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QUADRATIC ROTATIONAL WEINGARTEN SURFACES

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Abstract. The classification of rotational surfaces in Euclidean space satisfying a linear relation between their principal curvatures was completed in [9]. On the other hand, using the notion of geometric linear momentum of a planar curve with respect to a line introduced in [2] or [3], a new approach to rotational Weingarten surfaces was developed in [1]. Taking advantage of this study, we face the case that the principal curvatures satisfy a certain quadratic relation.

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1. Introduction

Following [4] or [10], Weingarten surfaces are those whose principal curvatures κ_1 and κ_2 satisfy a functional relation $\mathrm{W}(\kappa_1, \kappa_2) = 0$. In particular, those ones satisfying a linear relation $a\kappa_1 + b\kappa_2 = c$, $a^2 + b^2 \neq 0$, $c \in \mathbb{R}$, are called *linear Weingarten surfaces.*

In the case of rotational surfaces, the principal curvatures are reached along meridians and parallels and it is clear that rotational surfaces constitute a distinguish class of Weingarten surfaces. In the equation of linear rotational Weingarten surfaces, we can assume $a \neq 0$ without loss of generality and we just write $k_\mathrm{m} = p\, k_\mathrm{p} + q$, $p, q \in \mathbb{R}$, $p \neq 0$, where k_m and k_p denote the principal curvatures along meridians and parallels respectively. These surfaces have been recently classified in [9] by means of a qualitative study, providing closed (embedded and not embedded) surfaces and periodic (embedded and not embedded) surfaces with a geometric behaviour similar to Delaunay surfaces (cf. [5]). As a simple brief summary, they provide the following types of rotational surfaces (see [9] for details):

- $q = 0$: Planes, ovaloids and catenoid type.
- $q \neq 0, p > 0$: Ovaloids, vesicle type, pinched spheroid, immersed spheroid, cylindrical anti-nodoid type, anti-nodoid type and circular cylinders.

- $q \neq 0, p < 0$: Unduloid type, circular cylinders, spheres and nodoid type.

In [9] there is also a necessary distinction of cases according to $p = 1$ or $p \neq 1$. In fact, when $p \neq 1$, the generatrix curves are $\frac{p}{p-1}$- elastic curves generalizing classical elastic curves corresponding to $p = 2$ (see [8]).

On the other hand, Kühnel and Steller studied in [7] certain quadratic rotational Weingarten surfaces. Concretely, extending the method of Hopf in [6], they considered closed rotational surfaces satisfying $k_{\mathrm{m}} = c(k_{\mathrm{p}} - b)^2 + k_{\mathrm{p}} - a$, $ac > 0$, $b \in \mathbb{R}$, and were able to prove that there are explicit analytic solutions of genus zero with self-intersections.

In this article, our aim is to get a complete explicit local classification of the quadratic rotational Weingarten surfaces verifying $k_{\mathrm{m}} = \mu\, k_{\mathrm{p}}^2$, $\mu \neq 0$ (see Section 3), making use of the tools and the techniques developed in [1], which are collected in Section 2.

2. The geometric linear momentum of a rotational surface

In this section we deal with *rotational surfaces*, also called *surfaces of revolution*. They are surfaces globally invariant under the action of any rotation around a fixed line called *axis of revolution*. The rotation of a curve (called *generatrix*) around a fixed line generates a surface of revolution. The sections of a surface of revolution by half-planes delimited by the axis of revolution, called *meridians*, are special generatrices. The sections by planes perpendicular to the axis are circles called *parallels* of the surface.

We denote S_α the rotational surface in $\mathbb{R}^3$ generated by the rotation around the z-axis of a plane curve α in the xz-plane. That is, α is the generatrix curve that we can consider parameterized by arc-length, whose parametric equations are given by $x = x(s)$, $y = 0$, $z = z(s)$, $s \in I \subseteq \mathbb{R}$. The function $x = x(s)$, $s \in I \subseteq \mathbb{R}$, represents the (signed) distance from the point $\alpha(s)$ to the z- axis of revolution. Then S_α is parameterized by

$$S_\alpha \equiv X(s,\theta) = (x(s)\cos\theta, x(s)\sin\theta, z(s)), \ (s,\theta) \in I \times (-\pi,\pi).$$

Given any plane curve α in the xz-plane, we introduced in [1, Section 2] the *geometric linear momentum* of α (with respect to the z-axis) as a smooth function assuming values in $[-1,1]$ that completely determines it (up to translations in the z-direction). It is defined by $\mathcal{K}(s) = \dot{z}(s)$, where the dot $\dot{}$ means derivation with respect to the arc parameter s. Geometrically, $\mathcal{K}$ controls the angle of the Frenet frame of the curve with the coordinate axes. Moreover, in physical terms, $\mathcal{K} = \mathcal{K}(s)$ may be described as the linear momentum (with respect to the z-axis) of a particle of unit mass with unit speed and trajectory $\alpha(s)$. We point out that $\mathcal{K}$ is well defined, up to the sign, depending on the orientation of α.

If the plane curve $\alpha = (x,z)$ is not necessarily parameterized by arc length, i.e. $\alpha = \alpha(t)$, t being any parameter, one can compute the geometric linear momentum $\mathcal{K} = \mathcal{K}(t)$ by means of $\mathcal{K}(t) = z'(t)/|\alpha'(t)|$, where $'$ denotes derivation respect to t.

The importance of the geometric linear momentum $\mathcal{K}$ lies in the fact that it allows to determine by quadratures in a constructive explicit way the plane curves $\alpha = (x,z)$ such that its curvature depends on the distance to the z-axis, that is, it is given as a function of x, i.e. $\kappa = \kappa(x)$. In this case, $\mathcal{K} = \mathcal{K}(x)$ satisfies $\mathcal{K}'(x) = \kappa(x)$ and the algorithm to recover the curve $\alpha = (x,z)$ involves the following computations (see [2] and [3]):

(i) Arc-length parameter s of $\alpha = (x, z)$ in terms of x, defined – up to translations of the parameter – by the integral:

$$s = s(x) = \int \frac{dx}{\sqrt{1 - \mathcal{K}(x)^2}}, \tag{2.1}$$

where $-1 < \mathcal{K}(x) < 1$, and inverting $s = s(x)$ to get $x = x(s)$.

(ii) z-coordinate of the curve – up to translations along z-axis – by the integral:

$$z(s) = \int \mathcal{K}(x(s))\, ds. \tag{2.2}$$

Alternatively, if we eliminate ds in the above integrals, we obtain:

$$z = z(x) = \int \frac{\mathcal{K}(x)dx}{\sqrt{1 - \mathcal{K}(x)^2}}. \tag{2.3}$$

Thus we can summarize the determining role of the geometric linear momentum in the next result.

Corollary 2.1. [1, Corollary 1] *Any plane curve $\alpha = (x, z)$, with x non-constant, is uniquely determined by its geometric linear momentum $\mathcal{K}$ as a function of its distance to z-axis, that is, by $\mathcal{K} = \mathcal{K}(x)$. The uniqueness is modulo translations in the z-direction. Moreover, the curvature of α is given by $\kappa(x) = \mathcal{K}'(x)$.*

It is obvious that if we translate the generatrix curve α of a rotational surface S_α along z-axis, we obtain a congruent surface to S_α. An immediate consequence of Corollary 2.1 is then the following key result:

Corollary 2.2. [1, Corollary 2] *Any rotational surface S_α, with generatrix curve $\alpha = (x, z)$, is uniquely determined, up to z-translations, by the geometric linear momentum $\mathcal{K} = \mathcal{K}(x)$ of its generatrix curve, being x non-constant.*

We can confirm the result established in Corollary 2.2 when we study the geometry of S_α through its first and second fundamental forms, I and II, since a direct computation, using that $\kappa(x) = \mathcal{K}'(x)$, shows that both can be expressed only in terms of the geometric linear momentum $\mathcal{K}$ and, of course, the non constant distance x from the surface to the axis of revolution:

$$I \equiv ds^2 + x^2 d\theta^2, \quad II \equiv \mathcal{K}'(x)ds^2 + x\mathcal{K}(x)d\theta^2.$$

Therefore we get the following expressions for the principal curvatures κ_1 and κ_2, whose curvature lines are the meridians (m) and the parallels (p) respectively of the rotational surface S_α:

$$\kappa_1 \equiv k_{\mathrm{m}} = \mathcal{K}'(x), \quad \kappa_2 \equiv k_{\mathrm{p}} = \frac{\mathcal{K}(x)}{x}. \tag{2.4}$$

Making use of Corollary 2.2, we can list the following characterizations of some simple surfaces of revolution:

Example 2.3. [1, Propositon 1]

(i) Any (horizontal) plane is uniquely determined by the geometric linear momentum $\mathcal{K} \equiv 0$.
(ii) The circular cone with opening $\theta_0 \in (-\pi/2, \pi/2)$, given by $x^2 + y^2 = \cot^2 \theta_0\, z^2$, is uniquely determined by the geometric linear momentum $\mathcal{K} \equiv \sin \theta_0$.
(iii) The sphere of radius $R > 0$, given by $x^2 + y^2 + z^2 = R^2$, is uniquely determined by the geometric linear momentum $\mathcal{K}(x) = x/R$.

Now we can pay attention to *rotational Weingarten surfaces*. In general, we just simply write $\Phi(k_{\mathrm{m}}, k_{\mathrm{p}}) = 0$ for the Weingarten relation. But taking into account (2.4), we easily deduce that the above functional relation translates into a first-order differential equation $\hat{\Phi}(x, \mathcal{K}(x), \mathcal{K}'(x)) = 0$ for the geometric linear momentum $\mathcal{K} = \mathcal{K}(x)$ determining S_α according to Corollary 2.2.

Using this method, we proved in [1] that linear rotational Weingarten surfaces (see Section 1) are uniquely determined, up to z-translations, by the following geometric linear momenta:

$$p \neq 1: \quad \mathcal{K}(x) = \frac{q\,x}{1-p} + c\,x^p,\ c \in \mathbb{R},$$

and

$$p = 1: \quad \mathcal{K}(x) = q\,x \ln x + c\,x,\ c \in \mathbb{R}.$$

This can be a reasonable explanation of the commented distinction of cases in [9].

We will use the same simple idea in order to reach our main result of the paper in next section.

3. Rotational surfaces satisfying $k_{\mathrm{m}} = \mu\, k_{\mathrm{p}}^2,\ \mu \neq 0$

In this section, we are interested in the rotational Weingarten surfaces whose principal curvatures are related by the quadratic equation $k_{\mathrm{m}} = \mu\, k_{\mathrm{p}}^2$, $\mu \neq 0$. Taking into account (2.4), we arrive at the differential equation

$$\mathcal{K}'(x) = \mu\, \mathcal{K}^2(x)/x^2, \tag{3.1}$$

whose solutions determine the corresponding surfaces by Corollary 2.2.

Notice that the constant trivial solution $\mathcal{K} \equiv 0$ provides the plane in view of Example 2.3-(i). The general solution of (3.1) is given by

$$\mathcal{K}_{\mu,c}(x) = \frac{x}{\mu + cx}, \quad c \in \mathbb{R}. \tag{3.2}$$

Since $\mathcal{K}_{-\mu,-c}(x) = -\mathcal{K}_{\mu,c}(x)$, we may assume that $\mu > 0$ in the following. If $c = 0$, the geometric linear momentum (3.2) is $\mathcal{K}_{\mu,0}(x) = x/\mu$, and therefore the surface is locally congruent to the sphere of radius μ (see Example 2.3-(iii)).

Taking (2.3) into account, we deduce that the desired rotational surfaces are generated by the graphs

$$z_{\mu,c}(x) = \pm \int \frac{\mathcal{K}_{\mu,c}(x)}{\sqrt{1 - \mathcal{K}_{\mu,c}(x)^2}} dx = \pm \int \frac{x}{\sqrt{(c^2-1)x^2 + 2\mu c x + \mu^2}} dx. \tag{3.3}$$

Notice that $z_{\mu,-c}(-x) = z_{\mu,c}(x)$, hence we may also assume that $c > 0$.

Now, we compute the integrals in (3.3) depending on $c > 0$ and considering $\mu > 0$.

(a) If $0 < c < 1$, the generatrix curve is written, up to a constant, as the graph

$$z_{\mu,c}(x) = \frac{\pm 1}{c^2 - 1} \left(\sqrt{(c^2-1)x^2 + 2\mu c x + \mu^2} + \frac{\mu c}{\sqrt{1-c^2}} \arcsin\left(\frac{c^2-1}{\mu} x + c \right) \right), \tag{3.4}$$

with $-\dfrac{\mu}{c+1} \leq x \leq \dfrac{\mu}{1-c}$ (see Figure 1). We remark that if $c = 0$ in (3.4) we recover the circle of radius μ centered at the origin.

(b) If $c = 1$, the integral in (3.3) is immediate and the generatrix curve is given by

$$z_{\mu,1}(x) = \pm \frac{(x-\mu)\sqrt{2x+\mu}}{3\sqrt{\mu}}, \quad x > -\frac{\mu}{2}. \tag{3.5}$$

It is an algebraic curve of degree 3, since that $9\mu\, z_{\mu,1}^2 = 2x^3 - 3\mu\, x^2 + \mu^3$ (see Figure 2).

(c) If $c > 1$, the generatrix curve is given, up to a constant, by

$$(3.6)\quad z_{\mu,c}(x) = \pm\frac{\sqrt{(c^2-1)x^2+2\mu c x+\mu^2}}{c^2-1} \mp \frac{\mu c}{(c^2-1)^{3/2}}\log\left|2(c^2-1)x+2\mu c+2\sqrt{(c^2-1)((c^2-1)x^2+2\mu c x+\mu^2)}\right|,$$

with $x \leq \dfrac{\mu}{1-c}$ or $x \geq -\dfrac{\mu}{c+1}$. This curve has two connected components corresponding with these two intervals of variation for x (see Figure 3).

Therefore, we have proved the following local classification result for quadratic rotational Weingarten surfaces:

Theorem 3.1. *The only rotational surfaces whose principal curvatures satisfy $k_m = \mu\, k_p^2$, $\mu \neq 0$, are (open subsets of) the plane, the sphere of radius $|\mu|$ and the rotational surfaces generated by the graphs $z = z_{\mu,c}(x)$, $c > 0$, described in* (3.4), (3.5) *and* (3.6).

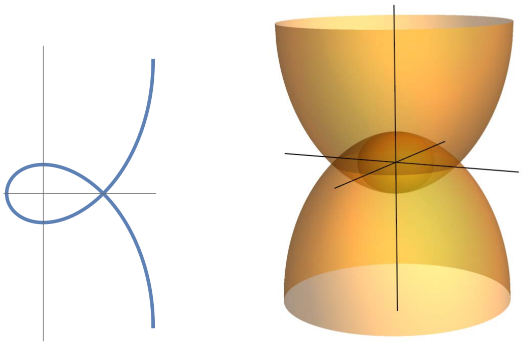

Figure 1. Curve $z_{\mu,c}$, $\mu > 0$, $0<c<1$, and the corresponding rotational surface ($\mu = 1, c = 0.5$).

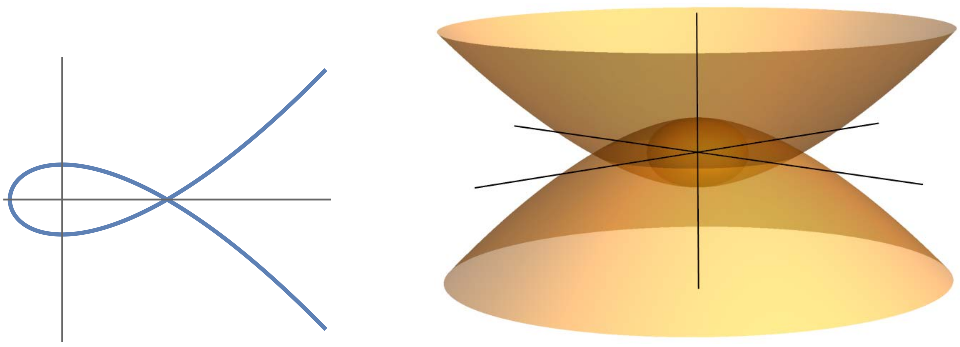

Figure 2. Curve $z_{\mu,1}$, $\mu > 0$, and the corresponding rotational surface ($\mu = 1$).

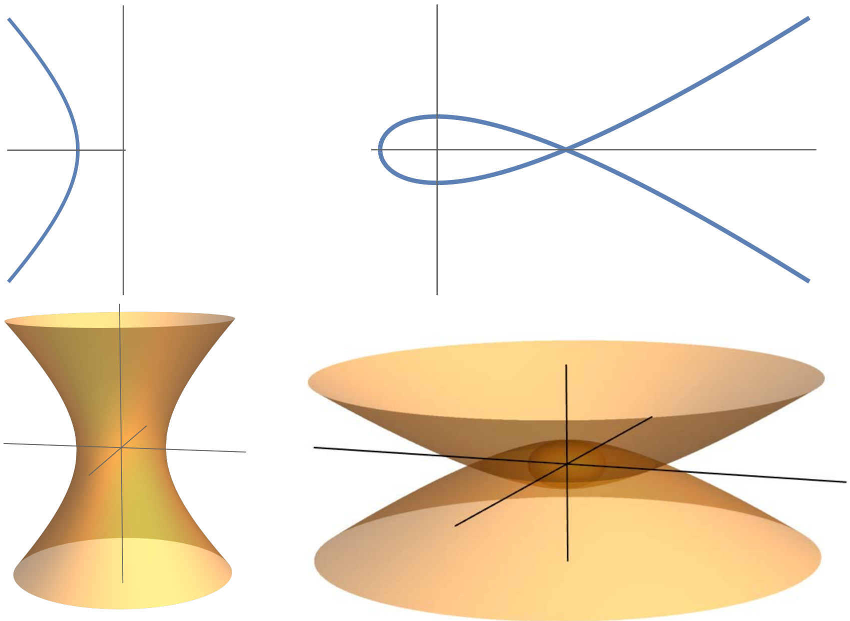

FIGURE 3. Curve $z_{\mu,c}$, $\mu > 0$, $c>1$ (left: $x \leq \dfrac{\mu}{1-c}$, right: $x \geq -\dfrac{\mu}{c+1}$), and the corresponding rotational surfaces ($\mu = 1, c = 1.5$).

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