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DISAFFINITY VECTOR FIELDS WITH APPLICATIONS

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Abstract. A disaffinity vector on a pseudo-Riemannian manifold is a vector field whose affinity tensor vanishes identically. In this article, we present several known results on disaffinity vector fields from two earlier joint papers. Furthermore, we present some of their applications to Riemannian geometry and soliton theory. In the last section of this article, we present ten important vector fields on Riemannian (or pseudo-Riemannian) manifolds. Then we explain their applications across various scientific and engineering disciplines.

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1. Introduction

This is a version of my talk delivered on May 23, 2025, at the International Conference on Riemannian Geometry and Applications — RIGA 2025, based on my two earlier joint papers [1, 15] *with Hanan Alohali, Sharief Deshmukh, and Amira Ishan.*

Let M^n be a connected, smooth Riemannian n-manifold endowed with a Riemannian metric g and the corresponding Levi-Civita connection ∇. We denote by $\mathfrak{X}(M^n)$ the space of smooth vector fields defined on M^n.

In [32, page 109], W. A. Poor defined the affinity tensor, denoted by $\mathcal{L}_\xi\nabla$, of a vector field ξ on M^n by

$$(1.1)\qquad \big(\mathcal{L}_\xi\nabla\big)\big(X,Y\big) = \mathcal{L}_\xi(\nabla_X Y) - \nabla_{\mathcal{L}_\xi X}Y - \nabla_X\mathcal{L}_\xi Y, \quad X,Y \in \mathfrak{X}(M),$$

where $\mathcal{L}$ represents the Lie derivative on M^n.

Note that affinity tensors have been used in [2, 5, 20] to obtain characterizations of trivial Ricci solitons, and also used to find some new characterizations of a sphere.

The notion of disaffinity vectors was defined in [15] as follows.

Definition 1.1. A vector field ξ on a Riemannian manifold M is called a *disaffinity vector* if its affinity tensor vanishes identically, i.e., $\mathcal{L}_\xi \nabla = 0$.

The simplest examples of disaffinity vectors are parallel vector fields on a flat Riemannian manifold. Indeed, disaffinity vectors exist abundantly, as we will explore. Furthermore, we observe that they play an important role in shaping the geometry of the Riemannian manifold.

The main purpose of this article is to present some relations between the disaffinity vector field, the Killing vector field, and the incompressible vector field. Also, I will present several applications of disaffinity vector fields to the theory of solitons. In the last section, we will present ten vector fields defined on a Riemannian (or pseudo-Riemannian) manifold. Then we explain their applications across various scientific and engineering disciplines.

2. Preliminaries

Recall that the Riemann curvature tensor R of a Riemannian manifold M^n is given by

$$R(X,Y)Z = [\nabla_X, \nabla_Y]\, Z - \nabla_{[X,Y]} Z, \qquad \forall X, Y, Z \in \mathfrak{X}(M). \tag{2.1}$$

Let $\{E_1, \ldots, E_n\}$ be a local orthonormal frame on M^n. Then the Ricci tensor, denoted by Ric, is defined by

$$Ric\big(X,Y\big) = \sum_l g\big(R\big(E_l, X\big)Y, E_l\big), \tag{2.2}$$

The corresponding Ricci operator Q is then given by

$$Ric\big(X,Y\big) = g\big(QX, Y\big), \tag{2.3}$$

and the scalar curvature ρ is

$$\rho = \sum_l Ric\big(E_l, E_l\big). \tag{2.4}$$

The following formula for the gradient $\nabla\rho$ of ρ is well known (cf. [8, 12, 14])

$$\frac{1}{2}\nabla\rho = \sum_l \big(\nabla_{E_l} Q\big)\big(E_l\big), \tag{2.5}$$

where

$$\big(\nabla_X Q\big)\big(Y\big) = \nabla_X QY - Q\big(\nabla_X Y\big).$$

For a function $\phi : M^n \to \mathbb{R}$, the *Laplacian* Δ of ϕ is

$$\Delta\phi = \operatorname{div}\big(\nabla\phi\big), \tag{2.6}$$

where $\nabla\phi$ denotes the gradient of ϕ. The *divergence* of a vector field $X \in \mathfrak{X}(M^n)$ is defined by

$$\operatorname{div} X = \sum_l g\big(\nabla_{E_l} X, E_l\big).$$

For a function $\phi : M^n \to \mathbb{R}$, the Hessian operator H_ϕ is defined by

$$H_\phi X = \nabla_X \nabla\phi, \qquad X \in \mathfrak{X}(M) \tag{2.7}$$

and the Hessian of ϕ is given by

$$Hess\big(\phi\big)\big(X,Y\big) = g\big(H_\phi X, Y\big). \tag{2.8}$$

When M^n is a compact manifold, the Stokes' Theorem gives

$$\int_{M^n} \big(\operatorname{div} X\big) dv = 0, \tag{2.9}$$

where dv represents the volume element of M^n.

3. Relations between disaffinity, Killing and incompressible vector fields

A vector field ξ on a Riemannian n-manifold M^n is said to be a *Killing vector field* if it satisfies (see, e.g., [7, 35, 36, 37])

$$\mathcal{L}_\xi g = 0. \tag{3.1}$$

Killing vector fields play a distinguished role in differential geometry and general relativity, which describe a vector field on a manifold that preserves the metric of the manifold under its associated flow.

In general relativity, Killing vector fields play a role in defining and understanding concepts like stationarity and staticity. For instance, a space-time is stationary if it admits a time-like Killing vector field, meaning that the metric components do not depend on the time coordinate in a suitable coordinate system. If, in addition, this time-like Killing vector field is hypersurface orthogonal, the space-time is called static. These distinctions are fundamental for analyzing the behavior of physical systems in gravitational fields.

The following result illustrates that disaffinity vector fields do exist abundantly.

Theorem 3.1. [15] *Every Killing vector field on a Riemannian manifold is a disaffinity vector field.*

A vector field ζ on a Riemannian manifold M^n is called a *incompressible vector field* if its divergence vanishes identically, representing flows where the density of the fluid remains constant within a given volume. This fundamental property leads to a wide array of applications across various scientific and engineering disciplines. For instance, in fluid mechanics, an incompressible vector field is a vector field that has no sources or sinks. For the importance of incompressible vector fields, we refer to [3, 9, 28, 29] and references therein.

For compact Riemannian manifolds, we have the following result from [1].

Theorem 3.2. *Every disaffinity vector field on a compact Riemannian manifold is an incompressible vector field.*

To state the next theorem, we mention the following useful lemma.

Lemma 3.3. [15] *A vector field ξ on a Riemannian manifold M^n is a disaffinity vector field if and only if the Riemann curvature tensor of M^n satisfies*

$$R\big(X, \xi\big)Y = \nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y} \xi, \quad \forall X, Y \in \mathfrak{X}(M^n). \tag{3.2}$$

Proof. This lemma follows the definition of disaffinty vector fields and the following:

$$\begin{aligned}
\left(\mathcal{L}_\xi \nabla\right)(X, Y) &= \mathcal{L}_\xi \nabla_X Y - \nabla_{\mathcal{L}_\xi X} Y - \nabla_X \mathcal{L}_\xi Y \\
&= [\xi, \nabla_X Y] - \nabla_{[\xi, X]} Y - \nabla_X [\xi, Y] \\
&= \nabla_\xi (\nabla_X Y) - \nabla_{\nabla_X Y} \xi - \nabla_{[\xi, X]} Y - \nabla_X \nabla_\xi Y + \nabla_X \nabla_Y \xi \\
&= R(\xi, X) Y + \nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y} \xi.
\end{aligned}$$

□

For a vector field ξ on a Riemannian n-manifold M^n, we define a symmetric operator B and a skew-symmetric operator F related to ξ by (see [1])

$$\big(\mathcal{L}_\xi g\big)(X,Y) = 2g(BX,Y), \quad d\eta(X,Y) = 2g(FX,Y), \tag{3.3}$$

where η is the 1-form dual to ξ.

In connection to Theorem 3.2, we also mention the following result.

Theorem 3.4. [1] *An incompressible vector field ξ on a compact Riemannian manifold M^n is a disaffinity vector field if it satisfies*

$$\int_{M^n} Ric(\xi,\xi)dv \geq \int_{M^n} \|F\|^2 dv$$

where F is the skew-symmetric operator associated with ξ defined by (3.3).

4. Disaffinity functions and Eikonal equations with applications

Let us recall the following definition.

Definition 4.1. [15]A function ϕ on a Riemannian manifold M^n is a disaffinity function if $\nabla\phi$ is a disaffinity vector.

The next result shows that the existence of a non-trivial disaffinity function on a Riemannian manifold M^n imposes a condition on the topology of M^n.

Theorem 4.2. [15] *If a Riemannian manifold M^n admits a non-trivial disaffinity function, then M^n is non-compact.*

Next, we use disaffinity functions to characterize a Euclidean space.

Theorem 4.3. [15] *A complete Riemannian manifold M^n is isometric to a Euclidean space if and only if it admits a non-harmonic disaffinity function ϕ that satisfies the inequality:*

$$Ric\big(\nabla\phi,\nabla\phi\big) \geq \Delta h - \frac{1}{n}\big(\Delta\phi\big)^2,$$

where $h = \frac{1}{2}\|\nabla\phi\|^2$

Now, we recall the following.

Definition 4.4. A function f on a Riemannian manifold is said to satisfy the Eikonal equation if we have $\|\nabla f\| = 1$.

Eikonal equations have been widely used in various scientific and engineering fields, such as image processing, seismology, computer graphics, and robotic path planning, due to their ability to model wave propagation in inhomogeneous media and compute shortest paths or travel times. (see, e.g., [16, 19, 31, 34]).

Dsaffinity functions and the Eikonal equation can be used to characterize Euclidean spaces as illustrated in the next theorem.

Theorem 4.5. [1] *Let M^n be a complete Riemannian manifold that admits a non-harmonic disaffinity function h satisfying the Eikonal equation. If the Ricci curvature of M^n satisfies*

$$Ric\big(\nabla h,\nabla h\big) \geq -\frac{1}{n}\big(\Delta h\big)^2, \tag{4.1}$$

then M^n is isometric to a Euclidean n-pace.

5. Disaffinity vectors and Ricci solitons

The concept of Ricci flow, defined by

$$\frac{\partial}{\partial t}g(t) = -2Ric(t),$$

was first introduced by R. Hamilton in 1982 (see, e.g., [23, 24]). A complete Riemannian manifold (M^n, g) is called a *Ricci soliton* if and only if there exists a vector field ζ ssatisfying the differential equation (cf. e.g., [4, 5, 12, 14])

$$\frac{1}{2}\mathcal{L}_\zeta g + Ric = \lambda g. \tag{5.1}$$

for some real number λ. In the following, we denote by $\big(M^n, g, \zeta, \lambda\big)$ a Ricci soliton, in which ζ is called the *potential field*. A Ricci soliton $\big(M^n, g, \zeta, \lambda\big)$ is called *trivial* if either $\zeta = 0$ or ζ is a Killing vector field.

There are three main types of Ricci solitons, classified by the constant λ. More precisely, the Ricci soliton is called *shrinking, steady*, or *expanding* according to $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively.

The Ricci soliton $\big(M^n, g, \zeta, \lambda\big)$ is called a *gradient Ricci soliton* if the potential vector field ζ is the gradient of a function ϕ, i.e., $\zeta = \nabla\phi$ for function ϕ (cf. [10, 12]).

Ricci solitons are significant in differential geometry and mathematical physics because they represent self-similar solutions to the Ricci flow equation. This characteristic makes Ricci solitons crucial for understanding the behavior of the Ricci flow, particularly in the study of singularities.

In this section, we are interested in the impact under the condition that the potential field ζ of the Ricci soliton $\big(M^n, g, \zeta, \lambda\big)$ is a disaffinity vector field. Note that if $\big(M^n, g, \zeta, \lambda\big)$ is a non-trivial Ricci soliton such that the potential field ζ is a disaffinity vector field, then the Ricci soliton is non-compact according to Theorem 4.2.

For a Ricci soliton $\big(M^n, g, \zeta, \lambda\big)$ with the potential field ζ a disaffinity vector field, we have following two results.

Theorem 5.1. [15] *If $\big(M^n, g, \zeta, \lambda\big)$ is a Ricci soliton with potential field ζ a disaffinity vector field, then the scalar curvature ρ of M^n is constant.*

Theorem 5.2. [15] *A Ricci soliton $\big(M^n, g, \zeta, \lambda\big)$ with ζ a disaffinity vector and the Ricci operator Q satisfying $Q\big(\zeta\big) = \lambda\zeta$ is trivial.*

6. Disaffinity vectors and Yamabei solitons

R. Hamilton [24] introduced the notion of Yamabe flow, in which the metric on a Riemannian manifold is deformed by evolving according to

$$\frac{\partial}{\partial t}g(t) = -\rho(t)g(t), \tag{6.1}$$

where $\rho(t)$ is the scalar curvature of the metric $g(t)$. It is well known that Yamabe solitons correspond to self-similar solutions of the Yamabe flow.

In dimension $n = 2$, the Yamabe flow is equivalent to the Ricci flow. However, in dimension $n > 2$, the Yamabe and Ricci flows do not agree, since the first one preserves the conformal class of the metric but the Ricci flow does not in general.

A connected Riemannian n-manifold (M^n, g) with $n \geq 2$ is called a *Yamabe soliton* if it admits a vector field ξ such that

$$\frac{1}{2}\mathcal{L}_\xi g = (\rho - \lambda)g, \tag{6.2}$$

where λ is a real number (cf. e.g., [14, 24, 11]). The vector field ξ in the definition is called a *soliton field.* A Yamabe soliton $\big(M^n, g, \xi, \lambda\big)$ is called trivial if its scalar curvature ρ is a constant. It is well-known that a compact gradient Yamabe soliton is always trivial (cf. [12]).

In this section, we are interested in the impact of the soliton field ξ being a disaffinity vector field on the geometry of the Yamabe soliton $\big(M^n, g, \xi, \lambda\big)$. In this respect, we mention the next result from [15].

Theorem 6.1. *A Yamabe soliton $\big(M^n, g, \xi, \lambda\big)$, $n \geq 2$, with potential field ξ a disaffinity vector field is trivial.*

7. Disaffinity vector field and trans-Sasakian manifold

Now, let us consider a contact metric manifold $(M^{2m-1}, \varphi, \xi, \eta, g)$, where φ is a $(1,1)$-tensor field, ξ a unit vector field and η the 1-form dual to ξ satisfying

$$\begin{aligned} &\varphi^2 = -I + \eta \otimes \xi, \\ &\varphi(\xi) = 0, \ \ \eta \circ \varphi = 0, \\ &g(\varphi U, \varphi V) = g(U, V) - \eta(U)\eta(V) \end{aligned}$$

for $U, V \in \mathfrak{X}(M^{2m-1})$ (cf. [6]). If there exist functions (α, β) on $(M^{2m-1}, \varphi, \xi, \eta, g)$ such that

$$(\nabla_U \varphi)(V) = \alpha\big(g(U, V)\xi - \eta(V)U\big) + \beta\big(g(\varphi U, V)\xi - \eta(V)\varphi U\big), \tag{7.1}$$

where

$$(\nabla\varphi)(U, V) = \nabla_U \varphi V - \varphi(\nabla_U V),$$

then M^{2m-1} is called a *trans-Sasakian manifold of type* (α, β) (cf., [13, 25, 20]). It follows that

$$\nabla_U \xi = -\alpha\varphi(U) + \beta(U - \eta(U)\xi), \quad U \in \mathfrak{X}(M). \tag{7.2}$$

A trans-Sasakian manifold is said to be *proper* if both $\alpha, \beta \neq 0$.

Since it is shown in [30] that there exist no proper trans-Sasakian manifolds of dimension $n \geq 5$, we are thus interested only in trans-Sasakian 3-manifolds.

We mention the following two lemmas on trans-Sasakian 3-manifolds.

Lemma 7.1. [13] *We have $\xi(\alpha) + 2\alpha\beta = 0$ on a trans-Sasakian manifold $(M^3, \varphi, \xi, \eta, g, \alpha, \beta)$ of type (α, β).*

Lemma 7.2. [13] *The Ricci operator Q of a trans-Sasakian manifold $(M^3, \varphi, \xi, \eta, g, \alpha, \beta)$ of type (α, β) satisfies*

$$Q(\xi) = \varphi\big(\nabla\alpha\big) - \nabla\beta + 2(\alpha^2 - \beta^2)\xi - g(\nabla\beta, \xi)\xi.$$

Also, we recall the following result.

Theorem 7.3. [37] *If a Riemannian manifold M^{2m-1} admits a Killing vector field ξ of constant length satisfying*

$$c^2\big(\nabla_X\nabla_Y\xi - \nabla_{\nabla_X Y}\xi\big) = g(Y, \xi)X - g(X, Y)\xi, \quad \forall X, Y \in \mathfrak{X}(M)$$

for a non-zero constant c, then M^{2m-1} is homothetic to a Sasakian manifold.

One of the interesting questions in the geometry of a trans-Sasakian 3-manifold M^3 is to find conditions under which the trans-Sasakian 3-manifold is homothetic to a Sasakian manifold. In this respect, we present the next result.

Theorem 7.4. [1] *If the Reeb vector field ξ of a trans-Sasakian 3-manifold $(M^3, \varphi, \xi, \eta, g, \alpha, \beta)$ of type (α, β) with $\alpha \neq 0$ is a disaffinity vector field and the sectional curvature of plane sections containing ξ is constant, then M^3 is homothetic to a Sasakian manifold.*

On trans-Sasakian 3-manifolds, we also have the following.

Theorem 7.5. [1] *If the Reeb vector field ξ of a trans-Sasakian 3-manifold $(M^3, \varphi, \xi, \eta, g, \alpha, \beta)$ of type (α, β) with $\alpha \neq 0$ is a disaffinity vector, then $\beta = 0$.*

8. Some other vector fields with applications

Space-time is a fundamental concept in physics that combines the three dimensions of space with the one dimension of time into a single 4-dimensional continuum. In terms of mathematics, a space-time is a time-oriented 4-dimensional Lorentz manifold.

Space-time symmetries are important features of space-times that can be described as exhibiting some form of symmetry, which are frequently employed in the study of exact solutions to Einstein's field equations of general relativity. In fact, physical problems are often investigated and solved by noticing features that have some form of symmetry. For instance, in the Schwarzschild solution of Einstein's field equations, the role of spherical symmetry is important in deriving the Schwarzschild solution and deducing the physical consequences of this symmetry. In cosmological problems, symmetry plays a role in the cosmological principle, which restricts the type of universes that are consistent with large-scale observations.

Space-time symmetries usually require some form of preserving property, the most important of which in general relativity include the following (see, e.g., [22]):

(a) Preserving geodesics of the space-time;
(b) Preserving the metric tensor;
(c) Preserving the curvature tensor.

In the following, we mention ten important vector fields that have important applications across various scientific and engineering disciplines.

8.1. **Killing vector fields.**

The name Killing refers to Wilhelm Killing (1847–1923), who first investigated Killing vector fields in the 19th century. Killing vector fields play a distinguished role in differential geometry and general relativity, which describes a vector field on a differentiable manifold that preserves the metric of the manifold under its associated flow.

Killing vector fields have some important geometric properties. For instance, the Lie bracket of two Killing vector fields is still a Killing field. Thus, the Killing vector fields on a manifold M form a Lie subalgebra of vector fields on M, which is the Lie algebra of the isometry group of M whenever M is complete.

Furthermore, for compact manifolds, we have:

(1) Negative Ricci curvature implies there are no non-trivial Killing vector fields.
(2) Non-positive Ricci curvature implies that any Killing vector field is parallel.
(3) If the sectional curvature is positive and the dimension of M is even, then every Killing vector field must have a zero point.

Moreover, in general relativity, a Killing vector field corresponds to a symmetry of space-time. If such a symmetry exists, it implies conservation laws. For instance:

(i) A time-like Killing vector field indicates conservation of energy.
(ii) A space-like Killing vector field may correspond to conservation of linear or angular momentum.

In physics, a conservation law is a fundamental principle stating that a specific measurable property of a physical system remains constant over time. This means that the total amount of this property does not change as the system evolves or undergoes processes.

8.2. **Conformal vector fields.** A conformal vector field on a manifold with (pseudo) Riemannian metric g is a vector field ζ whose flow defines conformal transformations, i.e., preserves g up to scale and preserves the conformal structure. In other words, it satisfies (cf. e.g., [35, 36])

$$\mathcal{L}_\zeta g = \lambda g$$

for some function λ on the manifold, where λ is called the conformal factor.

Conformal vector fields appear in various physical theories where angle-preserving symmetries are important. For examples, we have:

(a) In General Relativity, they describe space-times with specific symmetries.
(b) In Fluid Dynamics and Electromagnetism, they help analyze systems where scaling properties matter.
(c) In String Theory and Quantum Field Theory, they play roles in understanding symmetry groups like those in conformal field theory.

8.3. **Homothetic vector fields.** A homothetic vector field ζ (also known as homothetic collineation or homothety) is a projective vector field that satisfies

$$\mathcal{L}_\zeta g = cg$$

for a real number c.

(i) In General Relativity, homothetic vector fields are used in studying singularities in space-time geometry. Homothetic vector fields help identify self-similar solutions to Einstein's equations, which can simplify complex problems through symmetry reduction.
(ii) In Similarity Transformations, homothetic vector fields are instrumental in generating new solutions for differential equations (e.g., Einstein's equations) via similarity reductions.
(iii) In Conformal Geometry, homothetic vector fields play an essential role in conformal geometry.

8.4. **Concircular vector fields.** A concircular vector field on a (pseudo-) Riemannian manifold is a vector field ζ which satisfies (cf. e.g., [33, 35])

$$\nabla_X \zeta = \phi X$$

for an arbitrary vector field $X \in \mathfrak{X}(M)$, where ϕ is a scar function.

Concircular vector fields appear in various contexts within theoretical physics:

(a) In General Relativity, concircular vector fields are used to study space-time symmetries and conserved quantities in solutions to Einstein's equations.
(b) In Mathematical Physics, concircular vector fields play roles in understanding conserved currents and symmetry transformations under certain physical conditions.

8.5. **Concurrent vector fields.** A vector field ζ on a (pseudo) Riemannian manifold is called *concurrent* if it satisfies (cf. e.g., [10, 33, 35])

$$\nabla_X \zeta = X$$

for any vector field $X \in \mathfrak{X}(M)$.

(a) In General Relativity, concurrent vector fields can be used to model certain types of gravitational systems where space-time exhibits radial symmetry around a central mass or singularity (e.g., Schwarzschild space-time). These fields help describe how matter and energy are distributed radially around such points.
(b) In Cosmology, concurrent vector fields may represent flows converging toward an initial singularity (such as the Big Bang) or diverging from it during cosmic expansion. They are useful for studying isotropic and homogeneous universes.
(c) In Fluid Dynamics, concurrent vector fields can describe radial flow patterns, such as those seen in spherically symmetric explosions or implosions.
(d) In Electromagnetic Fields, radial electric field configurations around charged particles often exhibit properties analogous to those described by concurrent vector fields.

8.6. **Torse-forming vector fields.** A vector field ζ is called torse-forming if it satisfies (see [35])

$$\nabla_X \zeta = \phi X + \psi(X)\zeta$$

for any vector field $X \in \mathfrak{X}(M)$, where ϕ is a function, and ψ is a 1-form. Note that concircular vector fields form a special family of torse-forming vector fields.

The physical relevance of torse-forming vector fields arises from their ability to model real-world phenomena. For instance, we have:

(a) In Fluid Dynamics, torse-forming vectors can describe velocity fields with particular rotational or divergence properties. This is especially useful for studying incompressible flows or vorticity dynamics.
(b) In Electromagnetic Fields, certain configurations of electromagnetic fields can be described using torse-forming vector fields due to their ability to encode rotational symmetries.
(c) In Cosmology and Astrophysics, torse-forming vector fields have been applied in cosmology to study anisotropic models of the universe and other large-scale structures.

8.7. **Projective vector fields.** A projective vector field on M is a vector field whose flow preserves the geodesic structure of M without necessarily preserving the affine parameter of any geodesic (see. [25]). More intuitively, the flow of the projective vector field maps geodesics into geodesics (without preserving the affine parameter).

In cosmological models, projective vector fields can be used to describe large-scale symmetries of the universe.

(a) They may be used to analyze isotropic and homogeneous models like those described by Robertson-Walker space-times.
(b) Projective vector fields allow for the study of how test particles move along geodesics in curved space-time, which is critical for understanding gravitational effects.
(c) In quantum gravity or string theory, where space-time geometry becomes more complex, projective vector fields help explore higher-dimensional analogs or compactified spaces.

8.8. **Incompressible vector fields.** An incompressible vector field is a vector field where the divergence is zero everywhere (cf. e.g., [9, 27, 28, 29]).

The importance of incompressible vector fields lies in their ability to model real-world systems where conservation laws apply. For instance;

(a) Conservation of Mass: Incompressible vector fields ensure no loss or gain occurs within a system.
(b) Simplification: Incompressible vector fields reduce complexity by eliminating compressibility effects.
(c) Universality: Incompressible vector fields appear across disciplines from physics to engineering due to its foundational nature.

8.9. **Geodesic vector fields.** A *geodesic vector field* on a Riemannian manifold is characterized by its integral curves being geodesics, meaning that they satisfy the geodesic equation.

Geodesic vector fields have significant applications in various fields, such as general relativity, where they can describe trajectories of particles moving under gravity alone. In expanding universes modeled by de Sitter space, time-like geodesics describe how galaxies move apart due to cosmic expansion.

For engineering applications, we mention the following:

(a) In Optimal Path Planning: In robotics and aerospace engineering, finding optimal paths over complex surfaces often involves solving equations analogous to those defining geodesics.
(b) In Structural Design: Geometric principles based on minimizing energy, similar to geodesics, are often used for designing efficient structures like domes or airframes.

8.10. **Harmonic vector fields.** A vector field on a Riemannian manifold is called a *harmonic vector field* if its dual 1-form is a harmonic 1-form. Such vector fields are characterized by being both divergence-free (solenoidal) and curl-free (irrotational) within a given domain that satisfies the Laplace equation. This dual property makes them ideal for representing fields where there are no sources or sinks, and no rotational components.

Harmonic vector fields are of significant importance across various scientific and engineering disciplines due to their unique mathematical properties and their ability to model fundamental physical phenomena.

Harmonic vector fields appear in several physical theories. For instance, we have:

(a) In electrostatics and magnetostatics, where electric and magnetic fields in charge-free and current-free regions, respectively, are often harmonic (see e.g., [21]).
(b) In fluid dynamics, harmonic vector fields find applications particularly in the study of incompressible, irrotational flows.
(c) In elasticity, harmonic vector fields are used in the analysis of stress and strain in certain materials (see [26]).
(d) In signal processing and image analysis, harmonic functions and harmonic vector fields are utilized for tasks such as image reconstruction and denoising due to their smoothing properties.
(e) In differential geometry, particularly in Riemannian manifolds, harmonic forms and harmonic vector fields are fundamental concepts. The theory of harmonic maps has applications in various areas of mathematics and physics (see, e.g., [17, 18]).

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