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A STUDY ON FOUR-DIMENSIONAL RICCI SOLITONS

BAHAR KIRIK RÁCZ AND BURCU CINDIK

Abstract. The aim of this work is to examine Ricci solitons, which are among the popular research topics of differential geometry nowadays, by focusing on the 4-dimensional case. In this respect, 4-dimensional manifolds admitting different metric signatures are considered by expanding the studies in the literature. Several examples of Ricci solitons are given for 4-dimensional manifolds with Lorentz or neutral metric signatures. In the case of the manifold containing a parallel vector field, its relationship with the potential field is investigated and steady Ricci solitons are obtained under certain conditions. A similar analysis is made for recurrent vector fields. More explicitly, an example of a Ricci soliton whose potential field is recurrent is found and the Segre type of the Ricci tensor is determined. The results in question are interpreted by associating them with concepts such as holonomy theory and other types of vector fields on the relevant manifolds.

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1. Introduction

Ricci solitons have become one of the most remarkable research topics in the field of differential geometry, and their features have been investigated in different structures. In essence, they are self-similar solutions to the Ricci flow equation introduced by R. S. Hamilton [13] and their definition is applicable to semi-Riemannian manifolds that also have physical applications, such as the study of general relativity theory. On this theme, a considerable amount of work has been done in the field; see, for instance, [2–7], [14], [17], and the references therein, and others, while an exhaustive list of references is beyond the scope of this paper. Let M be a semi-Riemannian manifold with a smooth metric g of arbitrary signature. A Ricci soliton is characterized by the following equation

$$\frac{1}{2}\mathscr{L}_\xi g + Ric = \lambda g \tag{1.1}$$

where $\mathscr{L}_\xi$ symbolizes the Lie derivative along ξ referred to as a *potential field*, *Ric* is the Ricci tensor and λ denotes a constant. This structure will be indicated by (M, g, ξ, λ), and it is named as *shrinking*, *steady* and *expanding*, respectively, if $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$.

In this study, we will deal with 4-dimensional Ricci solitons and examine the cases where the metric in such a structure need not be positive definite. More specifically, the situation where g has neutral or Lorentz signature will be taken into account. The fact that 4-dimensional manifolds serve as a bridge between mathematics and physics raises the problem of studying Ricci solitons for different metric signatures. For example, since applications of symmetries, curvature, and tensor fields find effective use in 4-dimensional Lorentzian manifolds called *space-times* (for which g is of signature $(+,+,+,-)$), they are also naturally related to Ricci solitons. On the other hand, it is also significant to understand the geometry of 4-dimensional manifolds having signature $(+,+,-,-)$ (i.e., neutral metric or signature $(2,2)$). In particular, the classification of tensor fields on such manifolds is much more complicated. A comprehensive study of the relevant manifolds is available, for example, in [8] and [9].

This work also consists of investigating the connections of some special vector fields with Ricci solitons in the relevant metric signatures. Among these, parallel and recurrent vector fields are discussed, which have an essential position in the holonomy structure. The rest of the paper is arranged as follows: Section 2 briefly outlines some special vector fields examined in the study and subsequently focuses on certain basic information about 4-dimensional manifolds. In Section 3, the algebraic classification of the Weyl and Ricci tensors is briefly summarized for 4-dimensional manifolds with different metric signatures. The main results of the study, together with some supporting examples, are presented in Section 4.

2. Preliminary information

This section includes a brief overview of some special vector fields taken into account in our work, and then presents essential information about 4-dimensional manifolds with different metric signatures. Vector fields are one of the main objects of differential geometry, arising naturally in the study of Ricci solitons. Among them, torse-forming vector fields, introduced by K. Yano [20], are deeply involved in the study of geometric structure of manifolds, being defined as follows:

Definition 2.1. Consider a smooth n-dimensional manifold M. A vector field φ on M is called *torse-forming* if there exists a smooth function ρ and a 1-form μ such that the relation

$$\nabla_V \varphi = \rho\, V + \mu(V)\varphi, \tag{2.1}$$

holds for all $V \in \chi(M)$ with ∇ denoting the Levi-Civita connection of g, [20].

Well-known subclasses of torse-forming vector fields are also formed depending on certain cases of ρ and μ. One of the most important of these is the *recurrent vector field* that occurs when $\rho = 0$ in equation (2.1). The local coordinate form of such a vector field is expressed as

$$\nabla_i \varphi_j = \mu_i \varphi_j, \tag{2.2}$$

with φ_j and μ_i denoting the components of φ and μ. When $\mu = 0$ in (2.2), the vector field φ is said to be *parallel*. Recurrent and parallel vector fields are fundamental objects that have an important position in the holonomy theory of 4-dimensional manifolds (see, e.g., [8, 9]), and they are particularly relevant to our study. On the other hand, a torse-forming vector field φ is named as *concircular* if μ is a gradient. It can be deduced from (2.1) that if φ is concircular, one gets the property

$$\nabla_i \varphi_j = \rho g_{ij}. \tag{2.3}$$

Additionally, φ is referred to as *convergent* if ρ is constant in (2.3).

Now, consider the case where M is 4-dimensional having a smooth metric g. In this case, g is either positive definite, i.e., it is of signature $(+,+,+,+)$, or has Lorentz signature $(+,+,+,-)$, or has neutral signature $(+,+,-,-)$. One can set up a basis for T_pM (that is, the tangent space at $p \in M$) concerning each signature, see, e.g., [8, 10, 19], and we will adopt these notations accordingly. A non-zero vector $v \in T_pM$ is called *spacelike* if $g(v,v) > 0$, *timelike* if $g(v,v) < 0$ and *null* if $g(v,v) = 0$ at p. A 2-dimensional subspace of T_pM (called a 2-space) is categorized as follows: *spacelike*, if all non-zero elements are of the same (timelike or spacelike) type; *timelike*, if it has exactly two, null directions that are distinct; *null*, if it has exactly one null direction; or *totally null*, if all its non-zero vectors are null and mutually orthogonal.

For the positive definite signature, all 2-dimensional subspaces are spacelike, whereas totally null 2-spaces are only possible in neutral metric. Let Λ_pM denote the space of all bivectors (i.e., 2-forms). If the rank of a non-zero $F \in \Lambda_pM$ is 2 (or 4), F is referred to as *simple* (or *non-simple*). When the rank of F is 2, it can be expressed as $F^{ij} = u^i w^j - w^i u^j$ for $u, w \in T_pM$ where the 2-space generated by u and w is named as the *blade* of F, denoted by $u \wedge w$. More information about the classification of bivectors can be found in [8, 9].

3. Classifications of some tensor fields on 4-dimensional manifolds

When Ricci solitons are investigated on 4-dimensional manifolds with different metric signatures, it is useful to consider the classification of *Ric*. The Weyl conformal curvature tensor, denoted by C, also plays a central role in such manifolds. Understanding the classification of *Ric* and C is crucial when examining space-times, especially within the scope of general relativity theory. This section briefly reviews the classification of the relevant tensor fields on 4-dimensional manifolds with different metric signatures.

A symmetric tensor field of second-order, say $\mathcal{T}$, can be classified algebraically according to the structure of its eigenvalues and eigenspaces, which is known in the literature as the *Jordan-Segre classification*. More clearly, this classification relies on solving the eigenvalue problem $\mathcal{T}^i{}_j u^j = \alpha u^i$ (or equivalently, $\mathcal{T}_{ij} u^j = \alpha g_{ij} u^j$), where $\mathcal{T}^i{}_j$ (or $\mathcal{T}_{ij}$) denotes the components of $\mathcal{T}$, α are the eigenvalues, and u^i are the corresponding eigenvectors. In other words, taking into account real or complex eigenvectors with their eigenvalues, one gets the canonical forms of $\mathcal{T}$. The complete list of Segre types, valid for neutral, Lorentz and positive definite metric signatures respectively, is presented below.

- For $(+,+,-,-)$: $\{1111\}$, $\{11z\bar{z}\}$, $\{z\bar{z}w\bar{w}\}$, $\{211\}$, $\{2z\bar{z}\}$, $\{22\}$ (eigenvalues complex), $\{22\}$ (eigenvalues real), $\{31\}$ and $\{4\}$ (for details, see, e.g., [9, 11]),
- For $(+,+,+,-)$: $\{1,111\}$, $\{211\}$, $\{31\}$ and $\{z\bar{z}11\}$ (where the comma in type $\{1,111\}$ separates the eigenvalue associated with the timelike eigenvector from those related to spacelike ones) (for details, see, e.g., [8, 9]),
- For $(+,+,+,+)$: $\{1111\}$ (i.e., $\mathcal{T}$ is diagonalizable over $\mathbb{R}$) (for details, see, e.g., [8, 9]),

together with their possible degeneracies that may occur for each case. Given that *Ric* is a second-order symmetric tensor, it is classified according to one of the Segre types expressed above and some of these will be identified in the examples presented in Section 4.

Regardless of the metric signature, the Weyl tensor classification is also a fundamental concept in the study of various geometric structures on 4-dimensional manifolds. For space-times, it is famously known as the *Petrov classification* which categorizes the Weyl tensor into six algebraic types denoted by **I**, **II**, **III**, **D**, **N** and **O** (see [8, 18]). On the other hand, a comprehensive classification scheme for the neutral signature was provided in [12], which we adopt in this study. The Weyl tensor can be regarded as a linear map acting on the bivector space $\Lambda_p M$, and in a 4-dimensional manifold equipped with signature $(+,+,-,-)$, one has the following (unique) decomposition:

$$C_{hijk} = \overset{+}{W}_{hijk} + \overset{-}{W}_{hijk}, \tag{3.1}$$

where $\overset{+}{W}$ and $\overset{-}{W}$ are known as the self-dual and anti-self-dual components of C, which are defined by $\overset{+}{W} = \frac{1}{2}(C +^* C) = \overset{+*}{W}$, $\overset{-}{W} = \frac{1}{2}(C -^* C) = -\overset{-*}{W}$, and ${}^*C = C^*$ with ${}^*C$ and C^* denoting the left and right duals of C, respectively. The classification of C at $p \in M$ can then be achieved by analyzing the types of $\overset{+}{W}$ and $\overset{-}{W}$ individually. The canonical representations for $\overset{+}{W}$ and $\overset{-}{W}$ in this case are labelled as **I**, **II**, **III**, $\mathbf{D_1}$, $\mathbf{D_2}$, **N** and **O**, [12]. The canonical form of C then arises as a consequence of (3.1) such that its algebraic type at $p \in M$ can be written as label tuples $(\mathbf{X}, \mathbf{Y})$, where $\mathbf{X}$ (respectively, $\mathbf{Y}$) denotes the type of $\overset{+}{W}$ (respectively, $\overset{-}{W}$) (for details, see [12]).

Finally, when g is positive definite, C splits into self-dual and anti-self-dual parts as in the neutral case given by (3.1). But due to diagonalizability, its classification simplifies considerably, where $\overset{+}{W}$ (and $\overset{-}{W}$) takes one of the types **I**, **D** and **O** (see [1] for an example; additional references include [9, 15]).

4. Results on 4-dimensional Ricci solitons and Examples

In this section, the results of the study are presented and 4-dimensional Ricci soliton examples supporting them are constructed. Within this framework, we will concentrate on specific classes of torse-forming vector fields presented in Section 2 and examine their properties on Ricci solitons. Equation (1.1) can be locally expressed as

$$\frac{1}{2}(\nabla_i \xi_j + \nabla_j \xi_i) + R_{ij} = \lambda g_{ij}. \tag{4.1}$$

Firstly, suppose that a Ricci soliton (M, g, ξ, λ) contains a nowhere-zero parallel vector field η orthogonal to potential field ξ. In this case, $\nabla \eta = 0$, and from the Ricci identity, one has $R_{hijk}\eta^k = 0$, where R_{hijk} are components of the Riemann tensor of type $(0,4)$. This implies that $R_{ik}\eta^k = 0$. On the other hand, taking the covariant derivative of the orthogonality condition between η and ξ yields $\eta^i \nabla_j \xi_i = 0$. In that case, by contracting equation (4.1) with η^i and subsequently with η^j, we obtain

$$\lambda \eta_j \eta^j = 0. \tag{4.2}$$

If g is positive definite, then η must be non-null, which implies that λ must be zero from equation (4.2). Thus, a steady Ricci soliton is obtained. On the other hand, if the metric signature is neutral or Lorentzian, η can be null or non-null. However, if it is non-null, then λ must still be zero, and (M, g, ξ, λ) is a steady Ricci soliton. If η is null, λ may or may not be zero, and thus equation (4.2) is automatically satisfied. As a result, the subsequent theorem is established:

Theorem 4.1. *Let (M, g, ξ, λ) be a 4-dimensional Ricci soliton and assume that it admits a nowhere-zero parallel vector field η which is orthogonal to the potential field ξ. Accordingly, the following are valid:*

(i) *If g is of signature $(+,+,+,+)$, then such a Ricci soliton is steady.*

(ii) *If g is of signature $(+,+,+,-)$ or $(+,+,-,-)$, then either such a Ricci soliton is steady, or η is null, or both.*

Remark 4.2. Since the focus of this paper is on the 4-dimensional cases, Theorem 4.1 is stated accordingly, but note that its proof does not depend on the dimension.

Example 4.3. Let us consider $M = \mathbb{R}^4$ with coordinates (u, v, y, s) and the following metric:

$$u^2y^2du^2 + 2dudv + dy^2 + \epsilon ds^2 \tag{4.3}$$

where $u \neq 0$ and $y \neq 0$. In the case where $\epsilon = -1$ in (4.3), g has neutral signature. Let ξ be chosen as the vector field $\frac{u^3}{3}\frac{\partial}{\partial v}$. Then, ξ is null and recurrent since the conditions $g(\xi, \xi) = 0$ and $\nabla\xi = \xi \otimes \mu$ are satisfied where $\mu = \frac{3}{u}du$ is the recurrence 1-form. On the other hand, the Ricci tensor is calculated as $Ric = -u^2dudu = -u^2 l \otimes l$ where $l := \frac{\partial}{\partial v}$ and its Segre type is $\{(211)\}$ with eigenvalue zero. Moreover, one calculates

$$\mathscr{L}_\xi g = 2u^2dudu = 2u^2 l \otimes l. \tag{4.4}$$

With the help of (4.1) and (4.4), one gets

$$\frac{1}{2}\mathscr{L}_\xi g + Ric = u^2dudu - u^2dudu = 0,$$

which shows that (M, g, ξ, λ) is a steady Ricci soliton with the potential field ξ being null and recurrent. Due to the splitting form of (4.3), a timelike parallel vector field $\frac{\partial}{\partial s}$ is admitted, which is also orthogonal to ξ. Therefore, Theorem 4.1(ii) holds for (M, g, ξ, λ), where $\eta := \frac{\partial}{\partial s}$ is non-null (parallel) and $\lambda = 0$. Besides, $\frac{\partial}{\partial y}$ is also a spacelike vector field which is unit and orthogonal to ξ. Additionally, the Weyl type is $(\mathbf{N}, \mathbf{N})$. For $\epsilon = 1$ in (4.3), g is of Lorentz signature. Thus, (M, g, ξ, λ) is a space-time, similar outcomes as in the neutral case are achieved (but for this case, $\frac{\partial}{\partial s}$ is spacelike, parallel and orthogonal to ξ) and the Petrov type is $\mathbf{N}$.

Remark 4.4. Note that if the potential field ξ is recurrent, the equation (4.1) gives

$$R_{ij} = \lambda g_{ij} - \frac{1}{2}\left(\xi_i\mu_j + \mu_i\xi_j\right) \tag{4.5}$$

for some 1-form μ. If the Ricci tensor takes the form given in equation (4.5), (M, g) is known in the literature as a special kind of *quasi-Einstein manifold*, where ξ is typically assumed to be (unit and) non-null, and since it is usually considered on manifolds with positive definite metric signature, *Ric* is diagonalizable. However, one should be careful when the signature of the metric changes as the Ricci tensor may not be diagonalizable. Example 4.3 demonstrates a steady Ricci soliton whose potential vector field is null and recurrent but *Ric* is not diagonalizable having Segre type $\{(211)\}$ with eigenvalue zero. An additional example is given below:

Example 4.5. With a global coordinate system denoted by (u, v, y, s), consider $M = \mathbb{R}^4$ and the following metric:

$$(u^2 + s^2)du^2 + 2dudv + dy^2 + \epsilon ds^2 \tag{4.6}$$

where $u \neq 0$. For $\epsilon = -1$, the metric (4.6) has neutral signature. Then, the Ricci tensor is computed as $Ric = dudu = l \otimes l$, where $l := \frac{\partial}{\partial v}$ and it is of Segre type $\{(211)\}$ (zero eigenvalue). Moreover, $C_{hijk}l^k = 0$, and C is of type $(\mathbf{N}, \mathbf{N})$. Let $\xi := -u\frac{\partial}{\partial v}$. Then, ξ is a (null) recurrent vector field satisfying $\nabla \xi = \xi \otimes \mu$, where $\mu = \frac{1}{u}du$ is the recurrence 1-form. On the other hand, one gets

$$\mathscr{L}_\xi g = -2dudu = -2\, l \otimes l$$

and hence

$$\frac{1}{2}\mathscr{L}_\xi g + Ric = -dudu + dudu = 0. \tag{4.7}$$

Equation (4.7) reveals that (M, g, ξ, λ) is a steady Ricci soliton with the potential field ξ which is null and recurrent. This metric also admits a spacelike parallel vector field $\frac{\partial}{\partial y}$ orthogonal to ξ. Consequently, Theorem 4.1(ii) is also satisfied. It should be noted that for $\epsilon = 1$, the metric (4.6) is Lorentzian. The Ricci tensor, the potential field ξ, and $\mathscr{L}_\xi g$ differ from the previously obtained expressions only by a sign, and (4.7) holds. In addition, the Petrov type is $\mathbf{N}$.

Remark 4.6. It is possible to make further comments regarding Example 4.5. The analysis can also be interpreted in the context of the holonomy theory of 4-dimensional manifolds (for details, see, e.g., [8–10, 19]). Because for this example, l is a parallel, null vector field being an eigenvector of Ric. Moreover, $\nabla Ric = 0$ having type $\{(211)\}$. As the metric (4.6) also admits a non-null parallel vector field $\frac{\partial}{\partial y}$, the holonomy type is $1(c)$ (when g is of signature $(+, +, -, -)$, i.e., $\epsilon = -1$ in (4.6)) or R_3 (when g is of signature $(+, +, +, -)$, i.e., $\epsilon = 1$ in (4.6)). We refer to [8] and [9] for the representation of holonomy types and all other details.

According to the results reached above, it is worth examining whether similar situations hold for other special vector fields, such as convergent ones. Now suppose (M, g, ξ, λ) admits a nowhere-zero convergent vector field, say φ, orthogonal to ξ. It follows that φ satisfies the equation (2.3) for some constant ρ. Following a similar analysis as before, the Ricci identity yields $R_{hijk}\varphi^k = 0$, and so $R_{ik}\varphi^k = 0$. By taking the covariant derivative of the orthogonality condition between φ and ξ, one gets from (2.3) that

$$\varphi^i \nabla_j \xi_i + \rho \xi_j = 0. \tag{4.8}$$

Contracting (4.8) over φ^j, we get $\varphi^i\varphi^j\nabla_j\xi_i = 0$. In that case, multiplying equation (4.1) by $\varphi^i\varphi^j$, we find

$$\lambda \varphi_j \varphi^j = 0,$$

leading to conclusions analogous to those of equation (4.2). In addition, the case when φ is null, then it is known to reduce to a parallel vector field (see Theorem 3.3 (iii) in [16]). Hence, the following outcome is immediate.

Corollary 4.7. *Consider a 4-dimensional Ricci soliton (M, g, ξ, λ) and assume that it admits a nowhere-zero convergent vector field φ which is orthogonal to the potential field ξ. In this case, the results of Theorem 4.1 remain valid. Additionally, in the event that φ is null, it turns out to be parallel.*

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Department of Mathematics, Faculty of Science, Marmara University, Göztepe Campus 34722, İstanbul, Türkiye

Email address: bahar.kirik@marmara.edu.tr

Department of Mathematics, Institute of Pure and Applied Sciences, Marmara University, Göztepe Campus 34722, İstanbul, Türkiye

Email address: burcucindik@marun.edu.tr