Romanian Journal of Mathematics and Computer Science
available online at https://rjm-cs.utcb.ro
Issue 2, Vol. 15 (2025)

Proceedings of the International Conference Riemannian Geometry and Applications - RIGA 2025
Bucharest, Romania, May 23-25, 2025

INEQUALITY AND DIVERSITY: INSIGHTS FROM BIOLOGY

JOHAN GIELIS AND KARL J. NIKLAS

Abstract. Inequality is a significant topic in economics and human society, with parallels in biology at the levels of individuals, organizations, and ecosystems. It seems, however, that the mathematics underlying these phenomena is very similar, and a new field of research, Econobiology, evolves.

Mathematics Subject Classification (2020): 92-10, 92F05, 53Z10.
Key words: Lorenz curves, Gini coefficient, inequality, economics, diversity, trigonometry.

Article history:
Received: August 22, 2025
Received in revised form: September 9, 2025
Accepted: September 10, 2025

1. Lamé curves in biology and economy

Gabriel Lamé (1795-1860) introduced the curves named after him in order to apply Descartes' beautiful geometry to the study of crystals [31]. The set of Lamé curves is very general, but of particular interest are the closed curves, also known as supercircles and superellipses. In recent years, these superellipses have been used in a number of studies to examine natural forms, and these curves can be found everywhere, from the stomata of plant leaves [33], to the shape of leafs and fruits [34, 52, 66], the growth rings of trees [51, 26, 27], and the shape of the culms of square bamboos [25]. Lamé's fingerprint is everywhere, not just in plants.

In these studies, the polar form of superellipses was used, or rather a special case of a generalization, known as Gielis formula [16, 17, 59]. This geometric transformation has been applied to the shapes of starfish [53], diatoms [14, 15], and flowers [64, 63, 57], and has been widely used in technology, among others in antennas and communication [4, 67, 68, 32], in nanophotonics [46, 45, 47], laser technology [10], and many other fields [21, 20, 3]. This generic geometric transformation can be applied in almost any direction, including in geometry [30, 35, 58] and applied mathematics [36, 7, 6]. Recently, the Gielis formula inspired the generalization of inner and vector products, and of quaternions [40, 41].

Superellipses are also widely used in economics, especially in production functions, which have recently been studied by geometers [60, 61, 8]. The Constant Elasticity of Substitution production function which describes a relationship in which the ability to substitute one input for another (or one good for another) remains constant, regardless of the level of output or utility, is defined as:

$$Y = A\left[\alpha K^{\rho} + (1-\alpha)L^{\rho}\right]^{1/\rho} \tag{1.1}$$

where:

- Y is output,
- K is capital,
- L is labor,
- A is a productivity parameter,
- α is a distribution parameter (weighting capital vs. labor),
- ρ is a parameter related to the elasticity of substitution.

In the original publication [1], Arrow and co-authors showed that the Cobb-Douglas case occurs as a limiting form when capital and labor can be perfectly and proportionally substituted. When $\rho \to 0$, the function reduces to the Cobb-Douglas production function:

$$Y = AK^{\alpha}L^{1-\alpha} \tag{1.2}$$

In [1] three ways of proving this are shown, the most interesting of which is the one based on power means. In their book (*Inequalities*) [22] Hardy, Littlewood, and Polya show that as the order of the power mean converges to zero, it converges to the geometric mean. The theorem states that for positive numbers $x_1, x_2, \ldots, x_n$, the power mean of order r is defined as:

$$M_r(x_1, x_2, \ldots, x_n) = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^r\right)^{1/r} \tag{1.3}$$

As r approaches zero, M_r converges to the geometric mean:

$$\lim_{r\to 0} M_r(x_1, x_2, \ldots, x_n) = \left(\prod_{i=1}^{n} x_i\right)^{1/n} \tag{1.4}$$

A special case is $r = 1$, when the power mean reduces to the arithmetic mean, and the geometric mean. This coincides with our studies of superellipses and superparabolas, which result either from the addition, or from comparisons of x^n and y^m, ($y^n = x^m$, or $y = x^{m/n}$ with the parabola $x = y^2$). Power laws, which are omnipresent in nature and the sciences [5, 65, 37, 38], are superparabolas. According to the rules of arithmetic, x^n and y^n are the outermost entries in Pascal's triangle. These entries of type $x^n y^0$ or $x^0 y^n$ are the most unlikely outcomes that can occur when tossing a coin. But what is most unlikely is preferred by nature (Figure 1)[18].

2. Modeling inequality in economics

Inequality in the economy and in human society, manifests itself through disparities in income, in the distribution of resources and wealth, and in social structures [42, 39, 19]. The mathematical studies of inequality go back to Pareto, but especially since [42], the focus has been on the top 1 or the top 0.1 percent of income earners. However, it should be noted that this is a dynamic phenomenon. Students tend to have low incomes, but these rise over the course of their careers in industry or academia. At the end of their careers, they are usually in the top

Variables	Planar curves	Types	Special means
$x^n + y^n$	Supercircles & Superellipses	Lamé curves	Arithmetic mean $\frac{x+y}{2}$ (for $n = 1$)
$x^n \cdot y^m$	Superparabolas & Superhyperbolas	Power laws	Square of geometric mean $\sqrt{x \cdot y}$ (for $n = m = 1$)

Figure 1. Combining the most improbable of outcomes

deciles of income, but on retirement most fall back into the lower deciles. This dynamic is well documented [56, 2, 24].

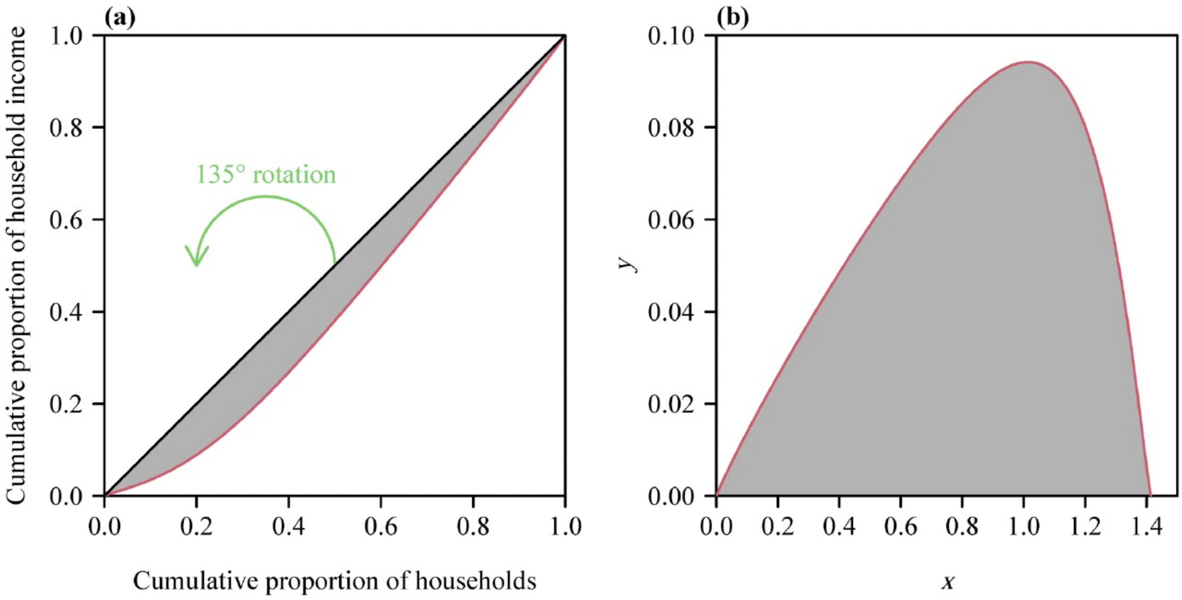

Figure 2. Left: Lorenz curve with cumulative proportion of households versus cumulative proportion of household incomes. Right: a 135° rotated and right shifted Lorenz curve [28]

The Lorenz curve graphically represents the cumulative distribution of income (Figure 2,[28]), with the Gini coefficient quantifying the area between the curve and the line of perfect equality. There are various ways to represent the Lorenz curve mathematically. *Modeling Income Distributions and Lorenz Curves* [9] provides a good overview of the various methods. Figure 3 shows examples of equality, Pareto and exponential functions, and Figure 4 shows a systematic extension of Pareto-type functions.

The Pareto type models and variations are related directly to superellipses [48]. The Lamé superellipse was used successfully as a one-parameter model for the study of inequality [23]

$$L(p) = \left(1 - (1-p)^k\right)^{1/k} \tag{2.1}$$

Distribution	C.D.F	Lorenz curve
Equal	$F(x) = \begin{cases} 0, x < \mu \\ 1, x \geq \mu \end{cases}$	$L(p) = p$
Pareto	$F(x) = 1 - (a/x)^a, x > a, a > 1$	$1 - (1-p)^{(a-1)/a}$
Exponential	$F(x) = 1 - e^{-\lambda x}, x > 0$	$p + (1-p)ln(1-p)$

Figure 3. Cumulative Density Functions and corresponding Lorenz curves

Using data from 89 countries sourced from the Luxembourg Income Study, this one-parameter model fit observed income distributions efficiently [23].

Lorenz curve	Gini
$L_0(p) = L_0(p;k) = 1 - (1-p)^k$	$G = \frac{1-k}{1+k}$
$L_1(p;k;\alpha) = p^\alpha[1 - (1-p)^k]$	$G = 1 - 2[B(\alpha+1,1) - B(\alpha+1,k+1)]$
$L_2(p;k;\gamma) = [1 - (1-p)^k]^\gamma$	$G = 1 - \frac{2}{k}\left[B\left(\frac{1}{k},\gamma+1\right)\right]$
$L_3(p;k;\alpha;\gamma) = p^\alpha[1 - (1-p)^k]^\gamma$	$G = 1 - 2\sum_{i=0}^{\infty} \frac{\Gamma(i-\gamma)}{\Gamma(i+1)\Gamma(-\gamma)} B(\alpha+1, ki+1)$

Figure 4. Variations on Lorenz curves of Pareto type and corresponding Gini Indices

3. Performance Equations in Biology

When considering Lorenz curves and Gini coefficients to model inequality in biology, Dr. Peijian Shi found that the curves that result when a Lorenz curve is rotated 135° and shifted to the right, with the starting point 0 and the end point $\sqrt{2}$ (Figure 2, right), are very similar to the curves that represent the performance equations in biology [19]. We refer to this transformation as Shi rotation [19]. Such performance curves are very common in biology. In the original paper, the performance of frogs was studied ("how far they jump as a function of temperature" [29]), and they are also found in the development rates of insects [54, 55, 50]. They are typically modeled as product exponentials:

$$c(1 - e^{-K_1(x-x_1)})(1 - e^{K_2(x-x_2)}), \quad (3.1)$$

with $x_1 = 0$ and $x_2 = \sqrt{2}$ as start and endpoints, respectively. This model and some variations, for example, the generalized performance equation

$$c(1 - e^{-K_1(x-x_1)})^a(1 - e^{K_2(x-x_2)})^b \tag{3.2}$$

were tested on bamboo leaves and melon fruits [62], and on the diversity of tree size in forests [69]. In all cases, the Shi-rotated Lorenz curve agrees well with the biological data based on dynamic processes (Figure 5). The Gini indices were 0.2379 for fruits, 0.1208 for Shibataea leaves, and 0.2684 for the size of trees in a forest. The values for leaves and tree size are lower than the lowest Gini indices for inequality in Western European countries. Gini indices for Europe, South-Asia, the Middle East, Northern Africa, USA and Canada range from 0.28 (Belgium) to 0.39 (US). The Gini indices for Sub-Saharan Africa and South America are higher than 0.4 [39, 19].

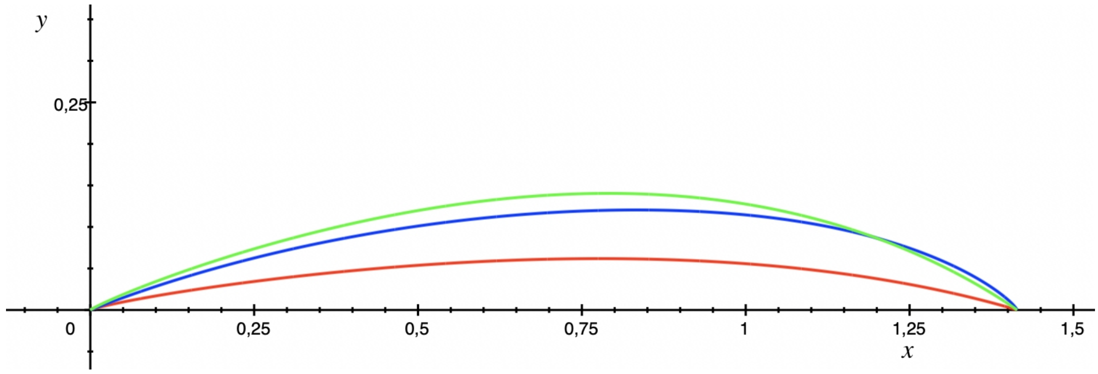

Figure 5. Three Lorenz-Shi curves with Equation (3.2) of the size of melon fruits (Blue) $c = 0.2033, K_1 = 1.5335, a = 0.9958, K_2 = 2.6217, b = 0.8053$; Shibataea leaves (Red) $c = 8.8721, K_1 = 0.0077, a = 0.8348, K_2 = 1.0619, b = 0.9716$; Tree size in forests (Green) $c = 2.4476, K_1 = 0.1203, a = 0.9044, K_2 = 1.0915, b = 0.9739$

When comparing various other models for Lorenz curves, GPE performed better than three other Lorenz equations in terms of goodness of fit, and the trade-off between goodness of fit and structural complexity of the model [49]. GPE is a powerful tool for quantifying size distributions over a wide range of organic units and can be used in a variety of ecological and evolutionary applications. Even for the simulation data of hypothetical, extremely skewed distribution curves, GPE still performed well [49]. The Gini coefficient, a measure of *inequality* both in economy and biology (resulting from our studies), is strongly correlated with other measures of *diversity* in biology, suggesting a mathematical unity across domains. Three other measures are the coefficient of variation CV (3.3), the Theil index T (3.4) and generalized entropy index $GPE(\alpha)$ (3.5):

$$CV = \frac{\sigma}{\mu} \tag{3.3}$$

$$T = \frac{1}{n}\sum_{i=1}^{n}\frac{A_i}{\mu}\ln\left(\frac{A_i}{\mu}\right) \tag{3.4}$$

$$GEI\,(\alpha) = \begin{cases} \frac{1}{n}\frac{1}{\alpha(\alpha-1)}\sum_{i=1}^{n}\left[\left(\frac{A_i}{\mu}\right)^n - 1\right] & \alpha \neq 0,1 \\ \frac{1}{n}\sum_{i=1}^{n}\frac{A_i}{\mu}\log\left(\frac{A_i}{\mu}\right) & \alpha = 1 \\ -\frac{1}{n}\sum_{i=1}^{n}\log\left(\frac{A_i}{\mu}\right) & \alpha = 0 \end{cases} \tag{3.5}$$

- The Gini index is proportional to the coefficient of variation (CV).
- Theil index is proportional to the square of the CV and to the square of the Gini index
- The Theil index is the generalized entropy index with $\alpha = 1$.
- Half of the CV^2 is the generalized entropy index with $\alpha = 2$.

4. Geometric relationships

4.1. **Areas and Optimization.** Overall, Lamé curves and product exponentials establish close links between inequalities in plants and nature, and phenomena in the economy. This will provide new and powerful methods to gain insights into how nature and societies are structured and function - a new era of Eco(no)biology. Comparing the Gini coefficients of size distribution in plants with income distribution in countries, one is tempted to conclude that plants are more successful than humans in size distribution and resource allocation. The class of Generalized Conic Sections which include supercircles and superellipses, as well as superparabolas and superhyperbolas, is pure geometry. In the same way that classic conic sections are related to areas (for example the Pythagorean Theorem, Kepler' law of areas, and the parabola as a machine turning rectangles into squares with the same area), for each of the Generalized Conic Sections, specific trigonometric functions and Pythagorean conservation laws can be developed. For supercircles, Pythagorean identies have already been derived [44], and in particular the trigonometry on the diamond and the related Fourier expansions [43]. But the recent results in [40, 41] also open the door for generalizing the notion of orthogonality tailored for generalized conic sections. Such generalizations will be possible for trigonometric functions on the hyperbola and the parabola as well. In the next section an overview of the basic functions is given. In further sections two examples are given, one on the relationship between product exponentials and hyperbolic functions (section 4.3), and one on generalized logistic functions (section 4.4).

4.2. **Trigonometric functions and Area.**

4.2.1. ***Circular functions.***

- $\cos\theta, \sin\theta$ satisfy the Pythagorean identity: $\sin^2\theta + \cos^2\theta = 1$
- An angle θ (in radians) defines a sector with area $A = \frac{1}{2}r^2\theta = \frac{1}{2}\theta$
- Thus, $\theta = 2A$, where A is the sector's area.

4.2.2. ***Hyperbolic functions.***

- $\cosh u, \sinh u$ satisfy the identity: $\cosh^2 u - \sinh^2 u = 1$.
- The parameter u (hyperbolic angle) corresponds to a sector from $(1,0)$ to $(\cosh u, \sinh u)$, with area $A = \frac{u}{2}$
- Thus, $u = 2A$.

4.2.3. ***Parabolic functions*** [13, 11].

- ${}_pc(\phi), {}_ps(\phi)$ satisfy the Parabolic-Pythagorean Identity: ${}_pc(\phi)^2 + {}_ps(\phi) = 1$.
- The parameter ϕ is defined as twice the area of the parabolic sector formed by:
  - The x-axis from $(1,0)$ to $({}_pc(\phi), 0)$,
  - The ray from the origin to $({}_pc(\phi), {}_ps(\phi))$,

  - The parabolic arc from $(1, 0)$ to $({}_pc(\phi), {}_ps(\phi))$.
- The area A is given by: $\frac{1}{2}{}_pc(\phi) \cdot {}_ps(\phi) + \int_{{}_pc(\phi)}^{1} (1 - \xi^2)\, d\xi = \frac{1}{2}\phi$.
- Thus, $\phi = 2A$.

4.3. **Product exponentials and hyperbolic functions.** Exponential functions are directly related to conic sections, with e defined on the hyperbola for $area = 1$. In the simplest form, the product exponentials from biology have K_1 and K_2 equal to 1. Figure 6 shows curves for $(1 - e^{-x})$ (red) and $((1 - e^{x})$ (blue) and two product exponentials (black).

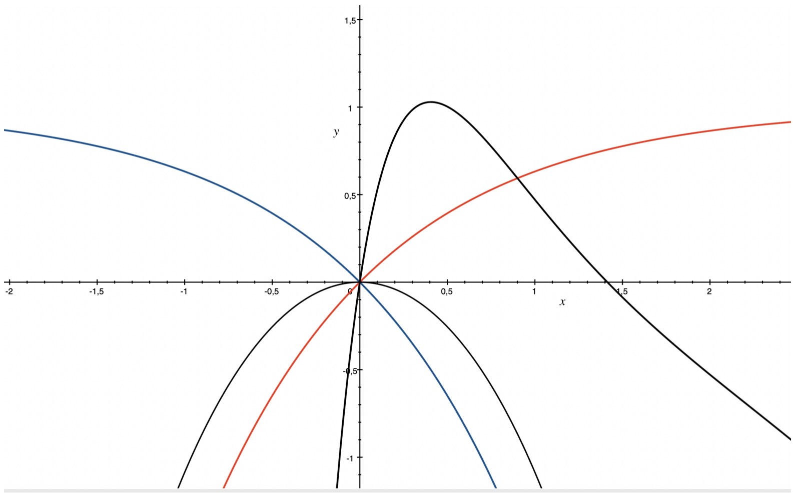

Figure 6. Graphs for $(1 - e^{-x})$ (red), $(1 - e^{+x})$(blue), and two product exponentials (black): One is $-4\sinh^2\left(\frac{x}{2}\right)$. The other product exponential is $y = 0.4(1 - e^{+0.5x})(1 - e^{-2,5(x-1.414)})$

One of the product exponential curves is actually directly related to hyperbolic functions, in particular $\sinh(x/2)$. Indeed:

$$(1 - e^{-x})(1 - e^{x}) = -4\sinh^2\left(\frac{x}{2}\right) \tag{4.1}$$

Further, for $((1 - e^{-x})(1 - e^{x}))^n$, the result is

$$-4^n \sinh^{2n}\left(\frac{x}{2}\right) \tag{4.2}$$

4.4. **Generalized Logistic Functions.** Performance equations are products of exponential functions of the type

$$(1 - e^{-x}) \tag{4.3}$$

which are directly related to the logistic equations or logistic functions (LF). The LF belongs to the family of sigmoid curves, which are ubiquitous in all areas of science. They play a central role in the study of all non-linear evolutionary processes that exhibit growth and saturation and cover many phenomena in physics, biology, ecology, economics and population growth. Different forms of logistic equations can be embedded in a single framework [12]. The logistic function (LF) is defined as:

$$F(x,r|K) = f_0 \frac{e^{rx}}{1+\frac{f_0}{K}(e^{rx}-1)} \tag{4.4}$$

This describes a growth process, counteracted by a reduction of the gain.

$$\hat{F}(x,r|K) = \frac{F}{K} = \frac{1}{1+\mu e^{-rx}}, \text{ with } \mu = -\alpha K. \tag{4.5}$$

For $\mu = -1$, we obtain the form used in the performance equations.

$$\frac{K}{F} = \frac{1}{\hat{F}} = (1-e^{-rx}) \tag{4.6}$$

A generalized logistic function LF can include other functions $P(x)$, which is called the characteristic of the LF:

$$Z = \frac{1}{1+\mu P(x)} \tag{4.7}$$

In this way it is possible to derive the associated differential equations. In [12] the example of $P(x) = \cosh(x)$ is discussed.

5. Conclusion

Tree rings build on last year's rings, integrating all the influences of the current year, and preparing for the next year, all in a two-parameter model. Similarly, supercircles are a one-parameter model for inequality in the economy. Lorenz curves turn out to be closely related to performance equations in biology. These findings are good evidence that very simple relationships can be found in nature and in the human economy, based on areas of actual shapes and form, or of curves. The Gini and related indices are simply different but related methods to measure inequality and diversity, which seem to be two sides of the same coin. Optimization in nature is closely linked to the area, and this provides ample opportunities for exploration of new trigonometric and Pythagorean Identities for each shape. The Gielis Formula provides a gateway as a continuous transformations between shape and optimization.

References

[1] K.J. Arrow, H.B. Chenery, B.S. Minhas and R.M. Solow, *Capital-labor substitution and economic efficiency*, Review of Economics and Statistics **43** (1961), 225–50, 196.

[2] G. Auten, G. Gee and N. Turner, *Income inequality, mobility, and turnover at the top in the US, 1987-2010*, American Economic Review **103(3)** (2013), 168–72.

[3] B. Beirinckx, *From superellipses to superformula and technology*, in Proceedings of the 1st International Symposium on Square Bamboos and the Geometree (ISSBG 2022), Athena Publishing, 2023, 123-142.

[4] P. Bia, D. Caratelli, L. Mescia and J. Gielis, *Electromagnetic characterization of supershaped lens antennas for high-frequency applications*, in Proceedings of the 2013 European Microwave Conference, Nuremberg, Germany, IEEE, 2013, 1679-1682.

[5] J.H. Brown, V.K. Gupta, B.L. Li, B.T. Milne, C. Restrepo and G.B. West, *The fractal nature of nature: power laws, ecological complexity and biodiversity*, Philosophical Transactions of the Royal Society of London. Series B: Biological Sciences **357(1421)** (2002), 619-626.

[6] D. Caratelli, J. Gielis, I. Tavkhelidze and P.E. Ricci, *The Dirichlet problem for the Laplace equation in supershaped annuli*, Boundary Value Problems **2013(1)** (2013), 113.

[7] D. Caratelli, P. Natalini and P.E. Ricci, *Fourier solution of the wave equation for a star-like-shaped vibrating membrane*, Computers & Mathematics with Applications **59(1)** (2010), 176-184.

[8] B.Y. Chen and G.E. Vîlcu, *Geometric classifications of homogeneous production functions*, Applied Mathematics and Computation **225** (2013), 345-351.
[9] D. Chotikapanich (Ed.), *Modeling Income Distributions and Lorenz Curves*, Springer, 2008.
[10] C.A. Codemard, A. Malinowski and M.N. Zervas, *Numerical optimisation of pump absorption in doped double-clad fiber with transverse and longitudinal perturbation*, in Proc. SPIE **10083**, Fiber Lasers XIV: Technology and Systems, 1008315 (2017), 208-214.
[11] G. Dattoli, E. Di Palma, J. Gielis and S. Licciardi, *Parabolic trigonometry*, International Journal of Applied and Computational Mathematics **6** (2020), 37.
[12] G. Dattoli and R. Garra, *A note on differential equations of logistic type*, Reports on Mathematical Physics **93(3)** (2024), 301-312.
[13] G. Dattoli, M. Migliorati, M. Quattromini and P.E. Ricci, *The parabolic-trigonometric functions*, preprint, arXiv:1105.4718, 2011.
[14] E. De Tommasi and A. Rogato, *The diatom frustule: morphogenesis and role in light manipulation*, Growth and Form **4(1-2)** (2023), 3-10.
[15] E. De Tommasi, A. Rogato, D. Caratelli, L. Mescia and J. Gielis, *Following the photons route: Mathematical models describing the interaction of diatoms with light*, in J.L. Pappas (Ed.), *The Mathematical Biology of Diatoms*, Scrivener Publishing, 2023, 343-391.
[16] J. Gielis, *A generic geometric transformation that unifies a wide range of natural and abstract shapes*, American Journal of Botany **90(3)** (2003), 333-338.
[17] J. Gielis, *The Geometrical Beauty of Plants*, Atlantis Press, Paris, 2017.
[18] J. Gielis, *Conquering mount improbable*, in Proceedings of the 1st International Symposium on Square Bamboos and the Geometree (ISSBG 2022), Athena Publishing, 2023, 153-173.
[19] J. Gielis, *Lorenz curves, performance equations and Shi rotations*, in Proceedings of the 2nd International Symposium on Square Bamboos and the Geometree (ISSBG 2023), Geniaal Press, 2024, 67-84.
[20] J. Gielis, D. Caratelli and I. Tavkelidze, *Rational science, unique and simple*, in Proceedings of the 2nd International Symposium on Square Bamboos and the Geometree (ISSBG 2023), Geniaal Press, 2024, 185-205.
[21] J. Gielis, P. Shi, B. Beirinckx, D. Caratelli and P.E. Ricci, *Lamé-Gielis curves in biology and geometry*, in Proceedings of the International Conference Riemannian Geometry and Applications (RIGA 2021), Bucharest, Romania (2021).
[22] G.H. Hardy, J.E. Littelwoord and G. Polya, *Inequalities*, The Macmillan Company, New York, 1934.
[23] J.M. Henle, N.J. Horton and S.J. Jakus, *Modeling inequality with a single parameter*, in D. Chotikapanich (Ed.) *Modeling Income Distributions and Lorenz Curves*, Springer, New York, 2008, 255–269.
[24] T.A. Hirschl and M.R. Rank, *The life course dynamics of affluence*, PLoS One **10(1)** (2015), e0116370.
[25] W. Huang, Y. Li, K.J. Niklas, J. Gielis, Y. Ding, L. Cao and P. Shi, *A superellipse with deformation and its application in describing the cross-sectional shapes of a square bamboo*, Symmetry **12(12)** (2020), 2073.
[26] W. Huang, K. Ma and D.K. Gladish, *Ellipse or superellipse for tree-ring geometries? Evidence from six conifer species*, Trees **38(6)** (2024), 1403-1413.
[27] W. Huang, K. Ma, J. Tan, M. Wei and Y. Lu, *Superellipse equation describing the geometries of Abies alba tree rings*, Plants **13(24)** (2024), 3487.
[28] L.C. Huang, D.A. Ratkowsky, C. Hui, J. Gielis, M. Lian, W.H. Yao, Q.Y. Li, L.Y. Zhang and P.J. Shi, *Inequality measure of leaf area distribution for a drought-tolerant landscape plant*, Plants **12** (2023), 3143.
[29] R.B. Huey, *Ecology of lizard thermoregulation*, Doctoral dissertation, Harvard University, 1975.
[30] M. Koiso and B. Palmer, *Equilibria for anisotropic surface energies and the Gielis formula*, Forma (Japanese Society on Form) **23** (2008), 1-8.
[31] G. Lamé, *Examen des Différentes Méthodes Employées pour Résoudre les Problèmes de Géométrie. Éd. 1818*, Hachette Bnf, 2021.
[32] K.W. Leung, C.T. Huang, Z. Fang, K. Lu and N. Yang, *Wideband circularly polarized horn antenna with twisted ridges of Gielis super-formula contour*, IEEE Transactions on Antennas and Propagation **73(2)** (2024), 1197-1202.
[33] Q. Li, K.J. Niklas, Ü. Niinemets, L. Zhang, K. Yu, J. Gielis, J. Gao and P. Shi, *Stomatal shape described by a superellipse in four Magnoliaceae species*, Botany Letters **171(1)** (2024), 93-101.

[34] Y. Li, K.J. Niklas, J. Gielis, Ü. Niinemets, J. Schrader, R. Wang and P. Shi, *An elliptical blade is not a true ellipse, but a superellipse – Evidence from two Michelia species*, Journal of Forestry Research **33(4)** (2022), 1341-1348.

[35] M. Matsuura, *Gielis' superformula and regular polygons*, Journal of Geometry **106(2)** (2015), 383-403.

[36] P. Natalini, R. Patrizi and P.E. Ricci, *The Dirichlet problem for the Laplace equation in a starlike domain of a Riemann surface*, Numerical Algorithms **49(1)** (2008), 299-313.

[37] K.J. Niklas, *Plant allometry: is there a grand unifying theory?*, Biological reviews **79(4)** (2004), 871-889.

[38] K.J. Niklas and U. Kutschera, *Kleiber's law: how the fire of life ignited debate, fueled theory, and neglected plants as model organisms*, Plant Signaling & Behavior **10(7)** (2015), e1036216.

[39] OECD, *In it together: Why less inequality benefits all*, OECD Publishing, Paris, 2015.

[40] Z. Özdemir and E. Parlak, *Superquadric motion and superquadric hyperbolic split quaternion algebra Via Gielis formula*, Mathematical Methods in the Applied Sciences, (2025), early view: https://doi.org/10.1002/mma.11229

[41] E. Parlak and Z. Özdemir, *Superelliptic motion and superelliptic quaternionsvia Gielis formula*, Physica Scripta (2025) (in press).

[42] T. Piketty, *Capital in the 21st Century*, Harvard University Press, Cambridge, MA, 2014.

[43] P.E. Ricci, *A Note on the D-trigonometry and the relevant D-Fourier expansions*, Growth and Form, **2(1)** (2021), 11-16.

[44] P.E. Ricci and J. Gielis, *From Pythagoras to Fourier and from Geometry to Nature*, Athena publishing, 2022.

[45] J.A. Rodrigo and T. Alieva, *Polymorphic beams and Nature inspired circuits for optical current*, Scientific reports **6(1)** (2016), 35341.

[46] R. Rodríguez-Oliveros and J.A. Sánchez-Gil, *Localized surface-plasmon resonances on single and coupled nanoparticles through surface integral equations for flexible surfaces*, Optics Express **19(13)** (2011), 12208-12219.

[47] M. Rück, A. Bandarenka, F. Calle-Vallejo and A. Gagliardi, *Fast identification of optimal pure platinum nanoparticle shapes and sizes for efficient oxygen electroreduction* Nanoscale Advances **1(8)** (2019), 2901-2909.

[48] J.M. Sarabia, V. Jordá and C. Trueba, *The Lamé class of Lorenz curves*, Communications in Statistics — Theory and Methods **46(11)** (2017), 5311–5326.

[49] P. Shi, L. Deng and K.J. Niklas, *Rotated Lorenz curves of biological size distributions follow two performance equations*, Symmetry **16(5)** (2024), 565.

[50] P.J. Shi, M.L. Fan and G.V. Reddy, *Comparison of thermal performance equations in describing temperature-dependent developmental rates of insects:(III) Phenological applications*, Annals of the Entomological Society of America **110(6)** (2017), 558–564.

[51] P.J. Shi, J.G. Huang, C. Hui, H.D. Grissino-Mayer, J.C. Tardif, L.H. Zhai, F.S. Wang and B.L. Li, *Capturing spiral radial growth of conifers using the superellipse to model tree-ring geometric shape*, Frontiers in Plant Science **6** (2015), 856.

[52] P. Shi, D.A. Ratkowsky, Y. Li, L. Zhang, S. Lin and J. Gielis, *A general leaf area geometric formula exists for plants – Evidence from the simplified Gielis equation*, Forests **9(11)** (2018), 714.

[53] P. Shi, D.A. Ratkowsky and J. Gielis, *The generalized Gielis geometric equation and its application*, Symmetry **12(4)** (2020), 645.

[54] P.J. Shi, G.V. Reddy, L. Chen and F. Ge, *Comparison of thermal performance equations in describing temperature-dependent developmental rates of insects:(I) empirical models*, Annals of the Entomological Society of America **109(2)** (2016), 211-215.

[55] P.J. Shi, G.V. Reddy, L. Chen and F. Ge, *Comparison of thermal performance equations in describing temperature-dependent developmental rates of insects:(II) two thermodynamic models*, Annals of the Entomological Society of America **110(1)** (2017), 113-120.

[56] T. Sowell, *Wealth, Poverty and Politics*, Hachette UK, 2016.

[57] L. Spíchal, *Superelipsa a superformule*, Matematika–fyzika–informatika **29(1)** (2020), 54-69.

[58] I. Tavkhelidze, J. Gielis and S. Pinelas, *About some methods of analytic representation and classification of a wide set of geometric figures with "complex" configuration*, in Proceedings of the International Conference on Differential & Difference Equations and Applications, Springer International Publishing, Cham, 2019, 347-359.

[59] L. Verstraelen, *Universal natural shapes*, Journal for Mathematics and Informatics of the Serbian Mathematical Society **40** (2004), 13-20.

[60] A.D. Vîlcu and G.E. Vîlcu, *On some geometric properties of the generalized CES production functions*, Applied Mathematics and Computation **218(1)** (2011), 124-129.

[61] A.D. Vîlcu and G.E. Vîlcu, *A survey on the geometry of production models in economics*, Arab Journal of Mathematical Sciences **23(1)** (2017), 18-31.

[62] L. Wang, K. He, C. Hui, D.A. Ratkowsky, W. Yao, M. Lian, J. Wang and P. Shi, *Comparison of four performance models in quantifying the inequality of leaf and fruit size distribution.* Ecology and Evolution **14(3)** (2024), e11072.

[63] L. Wang, Q. Miao, Ü. Niinemets, J. Gielis and P. Shi, *Quantifying the variation in the geometries of the outer rims of corolla tubes of Vinca major L*, Plants **11(15)** (2022), 1987.

[64] L. Wang, D.A. Ratkowsky, J. Gielis, P.E. Ricci and P. Shi, *Effects of the numerical values of the parameters in the Gielis equation on its geometries*, Symmetry **14(12)** (2022), 2475.

[65] G.B. West and J.H. Brown, *Life's universal scaling laws*, Physics Today **57(9)** (2004), 36-42.

[66] W. Yao, C. Hui, L. Wang, J. Wang, J. Gielis and P. Shi, *Comparison of the performance of two polar equations in describing the geometries of elliptical fruits*, Botany Letters **171(3)** (2024), 337-347.

[67] B. Zarghooni, A. Dadgarpour, J. Pourahmadazar and T.A. Denidni, *Supershaped metamaterial unit-cells using the Gielis formula*, in Proceedings of the 2015 IEEE International Symposium on Antennas and Propagation & USNC/URSI National Radio Science Meeting, Vancouver, BC, Canada, IEEE, 2015, 458-459.

[68] D. Zarifi, A. Farahbakhsh and M. Mrozowski, *Improved bandwidth of microstrip wide-slot antenna using Gielis curves*, IEEE Access **12** (2024), 74777-74783.

[69] L. Zhang, B.K. Quinn, C. Hui, M. Lian, J. Gielis, J. Gao and P. Shi, *New indices to balance α-diversity against tree size inequality*, Journal of Forestry Research **35(1)** (2024), 31.

Department of Biosciences Engineering, University of Antwerp, Belgium
Email address: johan.gielis@gmail.com

School of Integrative Plant Science, Cornell University, Ithaca, NY14853, USA
Email address: kjn2@cornell.edu