{B}(\varkappa,\varpi).$$

3. Let $p=1$ and $q\to 1^{-}$. Then, for $\tau\geq 1$, $0<\delta\leq 1$, $0\leq\gamma\leq 1$, and $\varkappa\in(-\frac{1}{3},\frac{1}{3})$, the class $\mathfrak{V}^{\tau,\delta}_{\sigma,p=1,q\to 1^{-}}(\gamma,\varkappa)$ consists of functions $\Phi\in\sigma$ satisfying

$$\frac{1}{2}\left\{\frac{\zeta(\Phi'(\zeta))^{\tau}}{(1-\gamma)\zeta+\gamma\Phi(\zeta)}+\left(\frac{\zeta(\Phi'(\zeta))^{\tau}}{(1-\gamma)\zeta+\gamma\Phi(\zeta)}\right)^{\frac{1}{\delta}}\right\}\prec \mathsf{B}(\varkappa,\zeta),$$

and

$$\frac{1}{2}\left\{\frac{\varpi(\Psi'(\varpi))^{\tau}}{(1-\gamma)\varpi+\gamma\Psi(\varpi)}+\left(\frac{\varpi(\Psi'(\varpi))^{\tau}}{(1-\gamma)\varpi+\gamma\Psi(\varpi)}\right)^{\frac{1}{\delta}}\right\}\prec \mathsf{B}(\varkappa,\varpi).$$

4. Let $\delta = 1$. Then, for $0 \le \gamma \le 1$, $\tau \ge 1$, and $\varkappa \in (-\frac{1}{3}, \frac{1}{3})$, the class $\mathfrak{V}^{\tau,1}_{\sigma,p,q}(\gamma,\varkappa)$ consists of elements $\Phi \in \sigma$ satisfying

$$\frac{\zeta(D_{p,q}\Phi(\zeta))^{\tau}}{(1-\gamma)\zeta+\gamma\Phi(\zeta)}\prec \mathsf{B}(\varkappa,\zeta), \quad \textit{and} \quad \frac{\varpi(D_{p,q}\Psi(\varpi))^{\tau}}{(1-\gamma)\varpi+\gamma\Psi(\varpi)}\prec \mathsf{B}(\varkappa,\varpi).$$

In Section 2, we find estimations for $|d_2|$, $|d_3|$, and the Fekete-Szegö functional $|d_3-\mu d_2^2|$, $\mu\in\mathbb{R}$, for functions in the class $\mathfrak{A}^{\tau,\delta}_{\sigma,p,q}(\nu,\varkappa)$. The limits for $|d_2|$, $|d_3|$, and the Fekete-Szegö functional $|d_3-\mu d_2^2|$, $\mu\in\mathbb{R}$, are obtained in Section 3 for functions in the class $\mathfrak{V}^{\tau,\delta}_{\sigma,p,q}(\gamma,\varkappa)$. We also present a number of outcomes of our results as special cases and draw attention to relevant connections with earlier findings.

2. Results of the class $\mathfrak{A}^{\tau,\delta}_{\sigma,p,q}(\nu,\varkappa)$

We first compute the coefficient estimates for members of the class $\mathfrak{A}^{\tau,\delta}_{\sigma,p,q}(\nu,\varkappa)$.

Theorem 2.1. *Let $\tau\ge 1$, $\nu \ge 1$, $0<\delta\le 1$, and $\varkappa\in(-\frac{1}{3},\frac{1}{3})$. If $\Phi\in\mathfrak{A}^{\tau,\delta}_{\sigma,p,q}(\nu,\varkappa)$, then*

$$|d_2| \le \frac{6\delta|\varkappa|\sqrt{3|\varkappa|}}{\sqrt{|(2\delta(\delta+1)(U+V)+(1-\delta)W^2)9\varkappa^2-(\delta+1)^2W^2(18\varkappa^2-1)|}}, \tag{2.1}$$

$$|d_3| \le \frac{6\delta|\varkappa|}{(\delta+1)|U|}+\frac{108\delta^2|\varkappa|^3}{|(2\delta(\delta+1)(U+V)+(1-\delta)W^2)9\varkappa^2-(\delta+1)^2W^2(18\varkappa^2-1)|}, \tag{2.2}$$

and for $\mu\in\mathbb{R}$

$$|d_3-\mu d_2^2| \le \begin{cases} \frac{6\delta|\varkappa|}{(\delta+1)|U|}; & |1-\mu|\le J \\ \frac{108\delta^2|\varkappa|^3\,|1-\mu|}{|(2\delta(\delta+1)(U+V)+(1-\delta)^2W^2)9\varkappa^2-(\delta+1)^2W^2(18\varkappa^2-1)|}; & |1-\mu|\ge J, \end{cases} \tag{2.3}$$

where

$$J=\left|\frac{(2\delta(\delta+1)(U+V)+(1-\delta)W^2)9\varkappa^2-(\delta+1)^2W^2(18\varkappa^2-1)}{18\delta(1+\delta)U\varkappa^2}\right|, \tag{2.4}$$

$$U=[3]_{p,q}(\nu\tau[3]_{p,q}-1), \tag{2.5}$$

$$V=[2]^2_{p,q}\left(1-\nu\tau[2]_{p,q}+\frac{\nu\tau(\tau-1)[2]^2_{p,q}}{2}\right), \tag{2.6}$$

and

$$W=[2]_{p,q}(\nu\tau[2]_{p,q}-1). \tag{2.7}$$

Proof. Let $\Phi\in\mathfrak{A}^{\tau,\delta}_{\sigma,p,q}(\gamma,\varkappa)$. Then, by Definition 1.3, it follows that

$$\frac{1}{2}\left\{\frac{1-\nu+\nu[D_{p,q}(\zeta D_{p,q}\Phi(\zeta))]^{\tau}}{D_{p,q}\Phi(\zeta)}+\left(\frac{1-\nu+\nu[D_{p,q}(\zeta D_{p,q}\Phi(\zeta))]^{\tau}}{D_{p,q}\Phi(\zeta)}\right)^{\frac{1}{\delta}}\right\}=\mathsf{B}(\varkappa,\mathfrak{m}(\zeta)), \tag{2.8}$$

and

$$\frac{1}{2}\left\{\frac{1-\nu+\nu[D_{p,q}(\varpi D_{p,q}\Psi(\varpi))]^{\tau}}{D_{p,q}\Psi(\varpi)}+\left(\frac{1-\nu+\nu[D_{p,q}(\varpi D_{p,q}\Psi(\varpi))]^{\tau}}{D_{p,q}\Psi(\varpi)}\right)^{\frac{1}{\delta}}\right\}=\mathsf{B}(\varkappa,\mathfrak{n}(\varpi)), \tag{2.9}$$

where

$$\mathfrak{m}(\zeta)=\mathfrak{m}_1\zeta+\mathfrak{m}_2\zeta^2+\mathfrak{m}_3\zeta^3+\cdots,\ \textit{and}\ \mathfrak{n}(\varpi)=\mathfrak{n}_1\varpi+\mathfrak{n}_2\varpi^2+\mathfrak{n}_3\varpi^3+\cdots, \tag{2.10}$$

are analytic in $\mathfrak{D}$ satisfying $|\mathfrak{m}(\zeta)|<1$, $|\mathfrak{n}(\varpi)|<1$, and $\mathfrak{m}(0)=\mathfrak{n}(0)=0$. We known that

$$|\mathfrak{m}_i|\le 1,\ \text{and}\ |\mathfrak{n}_i|\le 1, (i\in\mathbb{N}). \tag{2.11}$$

Substituting $\mathsf{B}(\varkappa,\zeta)$ from (1.5) into (2.8) and (2.9), and using (2.10), we obtain:

$$\mathsf{B}(\varkappa,\mathfrak{m}(\zeta))=1+\mathcal{B}_1(\varkappa)\mathfrak{m}_1\zeta+[\mathcal{B}_1(\varkappa)\mathfrak{m}_2+\mathcal{B}_2(\varkappa)\mathfrak{m}_1^2]\zeta^2+\cdots, \tag{2.12}$$

and

$$\mathsf{B}(\varkappa,\mathfrak{n}(\varpi))=1+\mathcal{B}_1(\varkappa)\mathfrak{n}_1\varpi+[\mathcal{B}_1(\varkappa)\mathfrak{n}_2+\mathcal{B}_2(\varkappa)\mathfrak{n}_1^2]\varpi^2+\cdots. \tag{2.13}$$

It follows from (2.8) and (2.9) that

$$\frac{1}{2}\left\{\frac{1-\nu+\nu[D_{p,q}(\zeta D_{p,q}\Phi(\zeta))]^{\tau}}{D_{p,q}\Phi(\zeta)}+\left(\frac{1-\nu+\nu[D_{p,q}(\zeta D_{p,q}\Phi(\zeta))]^{\tau}}{D_{p,q}\Phi(\zeta)}\right)^{\frac{1}{\delta}}\right\}=$$

$$1+\left(\frac{1+\delta}{2\delta}\right)Wd_2\zeta+\left[\frac{1+\delta}{2\delta}(Ud_3+Vd_2^2)+\frac{1-\delta}{4\delta^2}W^2d_2^2\right]\zeta^2+... \tag{2.14}$$

and

$$\frac{1}{2}\left\{\frac{1-\nu+\nu[D_{p,q}(\varpi D_{p,q}\Psi(\varpi))]^{\tau}}{D_{p,q}\Psi(\varpi)}+\left(\frac{1-\nu+\nu[D_{p,q}(\varpi D_{p,q}\Psi(\varpi))]^{\tau}}{D_{p,q}\Psi(\varpi)}\right)^{\frac{1}{\delta}}\right\}=$$

$$1-\left(\frac{1+\delta}{2\delta}\right)Wd_2\varpi+\left[\frac{1+\delta}{2\delta}(U(2d_2^2-d_3)+Vd_2^2)+\frac{1-\delta}{4\delta^2}W^2d_2^2\right]\varpi^2+... \tag{2.15}$$

where U, V and W are as mentioned in (2.5), (2.6) and (2.7), respectively.

Comparing (2.12) and (2.14), we have

$$\frac{(\delta+1)W}{2\delta}\,d_2=\mathcal{B}_1(\varkappa)\mathfrak{m}_1, \tag{2.16}$$

$$\left(\frac{\delta+1}{2\delta}\right)\left(Ud_3+Vd_2^2\right)+\left(\frac{1-\delta}{4\delta^2}\right)W^2d_2^2=\mathcal{B}_1(\varkappa)\mathfrak{m}_2+\mathcal{B}_2(\varkappa)\mathfrak{m}_1^2, \tag{2.17}$$

Comparing (2.13) and (2.15), we have

$$-\frac{(\delta+1)W}{2\delta}\,d_2=\mathcal{B}_1(\varkappa)\mathfrak{n}_1, \tag{2.18}$$

and

$$\left(\frac{\delta+1}{2\delta}\right)\left(U(2d_2^2-d_3)+Vd_2^2\right)+\left(\frac{1-\delta}{4\delta^2}\right)W^2d_2^2=\mathcal{B}_1(\varkappa)\mathfrak{n}_2+\mathcal{B}_2(\varkappa)\mathfrak{n}_1^2, \tag{2.19}$$

From (2.16) and (2.18), we easily obtain

$$\mathfrak{m}_1=-\mathfrak{n}_1, \tag{2.20}$$

and also

$$\frac{(\delta+1)^2W^2}{2\delta^2}\, d_2^2 = (\mathfrak{m}_1^2+\mathfrak{n}_1^2)(\mathcal{B}_1(\varkappa))^2. \tag{2.21}$$

The bound on $|d_2|$ is obtained by adding (2.17) and (2.19):

$$\left[\left(\frac{\delta+1}{\delta}\right)(U+V)+\left(\frac{1-\delta}{2\delta^2}\right)W^2\right] d_2^2 = \mathcal{B}_1(\varkappa)(\mathfrak{m}_2+\mathfrak{n}_2)+\mathcal{B}_2(\varkappa)(\mathfrak{m}_1^2+\mathfrak{n}_1^2). \tag{2.22}$$

The value of $\mathfrak{m}_1^2+\mathfrak{n}_1^2$ from (2.21) is substituted in (2.22), yielding

$$d_2^2 = \frac{2\delta^2\mathcal{B}_1^3(\varkappa)(\mathfrak{m}_2+\mathfrak{n}_2)}{(2\delta(\delta+1)(U+V)+(1-\delta)W^2)\mathcal{B}_1^2(\varkappa)-(\delta+1)^2W^2\mathcal{B}_2(\varkappa)}. \tag{2.23}$$

Applying (2.11) for the coefficients $\mathfrak{m}_2$ and $\mathfrak{n}_2$ yields (2.1).

We deduct (2.19) from (2.17) to get the bound $|d_3|$:

$$d_3 = d_2^2 + \frac{\mathcal{B}_1(\varkappa)(\mathfrak{m}_2-\mathfrak{n}_2)}{\left(\frac{\delta+1}{\delta}\right)U}. \tag{2.24}$$

This results in the inequality that follows:

$$|d_3| \le |d_2|^2 + \frac{|\mathcal{B}_1(\varkappa)||\mathfrak{m}_2-\mathfrak{n}_2|}{\left(\frac{\delta+1}{\delta}\right)|U|}. \tag{2.25}$$

Applying (2.11) for $\mathfrak{m}_2$ and $\mathfrak{n}_2$, we obtain (2.2) from (2.1) and (2.25).

Finally, we compute the bound on $|d_3-\mu d_2^2|$ for $\mu\in\mathbb{R}$, using the values of d_2^2 and d_3 from (2.23) and (2.24), respectively. Consequently, we have

$$|d_3-\mu d_2^2| = |\mathcal{B}_1(\varkappa)|\left|\left(\mathfrak{L}(\mu,\varkappa)+\frac{\delta}{(\delta+1)U}\right)\mathfrak{m}_2+\left(\mathfrak{L}(\mu,\varkappa)-\frac{\delta}{(\delta+1)U}\right)\mathfrak{n}_2\right|,$$

where

$$\mathfrak{L}(\mu,\varkappa) = \frac{2\delta^2(1-\mu)\mathcal{B}_1^2(\varkappa)}{(2\delta(\delta+1)(U+V)+(1-\delta)W^2)\mathcal{B}_1^2(\varkappa)-(\delta+1)^2W^2\mathcal{B}_2(\varkappa)}.$$

Clearly

$$|d_3-\mu d_2^2| \le \begin{cases} \frac{2\delta|\mathcal{B}_1(\varkappa)|}{(\delta+1)|U|} & ;\ 0\le|\mathfrak{L}(\mu,\varkappa)|\le\frac{\delta}{(\delta+1)|U|} \\ 2|\mathcal{B}_1(\varkappa)||\mathfrak{L}(\mu,\varkappa)| & ;\ |\mathfrak{L}(\mu,\varkappa)|\ge\frac{\delta}{(\delta+1)|U|}, \end{cases}$$

which leads us to the conclusion (2.3), with J as in (2.4). This concludes Theorem 2.1's proof. □

Taking $\tau=1$ and $\nu=1$ in the above theorem, respectively, yields the following results.

Corollary 2.2. *Let $\tau=1$. Then for $\Phi\in\mathfrak{A}_{\sigma,p,q}^{1,\delta}(\nu,\varkappa)$, the upper bounds of $|d_2|$, $|d_3|$ and $|d_3-\mu d_2^2|$, $\mu\in\mathbb{R}$, are given by (2.1), (2.2) and (2.3), respectively, with $U=U_1=[3]_{p,q}(\nu[3]_{p,q}-1)$, $V=V_1=[2]_{p,q}^2(1-\nu[2]_{p,q})$, and $W=W_1=[2]_{p,q}(\nu[2]_{p,q}-1)$. For J in (2.4), U, V, and W are to be replaceded by U_1, V_1, and W_1, respectively.*

Corollary 2.3. *Let $\nu=1$. Then for $\Phi\in\mathfrak{A}_{\sigma,p,q}^{\tau,\delta}(1,\varkappa)$, the upper bounds of $|d_2|$, $|d_3|$ and $|d_3-\mu d_2^2|$, $\mu\in\mathbb{R}$, are given by (2.1), (2.2) and (2.3), respectively, with $U=U_2=[3]_{p,q}(\tau[3]_{p,q}-1)$, $V=V_2=[2]_{p,q}^2\left(1-\tau[2]_{p,q}+\frac{\tau(\tau-1)}{2}[2]_{p,q}^2\right)$, and $W=W_2=[2]_{p,q}(\tau[2]_{p,q}-1)$. For J in (2.4), U, V, and W are to be replaced by U_2, V_2, and W_2, respectively.*

With $p=1$ and $q\to 1^-$, we obtain from Theorem 2.1:

Corollary 2.4. *Let $\nu \geq 1$, $0 < \delta \leq 1$, $\tau \geq 1$, and $\varkappa \in (-\frac{1}{3}, \frac{1}{3})$. If $\Phi \in \mathfrak{A}^{\tau,\delta}_{\sigma,p=1,q\to 1^-}(\nu,\varkappa)$, then*

$$|d_2| \leq \frac{3\delta|\varkappa|\sqrt{6|\varkappa|}}{\sqrt{|(\delta(\delta+1)(8\nu\tau^2 - 7\nu\tau + 1) + 2(1-\delta)(2\nu\tau-1)^2)9\varkappa^2 - 2(\delta+1)^2(2\nu\tau-1)^2(18\varkappa^2-1)|}},$$

$$|d_3| \leq \frac{2\delta|\varkappa|}{(\delta+1)(3\nu\tau-1)} + \frac{54\delta^2|\varkappa|^3}{|(\delta(\delta+1)(8\nu\tau^2 - 7\nu\tau + 1) + 2(1-\delta)(2\nu\tau-1)^2)9\varkappa^2 - 2(\delta+1)^2(2\nu\tau-1)^2(18\varkappa^2-1)|},$$

and for $\mu \in \mathbb{R}$

$$|d_3 - \mu d_2^2| \leq \begin{cases} \frac{2\delta|\varkappa|}{(\delta+1)(3\nu\tau-1)}; & |1-\mu| \leq J_1 \\ \frac{54\delta^2|\varkappa|^3\,|1-\mu|}{|(\delta(\delta+1)(8\nu\tau^2-7\nu\tau+1)+2(1-\delta)(2\nu\tau-1)^2)9\varkappa^2 - 2(\delta+1)^2(2\nu\tau-1)^2(18\varkappa^2-1)|}; & |1-\mu| \geq J_1, \end{cases}$$

where

$$J_1 = \left| \frac{(\delta(\delta+1)(8\nu\tau^2 - 7\nu\tau + 1) + 2(1-\delta)(2\nu\tau-1)^2)9\varkappa^2 - 2(\delta+1)^2(2\nu\tau-1)^2(18\varkappa^2-1)}{27\delta(\delta+1)(3\nu\tau-1)\varkappa^2} \right|.$$

To better understand the implications of the corollary, we now examine several special cases.
Case 2.1. In the case $\delta = 1$, the class $\mathfrak{A}^{\tau,1}_{\sigma,p=1,q\to 1^-}(\nu,\varkappa)$ denotes the set of functions $\Phi \in \sigma$ satisfying

$$\frac{\nu[(\zeta\Phi'(\zeta))']^{\tau} + (1-\nu)}{\Phi'(\zeta)} \prec \mathsf{B}(\varkappa,\zeta), \quad \text{and} \quad \frac{\nu[(\varpi\Psi'(\varpi))']^{\tau} + (1-\nu)}{\Psi'(\varpi)} \prec \mathsf{B}(\varkappa,\varpi),$$

where $\nu \geq 1$, $\tau \geq 1$, and $\varkappa \in (-\frac{1}{3}, \frac{1}{3})$.
Case 2.2. In the special case $\nu = 1$ in Example 2.1, the class $\mathfrak{A}^{\tau,1}_{\sigma,p=1,q\to 1^-}(1,\varkappa)$ denotes the set of functions $\Phi \in \sigma$ satisfying

$$\frac{[(\zeta\Phi'(\zeta))']^{\tau}}{\Phi'(\zeta)} \prec \mathsf{B}(\varkappa,\zeta), \quad \text{and} \quad \frac{[(\varpi\Psi'(\varpi))']^{\tau}}{\Psi'(\varpi)} \prec \mathsf{B}(\varkappa,\varpi),$$

where $\tau \geq 1$ and $\varkappa \in (-\frac{1}{3}, \frac{1}{3})$.

Corollary 2.5. *Let $\nu \geq 1$, $\tau \geq 1$, and $\varkappa \in (-\frac{1}{3}, \frac{1}{3})$. If $\Phi \in \mathfrak{A}^{\tau,1}_{\sigma,p=1,q\to 1^-}(\nu,\varkappa)$, then*

$$|d_2| \leq \frac{3|\varkappa|\sqrt{3|\varkappa|}}{\sqrt{|9(8\nu\tau^2 - 7\nu\tau + 1)\varkappa^2 - 4(2\nu\tau-1)^2(18\varkappa^2-1)|}},$$

$$|d_3| \leq \frac{|\varkappa|}{3\nu\tau-1} + \frac{27|\varkappa|^3}{|9(8\nu\tau^2 - 7\nu\tau + 1)\varkappa^2 - 4(2\nu\tau-1)^2(18\varkappa^2-1)|},$$

and for $\mu \in \mathbb{R}$

$$|d_3 - \mu d_2^2| \leq \begin{cases} \frac{|\varkappa|}{3\nu\tau-1}; & |1-\mu| \leq J_2 \\ \frac{27|\varkappa|^3\,|1-\mu|}{|9(8\nu\tau^2-7\nu\tau+1)\varkappa^2 - 4(2\nu\tau-1)^2(18\varkappa^2-1)|}; & |1-\mu| \geq J_2, \end{cases}$$

where

$$J_2 = \left| \frac{9(8\nu\tau^2 - 7\nu\tau + 1)\varkappa^2 - 4(2\nu\tau-1)^2(18\varkappa^2-1)}{27(3\nu\tau-1)\varkappa^2} \right|.$$

Corollary 2.6. *Let $\varkappa \in (-\frac{1}{3}, \frac{1}{3})$ and $\tau \geq 1$. If $\Phi \in \mathfrak{A}^{\tau,1}_{\sigma,p=1,q\to 1^-}(1, \varkappa)$, then*

$$|d_2| \leq \frac{3|\varkappa|\sqrt{3|\varkappa|}}{\sqrt{|(1-7\tau+8\tau^2)9\varkappa^2 - 4(2\tau-1)^2(18\varkappa^2-1)|}},$$

$$|d_3| \leq \frac{27|\varkappa|^3}{|9(1-7\tau+8\tau^2)\varkappa^2 - 4(2\tau-1)^2(18\varkappa^2-1)|} + \frac{|\varkappa|}{3\tau-1},$$

and for $\mu \in \mathbb{R}$

$$|d_3 - \mu d_2^2| \leq \begin{cases} \frac{|\varkappa|}{3\tau-1}; & |1-\mu| \leq J_3 \\ \frac{27|\varkappa|^3\,|1-\mu|}{|9(1-7\tau+8\tau^2)\varkappa^2-4(2\tau-1)^2(18\varkappa^2-1)|}; & |1-\mu| \geq J_3, \end{cases}$$

where

$$J_3 = \left|\frac{9(1-7\tau+8\tau^2))\varkappa^2 - 4(2\tau-1)^2(18\varkappa^2-1)}{27(3\tau-1)\varkappa^2}\right|.$$

Remark 2.7. *i) The result stated in Corollary 2.6, when specialized to $\beta = 1$, matches Corollary 2 of* [40], *provided the parameter $\varkappa$ lies in the interval $(-\frac{1}{3}, \frac{1}{3})$. ii) Likewise, setting $\tau = 1$ in Corollary 2.6 reproduces Corollary 2 of* [23], *under the same condition on $\varkappa$.*

Remark 2.8. *The condition $\varkappa \in (-\frac{1}{3}, \frac{1}{3})$ ensures the analyticity of the generating function within the unit disk $\mathfrak{D}$, a requirement not fully met by the domain $(-\frac{1}{2}, 1]$ as stated in* [23, 40], *which should therefore be corrected.*

Using $\delta = 1$ in the aforementioned theorem, we derive

Corollary 2.9. *Let $\tau \geq 1$, $\nu \geq 1$, and $\varkappa \in (-\frac{1}{3}, \frac{1}{3})$. If $\Phi \in \mathfrak{A}^{\tau,1}_{\sigma,p,q}(\nu, \varkappa)$, then*

$$|d_2| \leq \frac{3|\varkappa|\sqrt{3|\varkappa|}}{\sqrt{|(U+V)9\varkappa^2 - W^2(18\varkappa^2-1)|}},$$

$$|d_3| \leq \frac{3|\varkappa|}{|U|} + \frac{27|\varkappa|^3}{|(U+V)9\varkappa^2 - W^2(18\varkappa^2-1)|},$$

and for $\mu \in \mathbb{R}$

$$|d_3 - \mu d_2^2| \leq \begin{cases} \frac{3|\varkappa|}{|U|}; & |1-\mu| \leq J_4 \\ \frac{27|\varkappa|^3\,|1-\mu|}{|(U+V)9\varkappa^2-W^2(18\varkappa^2-1)|}; & |1-\mu| \geq J_4, \end{cases}$$

where

$$J_4 = \left|\frac{(U+V)9\varkappa^2 - W^2(18\varkappa^2-1)}{9U\varkappa^2}\right|,$$

and U, V, and W are as detailed in (2.5), (2.6), *and* (2.7), *respectively.*

3. Results of the class $\mathfrak{V}^{\tau,\delta}_{\sigma,p,q}(\gamma, \varkappa)$

We first determine the coefficient estimates for function Φ in the class $\mathfrak{V}^{\tau,\delta}_{\sigma,p,q}(\gamma, \varkappa)$ defined in Definition 1.4.

Theorem 3.1. *Let $\tau \geq 1$, $0 \leq \gamma \leq 1$, $0 < \delta \leq 1$, and $\varkappa \in (-\frac{1}{3}, \frac{1}{3})$. If $\Phi \in \mathfrak{V}^{\tau,\delta}_{\sigma,p,q}(\gamma, \varkappa)$, then*

$$|d_2| \leq \frac{6\delta|\varkappa|\sqrt{3|\varkappa|}}{\sqrt{|(2\delta(\delta+1)(A+S)+(1-\delta)B^2)9\varkappa^2-(\delta+1)^2B^2(18\varkappa^2-1)|}}, \tag{3.1}$$

$$|d_3| \leq \frac{6\delta|\varkappa|}{(\delta+1)|A|} + \frac{108\delta^2|\varkappa|^3}{|(2\delta(\delta+1)(A+S)+(1-\delta)B^2)9\varkappa^2-(\delta+1)^2B^2(18\varkappa^2-1)|}, \tag{3.2}$$

and for $\mu \in \mathbb{R}$

$$|d_3 - \mu d_2^2| \leq \begin{cases} \frac{6\delta|\varkappa|}{(\delta+1)|A|} & ; |1-\mu| \leq Q \\ \frac{108\delta^2|\varkappa|^3\,|1-\mu|}{|(2\delta(\delta+1)(A+S)+(1-\delta)B^2)9\varkappa^2-(\delta+1)^2B^2(18\varkappa^2-1)|} & ; |1-\mu| \geq Q, \end{cases} \tag{3.3}$$

where

$$Q = \left|\frac{(2\delta(\delta+1)(A+S)+(1-\delta)B^2)9\varkappa^2-(\delta+1)^2B^2(18\varkappa^2-1)}{18\delta(\delta+1)A\varkappa^2}\right|, \tag{3.4}$$

$$A = \tau[3]_{p,q} - \gamma, \tag{3.5}$$

$$S = \frac{\tau(\tau-1)[2]^2_{p,q}}{2} - \gamma\tau[2]_{p,q} + \gamma^2, \tag{3.6}$$

and

$$B = \tau[2]_{p,q} - \gamma. \tag{3.7}$$

Proof. Let $\Phi \in \mathfrak{V}^{\tau,\delta}_{\sigma,p,q}(\gamma, \varkappa)$. Next, as a result of Definition 1.4, we obtain

$$\frac{1}{2}\left\{\frac{\zeta(D_{p,q}\Phi(\zeta))^\tau}{(1-\gamma)\zeta+\gamma\Phi(\zeta)} + \left(\frac{\zeta(D_{p,q}\Phi(\zeta))^\tau}{(1-\gamma)\zeta+\gamma\Phi(\zeta)}\right)^{\frac{1}{\delta}}\right\} = \mathsf{B}(\varkappa, \mathfrak{m}(\zeta)), \tag{3.8}$$

and

$$\frac{1}{2}\left\{\frac{\varpi(D_{p,q}\Psi(\varpi))^\tau}{(1-\gamma)\varpi+\gamma\Psi(\varpi)} + \left(\frac{\varpi(D_{p,q}\Psi(\varpi))^\tau}{(1-\gamma)\varpi+\gamma\Psi(\varpi)}\right)^{\frac{1}{\delta}}\right\} = \mathsf{B}(\varkappa, \mathfrak{n}(\varpi)), \tag{3.9}$$

where, as described in (2.10), $\mathfrak{m}(\zeta)$ and $\mathfrak{n}(\varpi)$ are holomorphic functions.

It follows from (2.10), (3.8) and (3.9) that

$$\frac{1}{2}\left\{\frac{\zeta(D_{p,q}\Phi(\zeta))^\tau}{(1-\gamma)\zeta+\gamma\Phi(\zeta)} + \left(\frac{\zeta(D_{p,q}\Phi(\zeta))^\tau}{(1-\gamma)\zeta+\gamma\Phi(\zeta)}\right)^{\frac{1}{\delta}}\right\} =$$

$$1 + \left(\frac{\delta-1}{2\delta}\right)Bd_2\zeta + \left[\left(\frac{\delta+1}{2\delta}\right)(Ad_3+Sd_2^2) + \left(\frac{1-\delta}{4\delta^2}\right)B^2d_2^2\right]\zeta^2 + \cdots, \tag{3.10}$$

and

$$\frac{1}{2}\left\{\frac{\varpi(D_{p,q}\Psi(\varpi))^\tau}{(1-\gamma)\varpi+\gamma\Psi(\varpi)} + \left(\frac{\varpi(D_{p,q}\Psi(\varpi))^\tau}{(1-\gamma)\varpi+\gamma\Psi(\varpi)}\right)^{\frac{1}{\delta}}\right\} =$$

$$1 - \left(\frac{\delta-1}{2\delta}\right)Bd_2\varpi + \left[\left(\frac{\delta+1}{2\delta}\right)(A(2d_2^2-d_3)+Sd_2^2) + \left(\frac{1-\delta}{4\delta^2}\right)B^2d_2^2\right]\varpi^2 + \cdots. \tag{3.11}$$

Comparing (2.12) and (3.10), we have

$$\frac{(\delta+1)B}{2\delta}\, d_2 = \mathsf{B}_1(\varkappa)\mathfrak{m}_1, \tag{3.12}$$

and

$$\left(\frac{\delta+1}{2\delta}\right)\left(Ad_3 + Sd_2^2\right) + \left(\frac{1-\delta}{4\delta^2}\right) B^2 d_2^2 = \mathsf{B}_1(\varkappa)\mathfrak{m}_2 + \mathsf{B}_2(\varkappa)\mathfrak{m}_1^2. \tag{3.13}$$

Comparing (2.13) and (3.11), we have

$$-\frac{(\delta+1)B}{2\delta}\, d_2 = \mathsf{B}_1(\varkappa)\mathfrak{n}_1, \tag{3.14}$$

and

$$\left(\frac{\delta+1}{2\delta}\right)\left(A(2d_2^2 - d_3) + Sd_2^2\right) + \left(\frac{1-\delta}{4\delta^2}\right) B^2 d_2^2 = \mathsf{B}_1(\varkappa)\mathfrak{n}_2 + \mathsf{B}_2(\varkappa)\mathfrak{n}_1^2. \tag{3.15}$$

From (3.12) and (3.14), we easily obtain

$$\mathfrak{m}_1 = -\mathfrak{n}_1, \tag{3.16}$$

and also

$$\frac{(\delta+1)^2B^2}{2\delta^2}\, d_2^2 = (\mathfrak{m}_1^2 + \mathfrak{n}_1^2)(\mathsf{B}_1(\varkappa))^2. \tag{3.17}$$

We add (3.13) and (3.15) to obtain the bound on $|d_2|$:

$$\left(\left(\frac{\delta+1}{\delta}\right)(A+S) + \left(\frac{1-\delta}{2\delta^2}\right) B^2\right) d_2^2 = \mathsf{B}_1(\varkappa)(\mathfrak{m}_2 + \mathfrak{n}_2) + \mathsf{B}_2(\varkappa)(\mathfrak{m}_1^2 + \mathfrak{n}_1^2). \tag{3.18}$$

Substituting the value of $\mathfrak{m}_1^2 + \mathfrak{n}_1^2$ from (3.17) in (3.18), we get

$$d_2^2 = \frac{2\delta^2\mathsf{B}_1^3(\varkappa)(\mathfrak{m}_2 + \mathfrak{n}_2)}{(2\delta(\delta+1)(A+S) + (1-\delta)B^2)\mathsf{B}_1^2(\varkappa) - (\delta+1)^2B^2\mathsf{B}_2(\varkappa)}. \tag{3.19}$$

Applying (2.11) for the coefficients $\mathfrak{m}_2$ and $\mathfrak{n}_2$, we obtain (3.1).

The bound on $|d_3|$ is now obtained by subtracting (3.15) from (3.13) and using (3.16):

$$d_3 = d_2^2 + \frac{\mathsf{B}_1(\varkappa)(\mathfrak{m}_2 - \mathfrak{n}_2)}{\left(\frac{\delta+1}{\delta}\right) A}. \tag{3.20}$$

Then in view of (3.19), (3.20) becomes

$$d_3 = \frac{\delta\mathsf{B}_1(\varkappa)(\mathfrak{m}_2 - \mathfrak{n}_2)}{(\delta+1)|A|} + \frac{2\delta^2\mathsf{B}_1^3(\varkappa)(\mathfrak{m}_2 + \mathfrak{n}_2)}{(2\delta(\delta+1)(A+S) + (1-\delta)B^2)\mathsf{B}_1^2(\varkappa) - (\delta+1)^2B^2\mathsf{B}_2(\varkappa)}$$

and applying (2.11) for the coefficients $\mathfrak{m}_2$, and $\mathfrak{n}_2$ we get (3.2).

Finally, we compute the bound on $|d_3 - \mu d_2^2|$ for $\mu \in \mathbb{R}$, using the values of d_2^2 and d_3 from (3.19) and (3.20), respectively. Consequently, we have

$$|d_3 - \mu d_2^2| = |\mathsf{B}_1(\varkappa)| \left| \left(\mathfrak{L}_1(\mu,\varkappa) + \frac{\delta}{(\delta+1)A}\right)\mathfrak{m}_2 + \left(\mathfrak{L}_1(\mu,\varkappa) - \frac{\delta}{(\delta+1)A}\right)\mathfrak{n}_2 \right|,$$

where

$$\mathfrak{L}_1(\mu,\varkappa) = \frac{2\delta^2(1-\mu)\mathsf{B}_1^2(\varkappa)}{(2\delta(\delta+1)(A+S) + (1-\delta)B^2)\mathsf{B}_1^2(\varkappa) - (\delta+1)^2B^2\mathsf{B}_2(\varkappa)}.$$

Clearly

$$|d_3-\delta d_2^2|\le\begin{cases}\frac{2\delta|\mathsf{B}_1(\varkappa)|}{(\delta+1)|A|} & ;\ 0\le|\mathfrak{L}_1(\delta,\varkappa)|\le\frac{\delta}{(\delta+1)|A|}\\ 2|\mathsf{B}_1(\varkappa)||\mathfrak{L}_1(\delta,\varkappa)| & ;\ |\mathfrak{L}_1(\delta,\varkappa)|\ge\frac{\delta}{(\delta+1)|A|},\end{cases}$$

from which we conclude (3.3) with Q as in (3.4). □

Taking $\gamma=0$ and $\gamma=1$ in the above theorem, respectively, yields the following results.

Corollary 3.2. *Let $\gamma=0$. Then, for $\Phi\in\mathfrak{V}^{\tau,\delta}_{\sigma,p,q}(0,\varkappa)$, the upper bounds of $|d_2|$, $|d_3|$, and $|d_3-\mu d_2^2|$, $\mu\in\mathbb{R}$, are given by (3.1), (3.2), and (3.3), respectively, with $A=A_1=\tau[3]_{p,q}$, $S=S_1=\frac{\tau(\tau-1)}{2}[2]^2_{p,q}$, and $B=B_1=\tau[2]_{p,q}$. For Q in (3.4), A, S, and B are to be replaced by A_1, S_1, and B_1, respectively.*

Corollary 3.3. *Let $\gamma=1$. Then, for $\Phi\in\mathfrak{V}^{\tau,\delta}_{\sigma,p,q}(1,\varkappa)$, the upper bounds of $|d_2|$, $|d_3|$, and $|d_3-\mu d_2^2|$, $\mu\in\mathbb{R}$, are given by (3.1), (3.2), and (3.3), respectively, with $A=A_2=\tau[3]_{p,q}-1$, $S=S_2=\frac{\tau(\tau-1)}{2}[2]^2_{p,q}-\tau[2]_{p,q}+1$, and $B=B_2=\tau[2]_{p,q}-1$. For Q in (3.4), A, S, and B are to be substituted with A_2, S_2, and B_2, respectively.*

Taking $p=1$ and $q\to1^-$ in the Theorem 3.1, we get

Corollary 3.4. *Let $\tau\ge1$, $0\le\gamma\le1$, $0<\delta\le1$, and $\varkappa\in(-\frac{1}{3},\frac{1}{3})$. If $\Phi\in\mathfrak{V}^{\tau,\delta}_{\sigma,p=1,q\to1^-}(\gamma,\varkappa)$, then*

$$|d_2|\le\frac{6\delta|\varkappa|\sqrt{3|\varkappa|}}{\sqrt{|(2\delta(\delta+1)((\tau-\gamma)(2\tau+1)+\gamma^2)+(1-\delta)(2\tau-\gamma)^2)9\varkappa^2-(\delta+1)^2(2\tau-\gamma)^2(18\varkappa^2-1)|}},$$

$$|d_3|\le\frac{6\delta|\varkappa|}{(\delta+1)(3\tau-\gamma)}+\frac{108\delta^2|\varkappa|^3}{|(2\delta(\delta+1)((\tau-\gamma)(2\tau+1)+\gamma^2)+(1-\delta)(2\tau-\gamma)^2)9\varkappa^2-(\delta+1)^2(2\tau-\gamma)^2(18\varkappa^2-1)|},$$

and for $\mu\in\mathbb{R}$

$$|d_3-\mu d_2^2|\le\begin{cases}\frac{6\delta|\varkappa|}{(\delta+1)(3\tau-\gamma)}; & |1-\mu|\le Q_1\\ \frac{108\delta^2|\varkappa|^3\,|1-\mu|}{|(2\delta(\delta+1)((\tau-\gamma)(2\tau+1)+\gamma^2)+(1-\delta)(2\tau-\gamma)^2)9\varkappa^2-(\delta+1)^2(2\tau-\gamma)^2(18\varkappa^2-1)|}; & |1-\mu|\ge Q_1,\end{cases}$$

where

$$Q_1=\left|\frac{(2\delta(\delta+1)((\tau-\gamma)(2\tau+1)+\gamma^2)+(1-\delta)(2\tau-\gamma)^2)9\varkappa^2-(\delta+1)^2(2\tau-\gamma)^2(18\varkappa^2-1)}{18\delta(\delta+1)(3\tau-\gamma)\varkappa^2}\right|.$$

The following are specific instances derived from the preceding corollary.

Instance 3.1. In the case $\delta=1$, the class $\mathfrak{V}^{\tau,1}_{\sigma,p=1,q\to1^-}(\gamma,\varkappa)$ represents the collection of functions $\Phi\in\sigma$ satisfying

$$\frac{\zeta(\Phi'(\zeta))^\tau}{\gamma\Phi(\zeta)+(1-\gamma)\zeta}\prec\mathsf{B}(\varkappa,\zeta),\quad\textit{and}\quad\frac{\varpi(\Psi'(\varpi))^\tau}{\gamma\Psi(\varpi)+(1-\gamma)\varpi}\prec\mathsf{B}(\varkappa,\varpi),$$

where $\tau\ge1$, $0\le\gamma\le1$, and $\varkappa\in(-\frac{1}{3},\frac{1}{3})$.

Instance 3.2. In the special case $\gamma=1$ in Example 3.1, the class $\mathfrak{V}^{\tau,1}_{\sigma,p=1,q\to1^-}(1,\varkappa)$ represents the collection of functions $\Phi\in\sigma$ satisfying

$$\frac{\zeta(\Phi'(\zeta))^\tau}{\Phi(\zeta)}\prec\mathsf{B}(\varkappa,\zeta),\ \textit{and}\ \frac{\varpi(\Psi'(\varpi))^\tau}{\Psi(\varpi)}\prec\mathsf{B}(\varkappa,\varpi).$$

where $\tau \geq 1$ and $\varkappa \in (-\frac{1}{3}, \frac{1}{3})$.

Instance 3.3. Similarly, for $\gamma = 0$ in Instance 3.1, the class $\mathfrak{V}^{\tau,1}_{\sigma,p=1,q\to 1^-}(0,\varkappa)$ consists of functions $\Phi \in \sigma$ satisfying

$$(\Phi'(\zeta))^\tau \prec \mathsf{B}(\varkappa,\zeta), \quad \textit{and} \quad (\Psi'(\varpi))^\tau \prec \mathsf{B}(\varkappa,\varpi).$$

where $\tau \geq 1$ and $\varkappa \in (-\frac{1}{3}, \frac{1}{3})$.

Corollary 3.5. *Let $\tau \geq 1$, $0 \leq \gamma \leq 1$, and $\varkappa \in (-\frac{1}{3}, \frac{1}{3})$. If $\Phi \in \mathfrak{V}^{\tau,1}_{\sigma,p=1,q\to 1^-}(\gamma,\varkappa)$, then*

$$|d_2| \leq \frac{3|\varkappa|\sqrt{3|\varkappa|}}{\sqrt{|((\tau-\gamma)(2\tau+1)+\gamma^2)9\varkappa^2 - (2\tau-\gamma)^2(18\varkappa^2-1)|}},$$

$$|d_3| \leq \frac{3|\varkappa|}{3\tau-\gamma} + \frac{54|\varkappa|^3}{|((\tau-\gamma)(2\tau+1)+\gamma^2)9\varkappa^2 - (2\tau-\gamma)^2(18\varkappa^2-1)|},$$

and for $\mu \in \mathbb{R}$

$$|d_3 - \mu d_2^2| \leq \begin{cases} \frac{3|\varkappa|}{3\tau-\gamma}; & |1-\mu| \leq Q_2 \\ \frac{54|\varkappa|^3\,|1-\mu|}{|((\tau-\gamma)(2\tau+1)+\gamma^2)9\varkappa^2-(2\tau-\gamma)^2(18\varkappa^2-1)|}; & |1-\mu| \geq Q_2, \end{cases}$$

where

$$Q_2 = \left|\frac{((\tau-\gamma)(2\tau+1)+\gamma^2)9\varkappa^2 - (2\tau-\gamma)^2(18\varkappa^2-1)}{(3\tau-\gamma)9\varkappa^2}\right|,$$

Corollary 3.6. *Let $\tau \geq 1$ and $\varkappa \in (-\frac{1}{3}, \frac{1}{3})$. If $\Phi \in \mathfrak{V}^{\tau,1}_{\sigma,p=1,q\to 1^-}(1,\varkappa)$, then*

$$|d_2| \leq \frac{3|\varkappa|\sqrt{3|\varkappa|}}{\sqrt{|\tau(2\tau-1)9\varkappa^2 - (2\tau-1)^2(18\varkappa^2-1)|}},$$

$$|d_3| \leq \frac{3|\varkappa|}{3\tau-1} + \frac{54|\varkappa|^3}{|\tau(2\tau-1)9\varkappa^2 - (2\tau-1)^2(18\varkappa^2-1)|},$$

and for $\mu \in \mathbb{R}$

$$|d_3 - \mu d_2^2| \leq \begin{cases} \frac{3|\varkappa|}{3\tau-1}; & |1-\mu| \leq Q_3 \\ \frac{54|\varkappa|^3\,|1-\mu|}{|\tau(2\tau-1)9\varkappa^2-(2\tau-1)^2(18\varkappa^2-1)|}; & |1-\mu| \geq Q_3, \end{cases}$$

where

$$Q_3 = \left|\frac{\tau(2\tau-1)9\varkappa^2 - (2\tau-1)^2(18\varkappa^2-1)}{(3\tau-1)9\varkappa^2}\right|.$$

Remark 3.7. *i). The special case $\beta = 1$ in Corollary 1* [40] *corresponds with Corollary 3.6, provided the parameter $\varkappa$ lies in the interval $(-\frac{1}{3}, \frac{1}{3})$. ii). Likewise, setting $\tau = 1$ in Corollary 3.6 reproduces Corollary 2 of* [23]*, under the same condition on $\varkappa$.*

Corollary 3.8. *Let $\tau \geq 1$ and $\varkappa \in (-\frac{1}{3}, \frac{1}{3})$. If $\Phi \in \mathfrak{V}^{\tau,1}_{\sigma,p=1,q\to 1^-}(0,\varkappa)$, then*

$$|d_2| \leq \frac{3|\varkappa|\sqrt{3|\varkappa|}}{\sqrt{|(2\tau+1)9\tau\varkappa^2 - 4\tau^2(18\varkappa^2-1)|}},$$

$$|d_3| \leq \frac{|\varkappa|}{\tau} + \frac{54|\varkappa|^3}{|(2\tau+1)9\tau\varkappa^2 - 4\tau^2(18\varkappa^2-1)|},$$

and for $\mu \in \mathbb{R}$

$$|d_3 - \mu d_2^2| \leq \begin{cases} \frac{|\varkappa|}{\tau}; & |1-\mu| \leq Q_4 \\ \frac{54|\varkappa|^3\,|1-\mu|}{|(2\tau+1)9\tau\varkappa^2 - 4\tau^2(18\varkappa^2-1)|}; & |1-\mu| \geq Q_4, \end{cases}$$

where

$$Q_4 = \left|\frac{(2\tau+1)9\tau k^2 - 4\tau^2(18\varkappa^2-1)}{9\tau\varkappa^2}\right|.$$

Remark 3.9. *The special case* $\beta = 1$ *in Corollary 6* [40] *corresponds with Corollary 3.8, provided the parameter* $\varkappa$ *lies in the interval* $(-\frac{1}{3}, \frac{1}{3})$.

Corollary 3.10. *Let* $\tau \geq 1$, $0 \leq \gamma \leq 1$, *and* $\varkappa \in (-\frac{1}{3}, \frac{1}{3})$. *If* $\Phi \in \mathfrak{V}^{\tau,1}_{\sigma,p,q}(\gamma, \varkappa)$, *then*

$$|d_2| \leq \frac{3|\varkappa|\sqrt{3\varkappa}}{\sqrt{|9(A+S)\varkappa^2 - B^2(18\varkappa^2-1)|}},$$

$$|d_3| \leq +\frac{3|\varkappa|}{|A|} + \frac{27|\varkappa|^2}{|9(A+S)\varkappa^2 - B^2(18\varkappa^2-1)|},$$

and for $\mu \in \mathbb{R}$

$$|d_3 - \mu d_2^2| \leq \begin{cases} \frac{3|\varkappa|}{|A|} & ;\ |1-\mu| \leq \left|\frac{9(A+S)\varkappa^2 - B^2(18\varkappa^2-1)}{9A\varkappa^2}\right| \\ \frac{27|\varkappa|^3\,|1-\mu|}{|9(A+S)\varkappa^2 - B^2(18\varkappa^2-1)|} & ;\ |1-\mu| \geq \left|\frac{9(A+S)\varkappa^2 - B^2(18\varkappa^2-1)}{9A\varkappa^2}\right|, \end{cases}$$

where A, S, *and* B *are as mentioned in* (3.5), (3.6), *and* (3.7), *respectively.*

4. Conclusion

Upper bounds on $|d_2|$ and $|d_3|$ for functions in two subfamilies of σ associated with Lucas-Balancing polynomials are established in this study. Furthermore, for functions in these subfamilies, the Fekete-Szegö functional $|d_3 - \mu d_2^2|, \mu \in \mathbb{R}$ has been estimated. A number of implications have been revealed by varying the parameters in Theorem 2.1 and Theorem 3.1. Additionally, pertinent links to the ongoing research are found. This paper's examination of subfamilies may motivate researchers to focus on the (p, q)-operator. Subsequent research endeavors may involve investigating the expansion of acquired outcomes to higher-order Toeplitz determinants or Hankel determinants. The findings presented here demonstrate the importance of facts in the study of geometric function theory and offer a strong foundation for these advancements.

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