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ON A CERTAIN CLASS OF ROTATIONAL HYPERSURFACES SATISFYING $\Delta \mathbf{x} = \mathcal{A}\mathbf{x}$ IN THE SIX-DIMENSIONAL EUCLIDEAN SPACE

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Abstract. A specific class of rotational hypersurfaces $\mathbf{x}$ with five parameters in the six-dimensional Euclidean space $\mathbb{E}^6$ is investigated. The curvature functions associated with these hypersurfaces are explicitly computed, and their geometric properties are examined. Furthermore, the action of the Laplace–Beltrami operator on such hypersurfaces is analyzed, and the conditions under which the relation $\Delta \mathbf{x} = \mathcal{A}\mathbf{x}$ holds for a 6×6 matrix $\mathcal{A}$ are determined.

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1. Introduction

The concept of finite type submanifolds immersed in $\mathbb{E}^m$ or $\mathbb{E}^m_\nu$ was introduced by Chen [6, 7, 8, 9] using a finite number of eigenfunctions of the Laplacian, and was further developed and expanded up to the present day.

A related result was given by Takahashi [29], stating that a Euclidean submanifold was of 1-type if and only if it was minimal or minimal in some hypersphere of $\mathbb{E}^m$. Closed spherical submanifolds of 2-type were studied by Barros and Chen [5], and Chen [7]. Takahashi's theorem was further examined in $\mathbb{E}^m$ by Garay [20]. Hypersurfaces with constant curvature were considered by Cheng and Yau [13], while submanifolds with finite type Gauss maps in $\mathbb{E}^m$ were investigated by Chen and Piccinni [11]. Hypersurfaces with pointwise 1-type Gauss maps in $(n+1)$-dimensional space were introduced by Dursun [17]. The geometry of submanifolds was presented by Aminov [2].

In $\mathbb{E}^3$, minimal surfaces and spheres were characterized by Takahashi [29] as the only surfaces satisfying $\Delta r = \lambda r$, for some $\lambda \in \mathbb{R}$. The minimal helicoid possessing a pointwise 1-type Gauss map of the first kind was studied by Choi and Kim [14]. A class of finite type surfaces of revolution was obtained by Garay [19]. It was demonstrated by Dillen, Pas, and Verstraelen [15] that the only surfaces satisfying $\Delta r = Ar + B$, with $A \in \mathrm{Mat}(3,3)$ and $B \in \mathrm{Mat}(3,1)$, are minimal surfaces, spheres, and circular cylinders.

In $\mathbb{E}^4$, general rotational surfaces were investigated by Moore [27, 28]. Hypersurfaces possessing a harmonic mean curvature vector field were analyzed by Hasanis and Vlachos [23]. Complete hypersurfaces with constant mean curvature were studied by Cheng and Wan [12]. The fourth fundamental form and curvature formulas were investigated by Güler [21].

In the Minkowski 4-space $\mathbb{E}_1^4$, an analogue of the surfaces studied by Moore [27, 28] was presented by Ganchev and Milousheva [18]. The mean curvature vector field of M_1^3 satisfying $\Delta H = \alpha H$, for a constant α, was examined by Arvanitoyeorgos, Kaimakamis, and Magid [4]. Meridian surfaces of elliptic or hyperbolic type possessing a pointwise 1-type Gauss map were investigated by Arslan and Milousheva [3]. A family of right conoid hypersurfaces having a light-like axis was studied by Li, Güler, and Toda [26].

The twisted hypersurfaces in Euclidean 5-space $\mathbb{E}^5$ were studied by Li and Güler [25]. In addition, a family of helicoidal hypersurfaces in the same space was investigated by Güler [22].

A comprehensive survey of 1-type submanifolds and submanifolds with 1-type Gauss maps was provided by Chen et al. [10], covering developments over the last forty years.

In this work, a specific class of rotational hypersurfaces in six-dimensional Euclidean space $\mathbb{E}^6$ is investigated. In Section 2, the fundamental notions of six-dimensional Euclidean geometry are presented. In Section 3, curvature formulas for hypersurfaces in $\mathbb{E}^6$ are derived. The concept of a rotational hypersurface is formulated in Section 4. Finally, in the last section, rotational hypersurfaces satisfying the condition $\Delta \mathbf{x} = \mathcal{A}\mathbf{x}$, where $\mathcal{A}$ is a 6×6 matrix in $\mathbb{E}^6$, are introduced and examined.

2. Preliminaries

Let $\mathbb{E}^m$ denote the m-dimensional Euclidean space equipped with the standard Euclidean metric tensor $\widetilde{g} = \langle\ ,\ \rangle = \sum_{i=1}^{m} dx_i^2$, where $(x_1, x_2, \dots, x_m)$ is a global coordinate system on $\mathbb{E}^m$. Let M be an m-dimensional Riemannian submanifold of $\mathbb{E}^m$.

Let s_j denote the j-th elementary symmetric function of the principal curvatures $k_1, k_2, \dots, k_n$; that is,

$$s_j = \sigma_j(k_1, k_2, \dots, k_n),$$

where σ_j is defined by

$$\sigma_j(a_1, a_2, \dots, a_n) = \sum_{1 \le i_1 < i_2 < \dots < i_j \le n} a_{i_1} a_{i_2} \cdots a_{i_j}.$$

To express the elementary symmetric function excluding the i-th curvature, we introduce the notation

$$r_i^j = \sigma_j(k_1, k_2, \dots, k_{i-1}, k_{i+1}, \dots, k_n).$$

By definition, one has $r_i^0 = 1$, and $s_{n+1} = s_{n+2} = \dots = 0$.

Definition 2.1. The function s_k is referred to as the *k-th mean curvature* of the hypersurface M. In particular, the normalized mean curvature is given by $H = \frac{1}{n} s_1$, while the Gauss–Kronecker curvature is given by $K = s_n$. A hypersurface M for which $s_j \equiv 0$ is called a *j-minimal hypersurface.*

In $\mathbb{E}^{n+1}$, the i-th curvature functions $\mathfrak{C}_i$ (refer to [1] and [24] for further details), for $i = 0, 1, \ldots, n$, are obtained through the characteristic polynomial of the shape operator $\mathbf{S}$, given by

$$P_{\mathbf{S}}(\lambda) = \det(\mathbf{S} - \lambda \mathcal{I}_n) = \sum_{k=0}^{n} (-1)^k s_k \lambda^{n-k} = 0, \tag{2.1}$$

where $\mathcal{I}_n$ denotes the $n \times n$ identity matrix. Consequently, the curvature functions are related to the elementary symmetric functions by the formula $\binom{n}{i}\mathfrak{C}_i = s_i$.

An isometric immersion (M, x) into Euclidean space is said to be of *finite type* if the position vector field $x : M \longrightarrow \mathbb{E}^m$ can be expressed as a finite sum of eigenfunctions of the Laplacian Δ on M; that is,

$$x = x_0 + \sum_{i=1}^{k} x_i,$$

where x_0 is a constant map and $x_1, x_2, \ldots, x_k$ are non-constant maps satisfying $\Delta x_i = \lambda_i x_i$ for some $\lambda_i \in \mathbb{R}$, $i = 1, 2, \ldots, k$. If the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct, then the submanifold M is said to be of *k-type*. For more details, see [7].

Let $\mathbf{x} = \mathbf{x}(u, v, w, s, t)$ be an immersion from a 5-dimensional manifold $M^5 \subset \mathbb{E}^5$ into the six-dimensional Euclidean space $\mathbb{E}^6$. The quintuple vector product of five vectors $\overrightarrow{x} = (x_1, x_2, \ldots, x_6)$, $\overrightarrow{y} = (y_1, y_2, \ldots, y_6)$, $\overrightarrow{z} = (z_1, z_2, \ldots, z_6)$, $\overrightarrow{p} = (p_1, p_2, \ldots, p_6)$, and $\overrightarrow{q} = (q_1, q_2, \ldots, q_6)$ in $\mathbb{E}^6$ is defined by the determinant

$$\overrightarrow{x} \times \overrightarrow{y} \times \overrightarrow{z} \times \overrightarrow{p} \times \overrightarrow{q} = \begin{vmatrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ z_1 & z_2 & z_3 & z_4 & z_5 & z_6 \\ p_1 & p_2 & p_3 & p_4 & p_5 & p_6 \\ q_1 & q_2 & q_3 & q_4 & q_5 & q_6 \end{vmatrix}.$$

For a hypersurface $\mathbf{x}$ in $\mathbb{E}^6$, the first and second fundamental form matrices $(\mathbf{g}_{ij})_{5\times5}$ and $(\mathbf{h}_{ij})_{5\times5}$ are given, respectively, by

$$\mathbf{g}_{ij} = \langle \mathbf{x}_i, \mathbf{x}_j \rangle, \ \mathbf{h}_{ij} = \langle \mathbf{x}_{ij}, G \rangle, \quad i, j = 1, 2, \ldots, 5,$$

where $\mathbf{x}_i = \frac{\partial \mathbf{x}}{\partial u}$ when $i = 1$, $\mathbf{x}_{ij} = \frac{\partial^2 \mathbf{x}}{\partial u \partial v}$ when $i = 1$ and $j = 2$, and so on. The Gauss map G of the hypersurface is defined by

$$G = \frac{\mathbf{x}_u \times \mathbf{x}_v \times \mathbf{x}_w \times \mathbf{x}_s \times \mathbf{x}_t}{\|\mathbf{x}_u \times \mathbf{x}_v \times \mathbf{x}_w \times \mathbf{x}_s \times \mathbf{x}_t\|}, \tag{2.2}$$

where $\|\cdot\|$ denotes the standard Euclidean norm in $\mathbb{E}^6$.

3. Curvatures

In this section, we derive the curvature expressions for a general hypersurface given by $\mathbf{x} = \mathbf{x}(u, v, w, s, t)$ in the six-dimensional Euclidean space $\mathbb{E}^6$.

Theorem 3.1. *Let $\mathbf{x}$ be an immersion from a 5-dimensional manifold $M^5 \subset \mathbb{E}^5$ into the six-dimensional Euclidean space $\mathbb{E}^6$. Then, the curvature functions $\mathfrak{C}_i$ for $i = 0, 1, \ldots, 5$ satisfy the following relations*

$$\mathfrak{C}_0 = 1, \ 5\mathfrak{C}_1 = -\frac{\mathfrak{b}}{\mathfrak{a}}, \ 10\mathfrak{C}_2 = \frac{\mathfrak{c}}{\mathfrak{a}}, \ 10\mathfrak{C}_3 = -\frac{\mathfrak{d}}{\mathfrak{a}}, \ 5\mathfrak{C}_4 = \frac{\mathfrak{e}}{\mathfrak{a}}, \ \mathfrak{C}_5 = -\frac{\mathfrak{f}}{\mathfrak{a}}, \tag{3.1}$$

where the coefficients $\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d},\mathfrak{e},\mathfrak{f}$ *are obtained from the characteristic polynomial of the shape operator matrix* $\mathbf{S}$*, given by* $P_{\mathbf{S}}(\lambda) = \mathfrak{a}\lambda^5 + \mathfrak{b}\lambda^4 + \mathfrak{c}\lambda^3 + \mathfrak{d}\lambda^2 + \mathfrak{e}\lambda + \mathfrak{f} = 0$*. Here,* $\mathfrak{a}=\det(\mathbf{g}_{ij})$*,* $\mathfrak{f} = \det(\mathbf{h}_{ij})$*, where* $(\mathbf{g}_{ij})$ *and* $(\mathbf{h}_{ij})$ *denote the first and second fundamental form matrices of the hypersurface, respectively.*

Proof. The shape operator matrix $\mathbf{S}$ of the hypersurface $\mathbf{x}$ in $\mathbb{E}^6$ is obtained by multiplying the inverse of the first fundamental form matrix with the second fundamental form matrix. To determine the curvature functions $\mathfrak{C}_i$ for $i = 0, 1, \dots, 5$, we consider the characteristic polynomial associated with $\mathbf{S}$, given by

$$P_{\mathbf{S}}(\lambda) = \det(\mathbf{S} - \lambda\mathcal{I}_5) = \mathfrak{a}\lambda^5 + \mathfrak{b}\lambda^4 + \mathfrak{c}\lambda^3 + \mathfrak{d}\lambda^2 + \mathfrak{e}\lambda + \mathfrak{f} = 0,$$

where $\mathcal{I}_5$ indicates the 5×5 identity matrix. The coefficients $\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d},\mathfrak{e},\mathfrak{f}$ are derived from the entries of $(\mathbf{g}_{ij})$ and $(\mathbf{h}_{ij})$. These coefficients yield the curvature expressions in six-dimensional space. Accordingly, the mean curvature function is obtained as follows

$$\binom{5}{1}\mathfrak{C}_1 = k_1 + k_2 + k_3 + k_4 + k_5 = -\frac{\mathfrak{b}}{\mathfrak{a}}$$

□

For detailed treatments of the cases in $\mathbb{E}^4$ and $\mathbb{E}^5$, the reader is referred to the works of Güler [21], Li and Güler [25], respectively.

4. A Specific Rotational Hypersurface

In this section, the specific rotational hypersurface is introduced, and its differential geometric characteristics are investigated in the six-dimensional Euclidean space $\mathbb{E}^6$.

The rotational hypersurfaces in Riemannian space forms were presented by Do Carmo and Dajczer [16]. In Euclidean $(n+1)$-space, a rotational hypersurface M is constructed by rotating a profile curve γ, which does not intersect the axis ℓ, under the action of the orthogonal transformations that leave a fixed line $\mathfrak{r}$ invariant pointwise (see [16, Remark 2.3]).

The profile surface $\gamma(u, v) = (f(u, v), 0, g(u, v), 0, h(u, v), 0)$ is considered, and it is acted upon by the following block-diagonal rotation matrix

$$R = \operatorname{diag}\left(\; R_w, \quad R_s, \quad R_t \; \right),$$

where each submatrix R_j, for $j = w, s, t$, is defined as the planar rotation matrix

$$R_j = \begin{pmatrix} \cos j & -\sin j \\ \sin j & \cos j \end{pmatrix}.$$

Through this transformation, the rotational hypersurface is generated by the expression $\mathbf{x} = R.\gamma^T$, and its definition is presented below.

Definition 4.1. A *specific rotational hypersurface* $\mathbf{x} = \mathbf{x}(u, v, w, s, t)$ *in* $\mathbb{E}^6$ is defined by the parametrization

$$\mathbf{x} = (f \cos w, f \sin w, g \cos s, g \sin s, h \cos t, h \sin t)\,, \tag{4.1}$$

where the functions $f = f(u, v)$, $g = f(u, v)$, and $h = f(u, v)$ are assumed to be differentiable. The parameters, u, v vary over $\mathbb{R}$, while w, s, t are angular variables satisfying $0 \le w, s, t < 2\pi$.

By computing the first partial derivatives of the parametrization given in (4.1) with respect to the variables u, v, w, s, t, the components of the first fundamental form matrix $(\mathbf{g}_{ij})$ are obtained. The resulting matrix takes the form

$$(\mathbf{g}_{ij}) = \operatorname{diag}\left(\begin{array}{cccc} (\mathbf{g}_{ij})_{2\times 2} & \mathbf{g}_{33}, & \mathbf{g}_{44}, & \mathbf{g}_{55} \end{array} \right), \tag{4.2}$$

where the entries are given by

$$\begin{aligned}
\mathbf{g}_{11} &= f_u^2 + g_u^2 + h_u^2, \\
\mathbf{g}_{12} &= f_u f_v + g_u g_v + h_u h_v = \mathbf{g}_{21}, \\
\mathbf{g}_{22} &= f_v^2 + g_v^2 + h_v^2, \\
\mathbf{g}_{33} &= f^2, \\
\mathbf{g}_{44} &= g^2, \\
\mathbf{g}_{55} &= h^2.
\end{aligned}$$

with $W = \left(f_u^2 + g_u^2 + h_u^2\right)\left(f_v^2 + g_v^2 + h_v^2\right) - \left(f_u f_v + g_u g_v + h_u h_v\right)^2$, and the notations $f_u = \frac{\partial f}{\partial u}$, $f_v = \frac{\partial f}{\partial v}$, and similarly for g and h, are used. The determinant of the first fundamental form matrix $(\mathbf{g}_{ij})$ is computed as

$$\mathbf{g} = \det(\mathbf{g}_{ij}) = f^2 g^2 h^2 W.$$

By applying the Gauss map definition in (2.2) , the Gauss map G of the rotational hypersurface described in (4.1) is obtained as

$$G = (G_1 \cos w, G_1 \sin w, G_2 \cos s, G_2 \sin s, G_3 \cos t, G_3 \sin t), \tag{4.3}$$

where the component functions G_1, G_2, G_3 are given by

$$G_1 = \frac{g_v h_u - g_u h_v}{W^{1/2}}, \quad G_2 = \frac{f_u h_v - f_v h_u}{W^{1/2}}, \quad G_3 = \frac{f_v g_u - f_u g_v}{W^{1/2}}.$$

By utilizing the second-order partial derivatives of the rotational hypersurface described in (4.1), together with its Gauss map given in (4.3), the second fundamental form matrix $(\mathbf{h}_{ij})$ is obtained. This matrix takes the form

$$(\mathbf{h}_{ij}) = \operatorname{diag}\left(\begin{array}{cccc} (\mathbf{h}_{ij})_{2\times 2} & \mathbf{h}_{33}, & \mathbf{h}_{44}, & \mathbf{h}_{55} \end{array} \right), \tag{4.4}$$

with the entries computed as follows,

$$\begin{aligned}
\mathbf{h}_{11} &= G_1 f_{uu} + G_2 g_{uu} + G_3 h_{uu}, \\
\mathbf{h}_{12} &= G_1 f_{uv} + G_2 g_{uv} + G_3 h_{uv}, \\
\mathbf{h}_{22} &= G_1 f_{vv} + G_2 g_{vv} + G_3 h_{vv}, \\
\mathbf{h}_{33} &= G_1 f, \\
\mathbf{h}_{44} &= G_2 g, \\
\mathbf{h}_{55} &= G_3 h.
\end{aligned}$$

Accordingly, the determinant of the second fundamental form matrix is given by

$$\mathbf{h} = \det(\mathbf{h}_{ij}) = fghG_1G_2G_3\left(\mathbf{h}_{12}^2 - \mathbf{h}_{11}\mathbf{h}_{22}\right).$$

The shape operator matrix $\mathbf{S}$ associated with the rotational hypersurface defined in (4.1) is obtained by utilizing the first and second fundamental form matrices provided in (4.2) and (4.4), respectively. It is expressed as

$$\mathbf{S} = \operatorname{diag}\begin{pmatrix} \dfrac{\mathbf{g}_{22}\mathbf{h}_{11} - \mathbf{g}_{12}\mathbf{h}_{12}}{\mathbf{g}_{11}\mathbf{g}_{22} - \mathbf{g}_{12}^2} & \dfrac{\mathbf{g}_{22}\mathbf{h}_{12} - \mathbf{g}_{12}\mathbf{h}_{22}}{\mathbf{g}_{11}\mathbf{g}_{22} - \mathbf{g}_{12}^2} \\ \dfrac{\mathbf{g}_{11}\mathbf{h}_{12} - \mathbf{g}_{12}\mathbf{h}_{11}}{\mathbf{g}_{11}\mathbf{g}_{22} - \mathbf{g}_{12}^2} & \dfrac{\mathbf{g}_{11}\mathbf{h}_{22} - \mathbf{g}_{12}\mathbf{h}_{12}}{\mathbf{g}_{11}\mathbf{g}_{22} - \mathbf{g}_{12}^2} \end{pmatrix}, \frac{\mathbf{h}_{33}}{\mathbf{g}_{33}}, \frac{\mathbf{h}_{44}}{\mathbf{g}_{44}}, \frac{\mathbf{h}_{55}}{\mathbf{g}_{55}} \Bigg).$$

Finally, by applying the curvature identities provided in (3.1), together with the components of the first and second fundamental form matrices given in (4.2) and (4.4), respectively, the curvature functions of the trotational hypersurface described in (4.1) are determined as follows.

Theorem 4.2. *Let* $\mathbf{x} : M^5 \longrightarrow \mathbb{E}^6$ *be the immersion defined by the parametrization in* (4.1)*. Then, the hypersurface* $\mathbf{x}$ *is characterized by the following curvature function*

$$5\mathfrak{C}_1 = \frac{\left(f^2g^2\mathbf{h}_{55} + f^2h^2\mathbf{h}_{44} + g^2h^2\mathbf{h}_{33}\right)\left(\mathbf{g}_{11}\mathbf{g}_{22} - \mathbf{g}_{12}^2\right) + f^2g^2h^2\left(\mathbf{g}_{22}\mathbf{h}_{11} - 2\mathbf{g}_{12}\mathbf{h}_{12} + \mathbf{g}_{11}\mathbf{h}_{22}\right)}{f^2g^2h^2\left(\mathbf{g}_{11}\mathbf{g}_{22} - \mathbf{g}_{12}^2\right)}$$

where $f^2g^2h^2\left(\mathbf{g}_{11}\mathbf{g}_{22} - \mathbf{g}_{12}^2\right) \neq 0$.

Proof. The result follows from computing the trace of the shape operator: $\mathfrak{C}_1 = \frac{1}{5}\operatorname{Trace}(\mathbf{S})$. □

Theorem 4.3. *Let* $\mathbf{x} : M^5 \longrightarrow \mathbb{E}^6$ *be the immersion defined by the parametrization in* (4.1)*. If the numerator of the curvature function vanishes, that is,*

$$\left(f^2g^2\mathbf{h}_{55} + f^2h^2\mathbf{h}_{44} + g^2h^2\mathbf{h}_{33}\right)\left(\mathbf{g}_{11}\mathbf{g}_{22} - \mathbf{g}_{12}^2\right) + f^2g^2h^2\left(\mathbf{g}_{22}\mathbf{h}_{11} - 2\mathbf{g}_{12}\mathbf{h}_{12} + \mathbf{g}_{11}\mathbf{h}_{22}\right) = 0,$$

then the hypersurface $\mathbf{x}$ *is* 1-minimal*; that is, the mean curvature vanishes identically:* $\mathfrak{C}_1 = 0$.

Hence, the following example is presented to illustrate the result.

Example 4.4. Let $\mathbf{x} : M^5 \longrightarrow \mathbb{E}^6$ be the immersion defined by the parametrization given in (4.1). If the profile surface $\gamma(u, v)$ of $\mathbf{x}$ is parametrized on the unit sphere by

$$f(u, v) = \cos u \cos v, \quad g(u, v) = \sin u \cos v, \quad h(u, v) = \sin v,$$

then the shape operator matrix becomes the identity matrix, that is, $\mathbf{S} = \mathcal{I}_5$. Consequently, the rotational hypersurface has constant mean curvature given by $\mathfrak{C}_1 = 1$.

5. A Specific Rotational Hypersurface Satisfying $\Delta\mathbf{x} = \mathcal{A}\mathbf{x}$ in $\mathbb{E}^6$

In this section, the Laplace–Beltrami operator associated with a smooth function defined on a hypersurface in $\mathbb{E}^6$ is presented, and its explicit expression is computed in the context of a rotational hypersurface.

By employing the inverse of the first fundamental form matrix $(\mathbf{g}_{ij})_{5\times5}$, and denoting its determinant by $\mathbf{g} = \det(\mathbf{g}_{ij})$, the Laplace–Beltrami operator can be formulated as follows.

Definition 5.1. Let $\phi = \phi(x^1, x^2, x^3, x^4, x^5)$ be a smooth function of class C^5, defined on domain $\mathbf{D} \subset \mathbb{R}^5$. The *Laplace–Beltrami operator* $\Delta\phi$, depending on the first fundamental form, is defined by

$$\Delta\phi = \frac{1}{\mathbf{g}^{1/2}} \sum_{i,j=1}^{5} \frac{\partial}{\partial x^i}\left(\mathbf{g}^{1/2}\mathbf{g}^{ij}\frac{\partial\phi}{\partial x^j}\right), \tag{5.1}$$

where $(\mathbf{g}^{ij})$ is the inverse matrix of $(\mathbf{g}_{ij})$, and $\mathbf{g} = \det(\mathbf{g}_{ij})$.

For the tri-rotational hypersurface given in (4.1), the first fundamental form matrix $(\mathbf{g}_{ij})$ is diagonal except for the non-zero off-diagonal terms $\mathbf{g}_{12} = \mathbf{g}_{21} \neq 0$. Consequently, the Laplace–Beltrami operator for the parametrization $\mathbf{x} = \mathbf{x}(u, v, w, s, t)$ takes the form

$$\Delta \mathbf{x} = \frac{1}{|\mathbf{g}|^{1/2}} \begin{pmatrix} \frac{\partial}{\partial u}\left(|\mathbf{g}|^{1/2}\,\mathbf{g}^{11}\frac{\partial \mathbf{x}}{\partial u}\right) - \frac{\partial}{\partial u}\left(|\mathbf{g}|^{1/2}\,\mathbf{g}^{12}\frac{\partial \mathbf{x}}{\partial v}\right) \\ -\frac{\partial}{\partial v}\left(|\mathbf{g}|^{1/2}\,\mathbf{g}^{21}\frac{\partial \mathbf{x}}{\partial u}\right) + \frac{\partial}{\partial v}\left(|\mathbf{g}|^{1/2}\,\mathbf{g}^{22}\frac{\partial \mathbf{x}}{\partial v}\right) \\ +\frac{\partial}{\partial w}\left(|\mathbf{g}|^{1/2}\,\mathbf{g}^{33}\frac{\partial \mathbf{x}}{\partial w}\right) + \frac{\partial}{\partial s}\left(|\mathbf{g}|^{1/2}\,\mathbf{g}^{44}\frac{\partial \mathbf{x}}{\partial s}\right) \\ +\frac{\partial}{\partial t}\left(|\mathbf{g}|^{1/2}\,\mathbf{g}^{55}\frac{\partial \mathbf{x}}{\partial t}\right) \end{pmatrix}. \tag{5.2}$$

By computing the partial derivatives of the terms appearing in (5.2), with respect to u, v, w, s, and t, the explicit form of $\Delta \mathbf{x}$ is obtained.

Theorem 5.2. *The Laplace–Beltrami operator $\Delta \mathbf{x}$ of the tri-rotational hypersurface defined in* (4.1) *is expressed as*

$$\Delta \mathbf{x} = (\mathfrak{F}\cos w, \mathfrak{F}\sin w, \mathfrak{G}\cos s, \mathfrak{G}\sin s, \mathfrak{H}\cos t, \mathfrak{H}\sin t),$$

where the functions $\mathfrak{F} = \mathfrak{F}(u, v)$, $\mathfrak{G} = \mathfrak{G}(u, v)$, and $\mathfrak{H} = \mathfrak{H}(u, v)$ are given by

$$\begin{aligned} \mathfrak{F}(u,v) &= \left(\frac{\mathbf{g}_u}{2\mathbf{g}}\mathbf{g}^{11} + \left(\mathbf{g}^{11}\right)_u\right) f_u - \left(\left(\frac{\mathbf{g}_u + \mathbf{g}_v}{2\mathbf{g}}\right)\mathbf{g}^{12} + \left(\mathbf{g}^{12}\right)_u + \left(\mathbf{g}^{12}\right)_v - \frac{\mathbf{g}_v}{2\mathbf{g}}\mathbf{g}^{22} - \left(\mathbf{g}^{22}\right)_v\right) f_v \\ &\quad + \mathbf{g}^{11} f_{uu} - 2\mathbf{g}^{12} f_{uv} + \mathbf{g}^{22} f_{vv} - \frac{1}{f}, \\ \mathfrak{G}(u,v) &= \left(\frac{\mathbf{g}_u}{2\mathbf{g}}\mathbf{g}^{11} + \left(\mathbf{g}^{11}\right)_u\right) g_u - \left(\left(\frac{\mathbf{g}_u + \mathbf{g}_v}{2\mathbf{g}}\right)\mathbf{g}^{12} + \left(\mathbf{g}^{12}\right)_u + \left(\mathbf{g}^{12}\right)_v - \frac{\mathbf{g}_v}{2\mathbf{g}}\mathbf{g}^{22} - \left(\mathbf{g}^{22}\right)_v\right) g_v \\ &\quad + \mathbf{g}^{11} g_{uu} - 2\mathbf{g}^{12} g_{uv} + \mathbf{g}^{22} g_{vv} - \frac{1}{g}, \\ \mathfrak{H}(u,v) &= \left(\frac{\mathbf{g}_u}{2\mathbf{g}}\mathbf{g}^{11} + \left(\mathbf{g}^{11}\right)_u\right) h_u - \left(\left(\frac{\mathbf{g}_u + \mathbf{g}_v}{2\mathbf{g}}\right)\mathbf{g}^{12} + \left(\mathbf{g}^{12}\right)_u + \left(\mathbf{g}^{12}\right)_v - \frac{\mathbf{g}_v}{2\mathbf{g}}\mathbf{g}^{22} - \left(\mathbf{g}^{22}\right)_v\right) h_v \\ &\quad + \mathbf{g}^{11} h_{uu} - 2\mathbf{g}^{12} h_{uv} + \mathbf{g}^{22} h_{vv} - \frac{1}{h}. \end{aligned}$$

$\mathbf{g}^{ij}$ *are the components of the inverse matrix of* $(\mathbf{g}_{ij})$, *and* $\mathbf{g} = \det(\mathbf{g}_{ij})$.

Proof. By performing a direct computation of the Laplace–Beltrami operator as given in (5.2), the expressions for the functions $\mathfrak{F}(u, v)$, $\mathfrak{G}(u, v)$, and $\mathfrak{H}(u, v)$ can be simplified to the following clearer forms

$$\begin{aligned} \mathfrak{F}(u,v) &= \boldsymbol{\Omega} f_u - \boldsymbol{\Psi} f_v + \mathbf{g}_{11} f_{uu} - 2\mathbf{g}_{12} f_{uv} + \mathbf{g}_{22} f_{vv} - \frac{1}{f}, \\ \mathfrak{G}(u,v) &= \boldsymbol{\Omega} g_u - \boldsymbol{\Psi} g_v + \mathbf{g}_{11} g_{uu} - 2\mathbf{g}_{12} g_{uv} + \mathbf{g}_{22} g_{vv} - \frac{1}{g}, \\ \mathfrak{H}(u,v) &= \boldsymbol{\Omega} h_u - \boldsymbol{\Psi} h_v + \mathbf{g}_{11} h_{uu} - 2\mathbf{g}_{12} h_{uv} + \mathbf{g}_{22} h_{vv} - \frac{1}{h}, \end{aligned}$$

where

$$\boldsymbol{\Omega} = 2\zeta_6 + \left[\eta_2 + W^{-1}\left(\zeta_5 \mathbf{g}_{11} - (\zeta_3 + \zeta_4)\,\mathbf{g}_{12} + \zeta_6 \mathbf{g}_{22}\right)\right] \mathbf{g}_{11},$$

$$
\begin{aligned}
\mathbf{\Psi} &= \zeta_1-\zeta_2+\zeta_3+\zeta_4+\zeta_5+\eta_1+\eta_2 \\
&\quad -\left[\eta_1+W^{-1}\left(\zeta_2\mathbf{g}_{11}-(\zeta_1+\zeta_5)\,\mathbf{g}_{12}+\zeta_4\mathbf{g}_{22}\right)\right]\mathbf{g}_{22} \\
&\quad +W^{-1}\left[\begin{pmatrix} (\zeta_2+\zeta_5)\,\mathbf{g}_{11} \\ -(\zeta_1+\zeta_3+\zeta_4+\zeta_5)\,\mathbf{g}_{12} \\ +(\zeta_4+\zeta_6)\,\mathbf{g}_{22} \end{pmatrix}\right]\mathbf{g}_{12},
\end{aligned}
$$

$$
\begin{aligned}
W &= \mathbf{g}_{11}\mathbf{g}_{22}-(\mathbf{g}_{12})^2 \\
\mathbf{g}_{11} &= f_u^2+g_u^2+h_u^2, \\
\mathbf{g}_{12} &= f_uf_v+g_ug_v+h_uh_v=\mathbf{g}_{21}, \\
\mathbf{g}_{22} &= f_v^2+g_v^2+h_v^2, \\
\mathbf{g}_{33} &= f^2,\ \mathbf{g}_{44}=g^2,\ \mathbf{g}_{55}=h^2,
\end{aligned}
$$

$$
\begin{aligned}
\zeta_1 &= f_uf_{vv}+g_ug_{vv}+h_uh_{vv}, \\
\zeta_2 &= f_vf_{vv}+g_vg_{vv}+h_vh_{vv}, \\
\zeta_3 &= f_vf_{uu}+g_vg_{uu}+h_vh_{uu}, \\
\zeta_4 &= f_uf_{uv}+g_ug_{uv}+h_uh_{uv}, \\
\zeta_5 &= f_vf_{uv}+g_vg_{uv}+h_vh_{uv}, \\
\zeta_6 &= f_uf_{uu}+g_ug_{uu}+h_uh_{uu}, \\
\eta_1 &= \frac{f_v}{f}+\frac{g_v}{g}+\frac{h_v}{h}, \\
\eta_2 &= \frac{f_u}{f}+\frac{g_u}{g}+\frac{h_u}{h}.
\end{aligned}
$$

□

Therefore, we have the following.

Corollary 5.3. *Let* $\mathbf{x}: M^5 \longrightarrow \mathbb{E}^6$ *be a hypersurface immersion with position vector field* $\mathbf{x}$*, and let* $\mathfrak{C}_1$ *denote its first mean curvature. If* G *is the Gauss map of* $\mathbf{x}$ *as defined in* (2.2)*, then the Laplacian of* $\mathbf{x}$ *satisfies the identity*

$$\Delta\mathbf{x} = -5\mathfrak{C}_1\,G.$$

Example 5.4. Let $\mathbf{x}: M^5 \longrightarrow \mathbb{E}^6$ be the immersion defined as in (4.1). If the generating surface $\gamma(u,v)$ of $\mathbf{x}$ is parametrized by the unit sphere via

$$f(u,v)=\cos u\cos v,\quad g(u,v)=\sin u\cos v,\quad h(u,v)=\sin v,$$

then the rotational hypersurface satisfies $\Delta\mathbf{x}=\mathcal{A}\mathbf{x},$ where $\mathcal{A}=-5\mathcal{I}_6,$ and $\mathcal{I}_6$ denotes the identity matrix of order six.

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